Resolutions of Stanley-Reisner rings and a colorful Hochster formula

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- Hochster's Formula
- Colorful generalization
- CONJECTURE on canonical parameters

(If time permits ...

- CONJECTURE on depth sensitivity)

Hochster's Formula

Δ a simplicial complex on vertices $\{1,2,\dots,n\}$

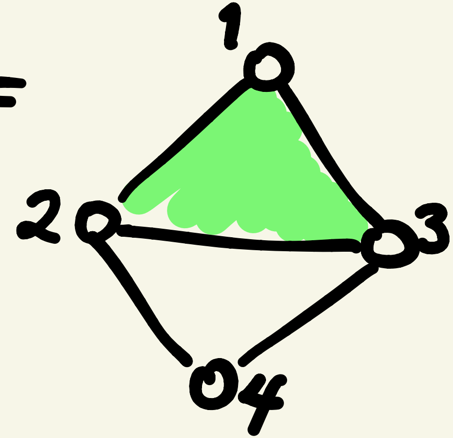
$\rightsquigarrow$ Stanley-Reisner ring

$$k[\Delta] := k[x_1,\dots,x_n] \big/ I_\Delta$$

ideal generated by $x^G := \prod_{i\in G} x_i$ for $G \notin \Delta$

EXAMPLE

$\Delta =$



$$k[\Delta] = k[x_1, x_2, x_3, x_4] / (x_1 x_4, x_2 x_3 x_4)$$

$$= k[x] / I_\Delta$$

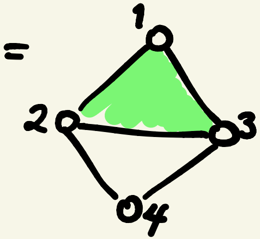
Regarding $k[\Delta]$ as a $k[\underline{x}]$-module, can write down a **minimal free resolution**

$$0 \leftarrow k[\Delta] \leftarrow k[\underline{x}] \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_n \leftarrow 0$$

which is **$\mathbb{N}^n$-graded**:

$$F_i = \bigoplus_{\alpha \in \mathbb{N}^n} k[\underline{x}](-\underline{x}^\alpha)^{\beta_{i,\alpha}}$$

where $\beta_{i,\alpha} = \dim_k \operatorname{Tor}_i^{k[\underline{x}]}(k[\Delta], k)_\alpha$

EXAMPLE $\Delta =$ 

$$0 \leftarrow k[\Delta] \leftarrow k[\underline{x}] \leftarrow \begin{matrix} k[\underline{x}](-x_1x_4) \\ \oplus \\ k[\underline{x}](-x_2x_3x_4) \end{matrix} \leftarrow k[\underline{x}](-x_1x_2x_3x_4) \leftarrow 0$$

Tor_0 $\qquad$ Tor_1 $\qquad$ Tor_2

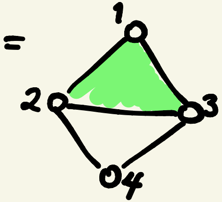
THEOREM (Hochster 1975)

$\mathrm{Tor}_i^{k[\underline{x}]}(k[\Delta], k)_\alpha$ vanishes unless $\underline{x}^\alpha = \underline{x}^S$ is squarefree, in which case

$$\mathrm{Tor}_i^{k[\underline{x}]}(k[\Delta], k)_S \cong \tilde{H}^{|S|-i-1}(\Delta|_S, k)$$

Δ restricted to vertices in S

EXAMPLE

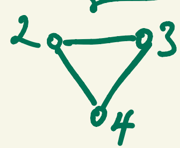
$\Delta =$ 

$$0 \leftarrow k[\Delta] \leftarrow k[\underline{x}] \leftarrow \begin{array}{c} k[\underline{x}](-x_1x_4) \\ \oplus \\ k[\underline{x}](-x_2x_3x_4) \end{array} \leftarrow k[\underline{x}](-x_1x_2x_3x_4) \leftarrow 0$$

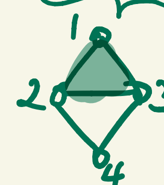
$$\tilde{H}^{0}(\underbrace{\Delta|_{\{1,4\}}}, k) = k$$

$$\tilde{H}^{-1}(\Delta|_{\underbrace{\emptyset}_{\{\emptyset\}}}, k) = k$$

$$\tilde{H}^{1}(\underbrace{\Delta|_{\{2,3,4\}}}, k) = k$$



$$\tilde{H}^{1}(\underbrace{\Delta|_{\{1,2,3,4\}}}, k) = k$$



It gives an $\mathbb{N}^M$-graded Hilbert series calculation

EXAMPLE

$$0 \leftarrow k[\Delta] \leftarrow k[\underline{x}] \leftarrow \begin{matrix} k[\underline{x}](-x_1x_4) \\ \oplus \\ k[\underline{x}](-x_2x_3x_4) \end{matrix} \leftarrow k[\underline{x}](-x_1x_2x_3x_4) \leftarrow 0$$

$$\Downarrow$$

$$\mathrm{Hilb}(k[\Delta], \underline{x}) = \frac{1 - x_1x_4 - x_2x_3x_4 + x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}$$

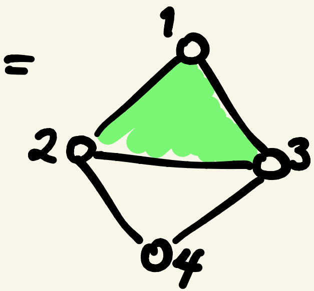
… which one can specialize, e.g. to a $\mathbb{N}$-grading, via $x_i \mapsto t$ for $i = 1, \dots, n$

$$\mathrm{Hilb}(k[\Delta], \underline{x}) = \frac{1 - x_1x_4 - x_2x_3x_4 + x_1x_2x_3x_4}{(1-x_1)(1-x_2)(1-x_3)(1-x_4)}$$

$$\Downarrow$$

$$\mathrm{Hilb}(k[\Delta], t) = \frac{1-t^2-t^3+t^4}{(1-t)^4}$$

$$= \frac{1+t-t^3}{(1-t)^3}$$

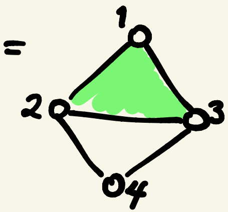
$\Delta =$ 

One issue:

The $k[\underline{x}]$-resolution of $k[\Delta]$ has length

$$n - \operatorname{depth} k[\Delta] \geq n - \dim k[\Delta]$$

EXAMPLE

$\Delta =$ 

$$0 \leftarrow k[\Delta] \leftarrow k[\underline{x}] \leftarrow \begin{matrix} k[\underline{x}](-x_1x_4) \\ \oplus \\ k[\underline{x}](-x_2x_3x_4) \end{matrix} \leftarrow k[\underline{x}](-x_1x_2x_3x_4) \leftarrow 0$$

0 1 $2 = n - \operatorname{depth} k[\Delta] = 4 - 2$

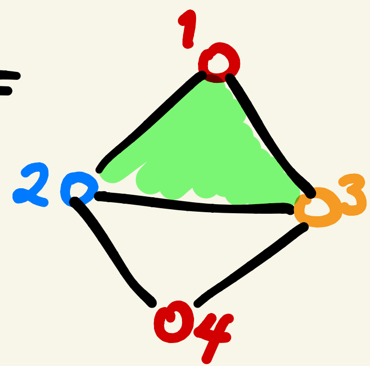
Colorful generalization

What if Δ has a proper d- coloring of its vertices ?

no two endpoints of an edge get same color

$\Rightarrow \dim k[\Delta] \leq d \leq n$

EXAMPLE

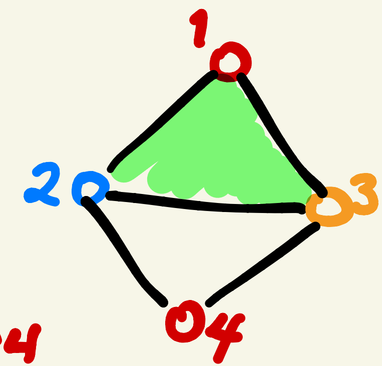
$\Delta =$ 

$d=3$ colors red blue orange

Then $k[\Delta]$ is finitely generated as a module over $A = k[z_1, z_2, \dots, z_d]$ via $z_i \mapsto \theta_i := \sum_{x_j \text{ of color } i} x_j$

by the squarefree monomials

EXAMPLE

$\Delta =$ 

$\theta_1 = x_1 + x_4$

$\theta_2 = x_2$

$\theta_3 = x_3$

$$x_1^a x_2^b x_3^c = x_1 x_2 x_3 \cdot \theta_1^{a-1} \theta_2^{b-1} \theta_3^{c-1}$$

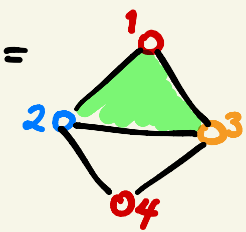
$$x_2^a x_4^b = x_2 x_4 \cdot \theta_2^{a-1} \theta_1^{b-1}$$

$k[\Delta]$ is an $\mathbb{N}^d$-graded Λ-module, with a minimal free Λ-resolution.

EXAMPLE

$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-z_1) \end{matrix} \longleftarrow A(-z_1 z_2 z_3) \leftarrow 0$$

Tor_0 $\qquad\qquad$ Tor_1

$\Delta =$ 

$z_1 \longmapsto \theta_1 = x_1 + x_4$

$z_2 \longmapsto \theta_2 = x_2$

$z_3 \longmapsto \theta_3 = x_3$

THEOREM (Adams–R.) (Colorful Hochster formula)

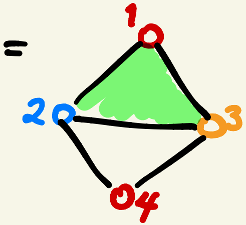
For $\alpha \in \mathbb{N}^d$, $\mathrm{Tor}_i^A(k[\Delta], k)_\alpha$ vanishes unless $\underline{z}^\alpha = \underline{z}^S$ for some $S \subseteq \{1, 2, \ldots, d\}$, in which case

$$\mathrm{Tor}_i^A(k[\Delta], k)_S \cong \tilde{H}_i^{|S|-i-1}(\Delta|_S, k)$$

Δ restricted to vertices whose color lies in S

EXAMPLE

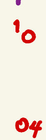
$\Delta =$



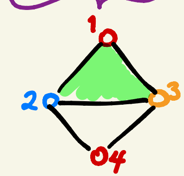
$$\tilde{H}^{-1}(\underbrace{\Delta_{\emptyset}}_{\{\emptyset\}}, k) = k$$

$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-z_1) \end{matrix} \leftarrow A(-z_1 z_2 z_3) \leftarrow 0$$

$$\tilde{H}^{0}(\underbrace{\Delta_{\{1\}}}, k) = k$$

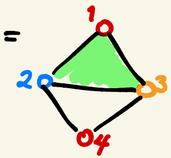


$$\tilde{H}^{1}(\underbrace{\Delta_{\{1,2,3\}}}, k) = k$$



II gives an $\mathbb{N}^d$-graded Hilbert series calculation

EXAMPLE

$\Delta =$ 

$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-z_1) \end{matrix} \leftarrow A(-z_1 z_2 z_3) \leftarrow 0$$

$$\Downarrow$$

$$\mathrm{Hilb}(k[\Delta], \underline{z}) = \frac{1 + z_1 - z_1 z_2 z_3}{(1-z_1)(1-z_2)(1-z_3)}$$

$\mathbb{N}^3$-graded

$$\Downarrow$$

$$\mathrm{Hilb}(k[\Delta], t) = \frac{1 + t - t^3}{(1-t)^3}$$

$\mathbb{N}$-graded

THEOREM (Colorful Hochster formula)

$\mathrm{Tor}_i^A(k[\Delta], k)_\alpha$ vanishes unless $\underline{z}^\alpha = \underline{z}^S$ for some $S \subseteq \{1,2,\ldots,d\}$,

in which case $\mathrm{Tor}_i^A(k[\Delta],k)_S \cong \tilde{H}_i^{|S|-i-1}(\Delta|_S, k)$

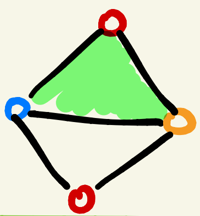
---

proof sketch: Compute Tor via Koszul complex for k over A tensored with $k[\Delta]$.

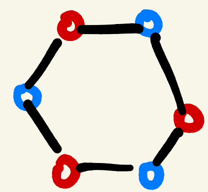
- Strands $\underline{z}^\alpha \neq \underline{z}^S$ are acyclic via chain-contraction.
- Strand $\underline{z}^\alpha = \underline{z}^S$ is isomorphic to $\tilde{C}_\cdot(\Delta|_S, k)$. $\square$

Two issues:

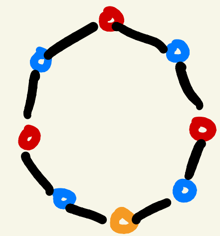
- One can't always properly color Δ with only $\dim k[\Delta]$ colors; such Δ are called **balanced**.



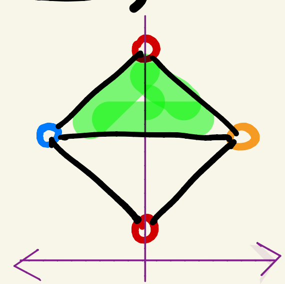
balanced



not balanced



- Even a balanced coloring can fail to be fixed by **symmetries** of Δ, so resolution lacks **equivariance**.



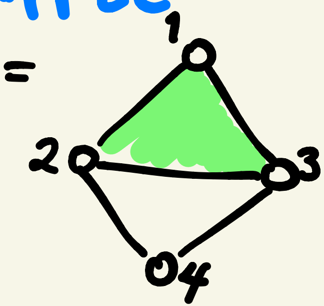
CONJECTURE on canonical parameters

PROPOSITION

$$\Theta_i := \sum_{\substack{(i+1)\text{-faces} \\ F \in \Delta}} x^F \quad \text{for } i = 1, 2, \ldots, \dim k[\Delta]$$

are always a system of parameters for $k[\Delta]$

EXAMPLE

$\Delta =$ 

$$\Theta_1 = x_1 + x_2 + x_3 + x_4$$

$$\Theta_2 = x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4$$

$$\Theta_3 = x_1x_2x_3$$

PROPOSITION

$$\theta_i := \sum_{\substack{(i+1)\text{-faces} \\ F \in \Delta}} x^F$$ are a system of parameters for $k[\Delta]$

---

proof: Elementary symmetric functions

$$e_1 = x_1 + x_2 + \cdots + x_n$$

$$e_2 = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n$$

$$\vdots$$

$$e_n = x_1x_2 \cdots x_n$$

are a system of parameters for $k[x_1, x_2, \ldots, x_n]$,

and $\theta_1, \theta_2, \ldots, \theta_{\dim k[\Delta]}$ are their (nonzero) images

$$k[\underline{x}] \twoheadrightarrow k[\Delta] = k[\underline{x}]/I_\Delta$$

$$e_i \longmapsto \theta_i$$

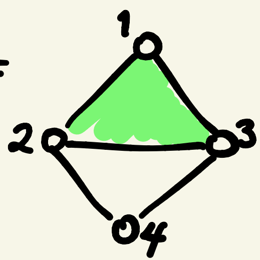
$\square$

Let $A = k[z_1, z_2, \ldots, z_d]$ for $d = \dim k[\Delta]$, $\deg(z_i) := i$

Then $z_i \mapsto \theta_i$ makes $k[\Delta]$ a fin. gen'd $\mathbb{N}$-graded A-module, so it has a minimal free resolution.

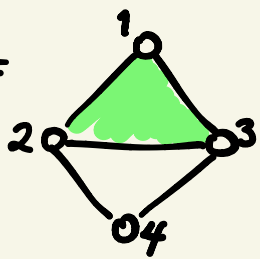
EXAMPLE

$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-1)^3 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^2 \end{matrix} \leftarrow \begin{matrix} A(-4) \\ \oplus \\ A(-5)^2 \\ \oplus \\ A(-6) \end{matrix} \leftarrow 0$$

$\Delta =$ 

This again gives the $\mathbb{N}$-graded Hilbert series (but can also be done equivariantly).

EXAMPLE

$\Delta =$ 

$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-1)^3 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^2 \end{matrix} \leftarrow \begin{matrix} A(-4) \\ \oplus \\ A(-5)^2 \\ \oplus \\ A(-6) \end{matrix} \leftarrow 0$$

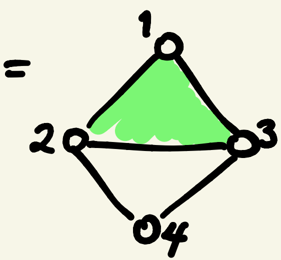
$$\Downarrow$$

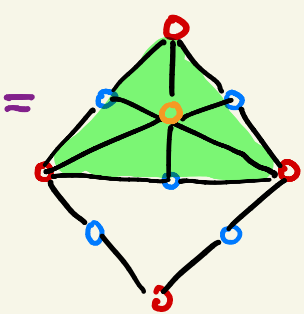
$$\mathrm{Hilb}(k[\Delta],t) = \frac{1+3t+4t^2+2t^3-t^4-2t^5-t^6}{(1-t)(1-t^2)(1-t^3)} = \frac{1+t-t^3}{(1-t)^3}$$

CONJECTURE: (Adams–R.) In this setting, for all $j \in \mathbb{N}$,

$$\operatorname{Tor}_i^A(k[\Delta], k)_j \cong \bigoplus_{\substack{S \subseteq \{1,2,\dots,d\}: \\ \sum_{s\in S} s = j}} \tilde{H}_i^{|S|-i-1}(\operatorname{Sd}\Delta|_S, k)$$

barycentric subdivision of Δ

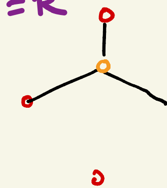
$\Delta =$ 

$\operatorname{Sd}\Delta =$ 

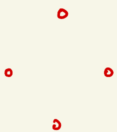
EXAMPLE

$\tilde{H}^{-1}(Sd\Delta|_{\emptyset}, k) = k$

$\tilde{H}^{0}(Sd\Delta|_{\{1,3\}}, k) = k$

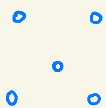


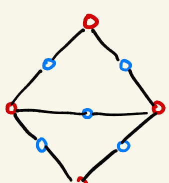
$\tilde{H}^{0}(Sd\Delta|_{\{1\}}, k) = k^3$



$$0 \leftarrow k[\Delta] \leftarrow \begin{matrix} A \\ \oplus \\ A(-1)^3 \\ \oplus \\ A(-2)^4 \\ \oplus \\ A(-3)^2 \end{matrix} \leftarrow \begin{matrix} A(-4) \\ \oplus \\ A(-5)^2 \\ \oplus \\ A(-6) \end{matrix} \leftarrow 0$$

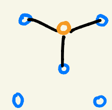
$\tilde{H}^{0}(Sd\Delta|_{\{2\}}, k) = k^4$





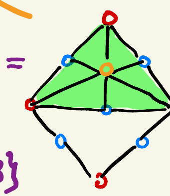
$\tilde{H}^{1}(Sd\Delta|_{\{1,2\}}, k) = k^2$

$\tilde{H}^{0}(Sd\Delta|_{\{2,3\}}, k) = k^2$



$Sd\Delta =$

$Sd\Delta|_{\{1,2,3\}}$



CONJECTURE

$$\mathrm{Tor}_i^A(k[\Delta],k)_j \cong \bigoplus_{\substack{S \subseteq \{1,2,\ldots,d\}: \\ \sum_{s\in S} s = j}} \tilde{H}_i^{|S|-i-1}(\mathrm{Sd}\Delta|_S, k)$$

PARAPHRASED CONJECTURE:

$\mathbb{N}$-graded Betti numbers for $k[\Delta]$ as $k[\Theta_1,\ldots,\Theta_d]$-mod

specialize $\mathbb{N}^d \longrightarrow \mathbb{N}$, $e_i \longmapsto i$

$\mathbb{N}^d$-graded Betti numbers for $k[\mathrm{Sd}\Delta]$ with balanced system of parameters, as in colorful Hochster formula.

EVIDENCE FOR THE CONJECTURE

One can present $k[\Delta] \cong k[y_F]_{\emptyset \neq F \in \Delta} / J_\Delta$

$k[\mathrm{Sd}\Delta] \cong k[y_F]_{\emptyset \neq F \in \Delta} / I_{\mathrm{Sd}\Delta}$

with $I_{\mathrm{Sd}\Delta}$ an initial ideal for J_Δ.

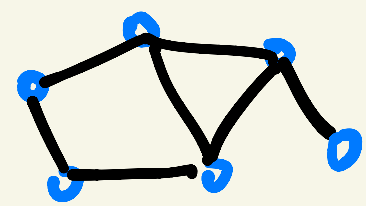
via Colorful Hochster Thm.

- correct as upper bound on $\mathrm{Tor}^A_\cdot(k[\Delta], k)$
- correctly predicts Hilbert series
- correct when Δ is Cohen-Macaulay

EVIDENCE FOR THE CONJECTURE

- correct as upper bound on $\mathrm{Tor}_{\cdot}^{A}(k[\Delta], k)$
- correctly predicts Hilbert series
- correct when Δ is Cohen-Macaulay

- ... and also checked correct for $\dim k[\Delta] = 2$
  i.e. Δ a graph



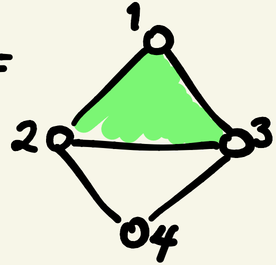
CONJECTURE on depth sensitivity

CONJECTURE

Letting $\Theta_i := \sum_{\substack{(i+1)\text{-faces} \\ F \in \Delta}} x^F$ for $i = 1, 2, \dots, \dim k[\Delta]$

as before, one has

$$\operatorname{depth} k[\Delta] = \max\left\{ \delta : \begin{array}{l} (\Theta_1, \Theta_2, \dots, \Theta_\delta) \text{ are a} \\ k[\Delta]\text{-regular sequence} \end{array} \right\}$$

EXAMPLE $\Delta =$ 

$k[\Delta] = k[x_1,x_2,x_3,x_4] \big/ (x_1x_4,\ x_2x_3x_4)$ has depth 2

and (θ_1, θ_2) as a regular sequence,

$\theta_1 = x_1+x_2+x_3+x_4$, $\theta_2 = x_1x_2+x_1x_3+x_2x_3+x_2x_4+x_3x_4$

but $\theta_3 = x_1x_2x_3$ is already a zero-divisor:

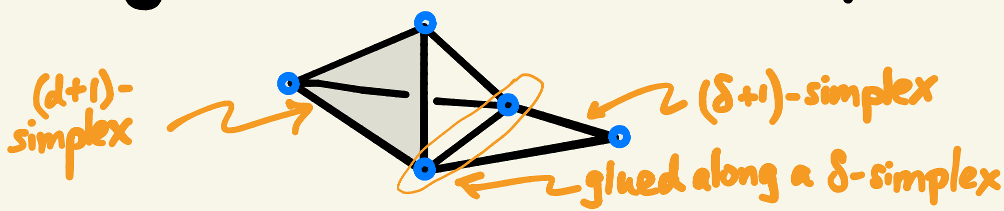
$$x_4 \cdot x_1x_2x_3 = 0$$

CONJECTURE

$$\operatorname{depth} k[\Delta] = \max\{\delta : (\theta_1, \theta_2, \dots, \theta_\delta) \text{ are a } k[\Delta]\text{-regular sequence}\}$$

EVIDENCE

- Proven by D.E. Smith (1990) when Δ is pure.
- Empirically tight for these Δ with $\dim k[\Delta] = d$, $\operatorname{depth} k[\Delta] = \delta$



which seem to have $(\theta_1, \theta_2, \dots, \theta_\delta)$ a $k[\Delta]$-regular sequence

but $\theta_{\delta+1}, \theta_{\delta+2}, \dots, \theta_n$ all zero-divisors on $k[\Delta]$.

Thanks for your attention!