(1)

On invariant theory for "coincidental" reflection groups

1. Invariant theory
2. Reflection groups
3. Relative invariants
4. CONJ / THM
5. Sample COROLLARY

---

1. Invariant theory...

... examines $W < GL(V) \circlearrowright S = k[\underbrace{x_1, \ldots, x_n}_{\substack{\text{dual basis}\\ \text{for } V^*}}] = \mathrm{Sym}(V^*)$

$V = k^n$ (basis $y_1, \ldots, y_n$), $GL(V) \cong GL_n(k)$

where $g \in W$ acts via $f(\underline{x}) \overset{g}{\longmapsto} g(f)(\underline{x}) = f(g^{-1}\underline{x})$

and the W-fixed or W-invariant subalgebra $S^W = \{ f \in S : g(f) = f \ \forall g \in W \} \hookrightarrow S$.

When W is finite, Hilbert, Noether showed (early 1900's)

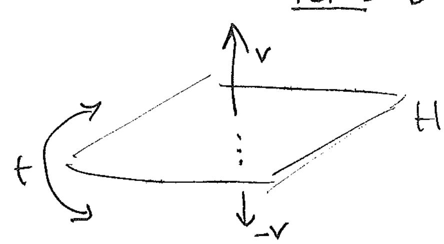
$$\begin{array}{ccc} S^W & \xrightarrow{\text{finite (integral)}} & S \\ \| & & \| \\ k[V/W] & & k[V] \\ V/W & \xleftarrow{\pi} & V \end{array}$$

, S^W is finitely gen'd $S^W \cong k[f_1, \ldots, f_m]/I$, $m \geq n$, I = syzygies, relations

THM (Shephard-Todd 1955, Chevalley 1955) For W finite, and $\mathrm{char}(k) = 0$, one has $S^W = k[f_1, \ldots, f_n]$ polynomial (i.e. $m = n$ and no relations)

$\iff$ W is a reflection group, i.e. gen'd by refl'ns t having $V^t = H$ a hyperplane

e.g.



if $k = \mathbb{R}$

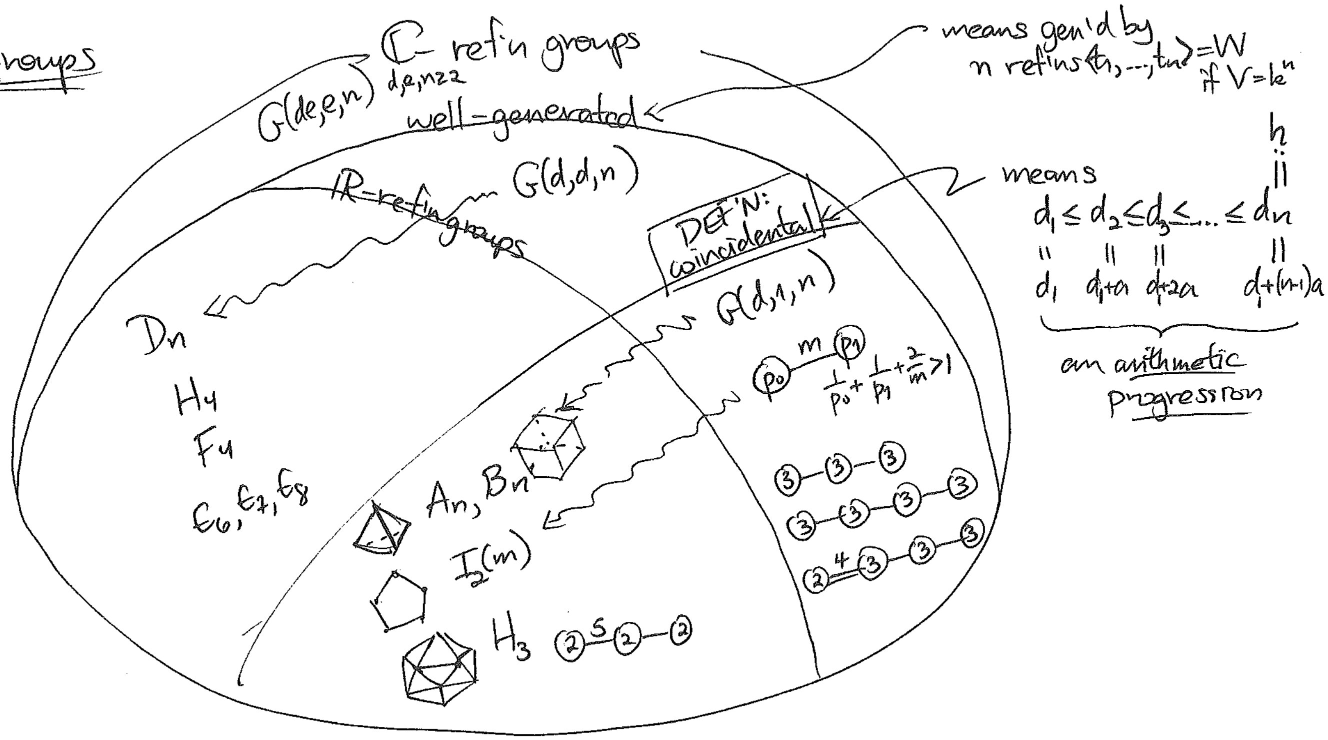
or $t \sim \begin{bmatrix} e^{\frac{2\pi i}{d}} & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$ if $k = \mathbb{C}$

In this case, can choose $f_1, \ldots, f_n$ homogeneous (basic invariants)

and their degrees $d_1 \leq \ldots \leq d_n$ control huge swaths of topology/geometry/combinatorics of W...

2. Refl'n groups

a taxonomy: (Shephard-Todd 1955)



3. Relative invariants

What about the rest of $W \circlearrowright S = S^W \oplus \bigoplus_{\substack{W\text{-irreducibles}\\ U}} \overbrace{S^{W,U}}^{U\text{-isotypic component of } S}$?

if $\mathrm{char}(k)=0$

THM (Eagon-Hochster 1971) if $\mathrm{char}(k)=0$, for any finite W, S^W is a Cohen-Macaulay ring and each $S^{W,U}$ is a C-M S^W-module

(uses S^W-mod splitting $S^W \hookrightarrow S$, $\xleftarrow{\pi = \text{Reynolds operator}}$

$$\pi(f) = \frac{1}{|W|}\sum_{g\in W} g(f)\ )$$

$\Downarrow$

COR For refl'n groups $W < GL_n(\mathbb{C})$, each $S^{W,U}$ is a free S^W-module.
$S^W = \mathbb{C}[f_1,\ldots,f_n]$.

Similarly, for any W-rep'n U, not nec. irreducible
$\mathrm{Hom}_W(U,S) \cong (S\otimes U^*)^W$ are free S^W-modules

PROBLEM: Compute an explicit homogeneous S^W-basis $\{b_1,\ldots,b_t\}$ for $M=(S\otimes U^*)^W$, thereby giving

$$\mathrm{Hilb}(M,q) := \sum_{d\geq 0} \dim_k M_d \cdot q^d = \mathrm{Hilb}(S^W,q)\cdot \sum_{i=1}^t q^{\deg(b_i)}$$

$$= \frac{\sum_{i=1}^t q^{\deg(b_i)}}{\prod_{i=1}^n (1-q^{d_i})}$$

EXAMPLES: ①

~~A~~ famous solved instance...

THM (Solomon 1963) For any $\mathbb{C}$-refl'n group W,
$$(S\otimes\wedge^{\bullet}V^*)^W \cong \wedge_{S^W}\{df_1,\dots,df_n\}$$
exterior algebra over S^W

where $df := \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\otimes x_i \in S\otimes\wedge^1 V^*$

i.e. $(S\otimes\wedge^k V^*)^W$ has free S^W-basis $\{df_{i_1}\wedge\cdots\wedge df_{i_k}\}_{1\le i_1<\cdots<i_k\le n}$

② Any S^W-basis $\{\theta_1,\dots,\theta_n\}$ for $(S\otimes V)^W$ are called basic derivations;

their S-degrees $e_1^*\le\dots\le e_n^*$ are called co-exponents for W

FACTS: (a) For $\mathbb{R}$-refl'n groups, $e_i^*=e_i=d_i-1$

(b) For well-gen'd $\mathbb{C}$-refl'n groups, $e_i^*=h-e_{n-i+1}$

③ In 2016, Shepler & I gave an S^W-basis for $(S\otimes\wedge^{\bullet}V^*\otimes V)^W$ for all well-gen'd $\mathbb{C}$-refl'n groups

4. CONJ (Shepler-Sommers-R. 2018) For W a coincidental $\mathbb{C}$-refl'n group, one can choose basic derivations $\theta_1,\dots,\theta_n\in(S\otimes V)^W \cong (S\otimes\wedge^{\bullet}V^*\otimes V)^W$

so that $(S\otimes\wedge^{\bullet}V^*\otimes\wedge^r V)^W$ is a free R-module on R-basis
$$R = \wedge_{S^W}\{df_1,\dots,df_{n-r}\}$$
$$\left\{\tilde\theta_{i_1}\cdots\tilde\theta_{i_k}\left(\theta_{j_1}\wedge\cdots\wedge\theta_{j_r}\right)\right\}_{\substack{1\le i_1<\cdots<i_k<r\\ 1\le j_1<\cdots<j_r\le n}}$$

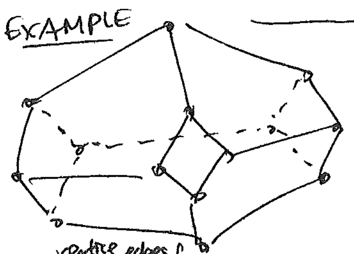
in which $\theta=\sum_{\ell=1}^{n}\theta^{(\ell)}(x)\otimes y_\ell$ has
$$\tilde\theta\big(f(x)\otimes x_I\otimes y_J\big) := \sum_{\ell=1}^{n}\theta^{(\ell)}\left(\tfrac{\partial}{\partial x_1},\dots,\tfrac{\partial}{\partial x_n}\right)(f)\otimes x_\ell\wedge x_I\otimes y_J$$
$x_I = x_{i_1}\wedge\cdots\wedge x_{i_p}$, $y_J = y_{j_1}\wedge\cdots\wedge y_{j_q}$

THM (SSR 2018) This CONJ does give the correct Hilbert series!

5. Sample COROLLARY: For real coincidental W, the f-vector of the W-associahedron has product formula
$$f_r = \binom{n}{r}\frac{\prod_{j=0}^{n-r-1}(h+d_1+(i-1)a)\cdot\prod_{j=0}^{r-1}(h-ia)}{\prod_{j=0}^{n-1}(d_1+a)}$$

Armstrong-Rhoades-R. 2012
$$= \lim_{q\to1}\left[\mathrm{Hilb}\left((S\otimes\wedge^{\bullet}V^*\otimes\wedge^r V)^W; q,t\right)\right]_{t=-q^{h+1}}$$

EXAMPLE



vertices edges faces

$f=(f_0,f_1,f_2,f_3) = (14,21,9,1)$ has $f_r=\binom{3}{r}\frac{\prod_{i=0}^{2-r}(6+i)\prod_{i=0}^{r-1}(4-i)}{2\cdot3\cdot4}$

$W=A_3$, $(d_1,d_2,d_3)=(2,3,4)$, $h=d_3$

$f_0=\binom{3}{0}\cdot\frac{6\cdot7\cdot8}{2\cdot3\cdot4}=14$

$f_1=\binom{3}{1}\cdot\frac{(6\cdot7)(4)}{2\cdot3\cdot4}=21$

$f_2=\binom{3}{2}\cdot\frac{(6)(4\cdot3)}{2\cdot3\cdot4}=9$

$f_3=\binom{3}{3}\frac{4\cdot3\cdot2}{2\cdot3\cdot4}=1$