Combinatorics of configuration spaces - recent progress

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Math Colloquium - Univ. of Rome, Tor Vergata
November 3, 2023

PLAN:

1st half

Symmetric group $\mathfrak{S}_n$ acting on
$H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$ for $d=1,3,5,7,\ldots$ odd
versus Eulerian idempotents

a 20th century story,
so now CLASSICAL,
but mysterious!

2nd

reflection groups W
replacing
$W=\mathfrak{S}_n$

3rd

"hidden" action
of $\mathfrak{S}_{n+1}$

4th

peak
idempotents

We'll examine topology of

$$\mathrm{Conf}_n(\mathbb{R}^d) := \{(p_1, p_2, \ldots, p_n) \in (\mathbb{R}^d)^n : p_i \neq p_j \text{ for } 1 \leq i < j \leq n\}$$

$$= (\mathbb{R}^d)^n - \underbrace{\bigcup_{1 \leq i < j \leq n} \{p_i = p_j\}}_{\text{thick diagonal}}$$

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focussing on its cohomology ring $H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$ with $\mathbb{k}$ coefficients

usually of field of char 0, sometimes $\mathbb{Z}$

and the action of $\mathfrak{S}_n$ = symmetric group on n letters

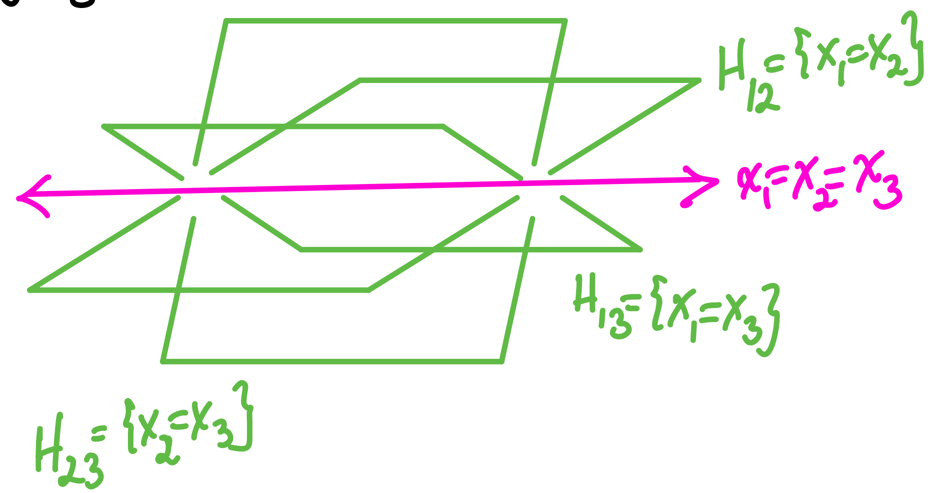
via $(p_1, p_2, \ldots, p_n) \xmapsto{\omega} (p_{\omega_1^{-1}}, p_{\omega_2^{-1}}, \ldots, p_{\omega_n^{-1}})$

Let's analyze ...

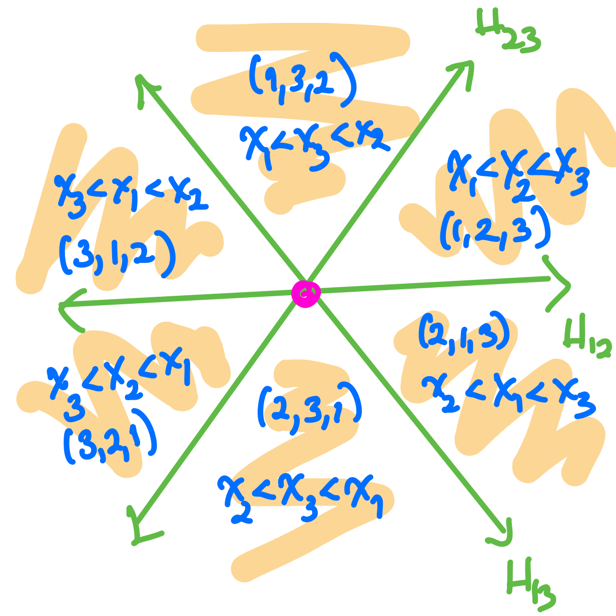
$\mathrm{Conf}_n(\mathbb{R}^1)$ = complement of hyperplane arrangement

$$\mathcal{A} = \{ H_{ij} = \{x_i = x_j\} : 1 \le i < j \le n \} \subset \mathbb{R}^n$$

$\mathrm{Conf}_3(\mathbb{R}^1)$



project out $x_1 = x_2 = \dots = x_n$ $\rightsquigarrow$



$H^\bullet \mathrm{Conf}_n(\mathbb{R}^1) = H^0 \mathrm{Conf}_n(\mathbb{R}^1) \cong \Bbbk \mathfrak{S}_n$ regular representation as $\mathfrak{S}_n$-reps

e.g. $H^0 \mathrm{Conf}_3(\mathbb{R}^1) \cong H^0 \mathrm{Conf}_3(\mathbb{R}^1) \cong \Bbbk \mathfrak{S}_n = \underbrace{\chi^{\tiny\square\square\square}}_{\text{trivial rep}} + 2\chi^{\tiny\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} + \underbrace{\chi^{\tiny\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}}_{\text{sign rep}}$

where χ^λ = irreducible $\mathfrak{S}_n$-character indexed by $\underset{(\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n)}{\lambda \,\Vert} \vdash n$

THEOREM

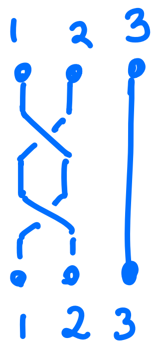
Fadell–Fox–Neuwirth 1962

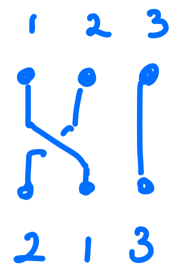
$$\mathrm{Conf}_n(\mathbb{R}^2) = \text{pure braid space}$$
$$= \text{Eilenberg-MacLane } K(\pi,1)\text{-space for}$$

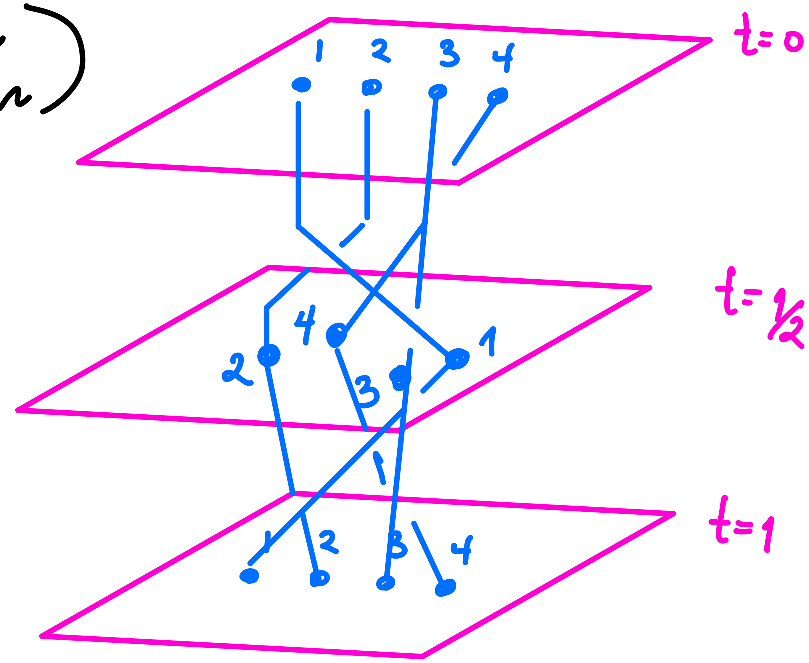
$$\pi = \pi_1 \mathrm{Conf}_n(\mathbb{R}^2) = PBr_n := \ker\left(Br_n \twoheadrightarrow \mathfrak{S}_n\right)$$

PBr_n: pure braid group

Br_n: braid group on n strands

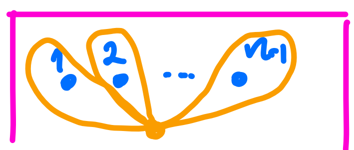






Proof via "forgetting fibration":

$$\mathbb{R}^2 - \{n-1 \text{ points}\} \longrightarrow \mathrm{Conf}_n \mathbb{R}^2 \longrightarrow \mathrm{Conf}_{n-1} \mathbb{R}^2$$
$$(p_1, \dots, p_{n-1}, p_n) \longmapsto (p_1, \dots, p_{n-1})$$



COROLLARY: $H^\bullet \mathrm{Conf}_n(\mathbb{R}^2) \cong$ group cohomology $H^\bullet(PBr_n)$

THEOREM

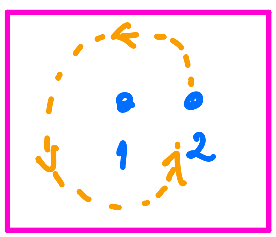
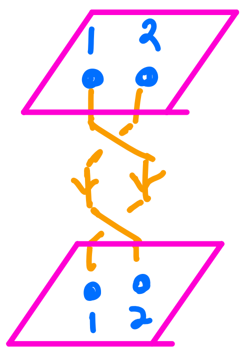
V.I. Arnol'd 1969

$$H^\bullet \operatorname{Conf}_n(\underset{\overset{\|}{\mathbb{C}}}{\mathbb{R}^2}, \mathbb{Z}) \cong \underbrace{\Lambda_{\mathbb{Z}}\{u_{ij}\}_{1\le i<j\le n}}_{\text{exterior algebra}} \Big/ (u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk})_{1\le i<j<k\le n}$$

$$u_{ij}^2 = 0$$

$$u_{ij}\,u_{kl} = -u_{kl}\,u_{ij}$$

where u_{ij} = pullback of $u_{12} \in H^1 \operatorname{Conf}_2(\mathbb{R}^2)$,
dual to homology cycle $z_{12} \in H_1 \operatorname{Conf}_2(\mathbb{R}^2) \cong \mathbb{Z}$

$z_{12} =$  $=$ 

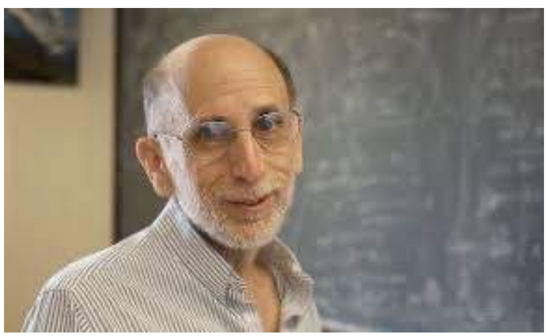
More generally ...

THEOREM
F. Cohen 1973

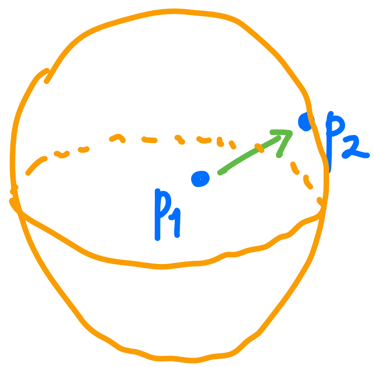
$$H^\bullet \mathrm{Conf}_n(\mathbb{R}^d, \mathbb{Z}) \cong$$

$$\begin{cases} \overbrace{\Lambda_{\mathbb{Z}}\{u_{ij}\}_{1\le i<j\le n}}^{\text{exterior algebra}} \Big/ \left(u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk}\right)_{1\le i<j<k\le n} & d = 2,4,6,\ldots \text{ even} \\ \overbrace{\mathbb{Z}[u_{ij}]_{1\le i<j\le n}}^{\text{commutative polynomial}} \Big/ \left(u_{ij}^2,\ u_{ij}u_{ik} - u_{ij}u_{jk} + u_{ik}u_{jk}\right)_{1\le i<j<k\le n} & d = 3,5,7,\ldots \text{ odd} \end{cases}$$

where u_{ij} = pullback of $u_{12} \in H^{d-1}\mathrm{Conf}_2(\mathbb{R}^d) \cong \mathbb{Z}$



Fred Cohen (1945-2022)



$$\begin{array}{ccc} (p_1, p_2) & & \mathrm{Conf}_2(\mathbb{R}^d) \\ \big\downarrow & & \wr\wr \ \text{homotopy equiv.} \\ p_2 - p_1 & & \mathbb{R}^d - \{0\} \\ & & \wr\wr \\ & & S^{d-1} \end{array}$$

$H^\bullet \mathrm{Conf}_3(\mathbb{R}^d)$ for $d = 3, 5, 7, \ldots$ odd

	H^0	H^{d-1}	$H^{2(d-1)}$
dimension	1	3	2
Cohen's $\Bbbk$-basis (= nbc monomials)	1	u_{12}, u_{13}, u_{23}	$u_{12}u_{13}$, $u_{13}u_{23}$
$\mathfrak{S}_3$-rep if $\mathrm{char}(\Bbbk)=0$	$\chi^{(3)}$	$\chi^{(2,1)} + \chi^{(1,1,1)}$	$\chi^{(2,1)}$

$\rightsquigarrow$ Poincaré-Hilbert series

$1+3t+2t^2 = (1+t)(1+2t)$

Follows from Cohen's proof

$\rightsquigarrow$ total rep $= \chi^{(3)} + 2\chi^{(2,1)} + \chi^{(1,1,1)}$

$\cong \Bbbk\mathfrak{S}_3$ regular rep

(like $H^\bullet \mathrm{Conf}_3(\mathbb{R}^1)$)

$H^\bullet \mathrm{Conf}_4(\mathbb{R}^d)$ for $d = 3, 5, 7, \ldots$ odd

	H^0	H^{d-1}	$H^{2(d-1)}$	$H^{3(d-1)}$
dimension	1	6	11	6
$\mathfrak{S}_4$-rep if $\mathrm{char}(\mathbb{k})=0$	$\chi^{(4)}$	$\chi^{(3,1)} + \chi^{(2,1,1)}$	$\chi^{(2,2)} + \chi^{(3,1)} + \chi^{(1,1,1,1)} + \chi^{(2,2)} + \chi^{(2,1,1)}$	$\chi^{(3,1)} + \chi^{(2,1,1)}$

$\rightsquigarrow$ Poincaré-Hilbert series

$1+6t+11t^2+6t^3$
$=(1+t)(1+2t)(1+3t)$

$\rightsquigarrow$ total rep $=$

$\chi^{(4)} + 3\chi^{(3,1)} + 2\chi^{(2,2)} + 3\chi^{(2,1,1)} + \chi^{(1,1,1,1)}$

$\cong \mathbb{k}\mathfrak{S}_4$ regular rep

Why? THEOREM (Folklore, known at least to S. Sundaram!)

There exist complete orthogonal idempotents $e_0, e_1, \dots, e_{n-1}$ in $\Bbbk\mathfrak{S}_n$ $(\mathrm{char}(\Bbbk)=0)$ called Eulerian idempotents with

$e_i^2 = e_i$
$e_ie_j = e_je_i = 0$
$1 = e_0 + e_1 + \dots + e_{n-1}$

$$H^{k(d-1)}\,\mathrm{Conf}_n(\mathbb{R}^d) \cong \Bbbk\mathfrak{S}_n \cdot e_{n-1-k}$$

e.g. $n=3$

$$e_2 = \frac{1}{6}\Big(\underbrace{(1,2,3)}_{\text{0 descents}} + \underbrace{(1,3;2) + (2;1,3) + (2,3;1) + (3;1,2)}_{\text{permutations with 1 descent } w_i > w_{i+1}} + \underbrace{(3;2;1)}_{\text{2 descents}}\Big)$$

$$e_1 = \frac{1}{2}\Big((1,2,3) - (3;2;1)\Big)$$

$$e_0 = \frac{1}{6}\Big(2\cdot(1,2,3) - (1,3;2) - (2;1,3) - (2,3;1) - (3;1,2) + 2\,(3;2;1)\Big)$$

$\Bbbk\mathfrak{S}_3 \cdot e_2 = \chi^{(3)} \cong H^0$

$\Bbbk\mathfrak{S}_3 \cdot e_1 = \chi^{(2,1)} + \chi^{(1,1,1)} \cong H^{d-1}$

$\Bbbk\mathfrak{S}_3 \cdot e_0 = \chi^{(2,1)} \cong H^{2(d-1)}$

The Eulerian idempotents $e_0, e_1, \ldots, e_{n-1}$ in $\Bbbk\mathfrak{S}_n$ (up to twisting by $\Bbbk\mathfrak{S}_n \longrightarrow \Bbbk\mathfrak{S}_n$, $w \longmapsto \operatorname{sgn}(w)\cdot w$)

- intertwine Hochschild homology boundary maps

$$M \otimes A^{\otimes n} \xrightarrow{\partial_i} M \otimes A^{\otimes (n-1)}$$

(leading to Hodge decomposition)

Gerstenhaber-Schack 1987

- are Lagrange interpolation projectors onto the eigenspaces of Barr's element (Barr 1968)

$$B := \sum_{i=1}^{n-1} \sum_{\substack{w \in \mathfrak{S}_n: \\ \operatorname{Des}(w) \subset \{i\}}} w$$

where $w = (w_1, w_2, \ldots, w_n)$ has descent set $\operatorname{Des}(w) = \{i : w_i > w_{i+1}\}$

- can be defined via generating functions of Loday 1989, Garsia-Reutenauer 1989

$$\sum_{k=1}^{n} t^k e_{k-1} = \sum_{w \in \mathfrak{S}_n} \binom{t + (n-1-\operatorname{des}(w))}{n} \cdot w$$

The folklore $H^{k(d-1)}\mathrm{Conf}_n(\mathbb{R}^d) \cong \Bbbk\mathfrak{S}_n \cdot e_{n-1-k}$ was first proven by comparing two finer $\mathfrak{S}_n$-character computations.

Garsia-Reutenauer 1989 had defined a finer complete orthogonal family called higher Lie idempotents $\{e_\lambda\}_{\lambda \vdash n}$ generating higher Lie $\mathfrak{S}_n$-reps $\mathrm{Lie}_\lambda := \Bbbk\mathfrak{S}_n \cdot e_\lambda$

and satisfying $e_k = \sum_{\substack{\lambda \vdash n: \\ \lambda \text{ has } k+1 \text{ parts}}} e_\lambda$

$H^\bullet \mathrm{Conf}_4(\mathbb{R}^d)$ for $d = 3, 5, 7, \ldots$ odd

H^0	H^{d-1}	$H^{2(d-1)}$		$H^{3(d-1)}$
$\chi^{(4)}$	$\chi^{(3,1)} + \chi^{(2,1,1)}$	$\chi^{(2,2)} + \chi^{(1,1,1,1)}$	$+\chi^{(3,1)} + \chi^{(2,2)} + \chi^{(2,1,1)}$	$\chi^{(3,1)} + \chi^{(2,1,1)}$
$\uparrow$ $\mathrm{Lie}_{(1,1,1,1)}$	$\uparrow$ $\mathrm{Lie}_{(2,1,1)}$	$\uparrow$ $\mathrm{Lie}_{(2,2)}$	$\uparrow$ $\mathrm{Lie}_{(3,1)}$	$\uparrow$ $\mathrm{Lie}_{(4)}$

- P. Hanlon 1990 calculated (by brute force!) the $\mathfrak{S}_n$-character of

$$\mathrm{Lie}_\lambda := \Bbbk\mathfrak{S}_n \cdot e_\lambda$$

- Sundaram-Welker 1997 proved an equivariant Goresky-MacPherson formula 1988 for cohomology of linear subspace complements in $\mathbb{R}^m$; applied it to calculate

$$H^{(n-1-k)(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) \cong \bigoplus_{\substack{\lambda \vdash n: \\ \lambda \text{ has } k+1 \text{ parts}}} W_\lambda$$

with same character for W_λ as Hanlon calculated for Lie_λ

COROLLARY: $H^{(n-1-k)(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) = \bigoplus_{\substack{\lambda \vdash n \\ \lambda \text{ has } k+1 \text{ parts}}} \mathrm{Lie}_\lambda = \Bbbk\mathfrak{S}_n \cdot e_k$

Even more mysterious ....

Eulerian idempotents

$\{e_k\}_{k=0,1,2,\ldots,n-1}$

give a primitive family of idempotents for

$$\left\{ \sum_{w \in W} c_w \cdot w \; : \; c_w \text{ depends only on } \mathrm{des}(w) \right\} = \text{Eulerian subalgebra of } \Bbbk\mathfrak{S}_n \quad \mathcal{E}_n$$

Higher Lie idempotents

$\{e_\lambda\}_{\lambda \vdash n}$

give a primitive family of idempotents for

$$\left\{ \sum_{w \in W} c_w \cdot w \; : \; c_w \text{ depends only on } \mathrm{Des}(w) \right\} \subset \Bbbk\mathfrak{S}_n$$

$$\left\{ \sum_{w \in W} c_w \cdot w \; : \; c_w \text{ depends only on } \mathrm{Des}(w) \right\} = \text{L. Solomon's descent algebra (1976)} \quad \mathrm{Sol}_n$$

$\Bbbk\mathfrak{S}_n$ = group algebra of $\mathfrak{S}_n$

Eulerian subalgebra $\subset$ Solomon's descent algebra $\subset \Bbbk\mathfrak{S}_n$

not obviously subalgebras!

Reflection group generalization

Finite reflection groups W acting irreducibly on $\mathbb{R}^n$

have Coxeter generators $S = \{s_1, \ldots, s_n\}$ with a nice Coxeter presentation

$$s_i^2 = 1$$
$$(s_i s_j)^{m_{ij}} = 1$$

$\leftarrow$ reflections through walls of a fixed chamber in the complement of the reflecting hyperplane arrangement $\mathcal{A}_W$

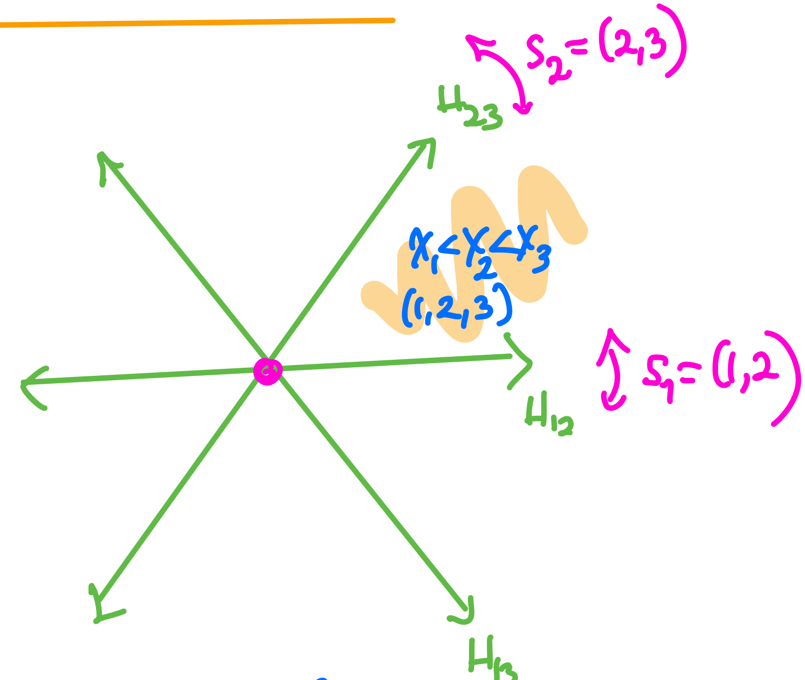
---

e.g. $W = \mathfrak{S}_n$ acting on $\mathbb{R}^{n-1}$

$$S = \{s_1, s_2, \ldots, s_{n-1}\}$$
$$s_1 = (1,2), \quad s_2 = (2,3), \quad \ldots, \quad s_{n-1} = (n-1,n)$$

adjacent transpositions

$n=3$



$W = \mathfrak{S}_3 \qquad S = \{s_1, s_2\}$

$s_1 = (1,2), \quad s_2 = (2,3)$

e.g. $W = \mathfrak{S}_n^{\pm} = \{\text{signed } n\times n \text{ permutation matrices}\}$ acting on $\mathbb{R}^n$, has reflecting hyperplanes

$= \text{hyperoctahedral group}$

$H_{ij}^{+} = \{x_i = x_j\}$

$H_{ij}^{-} = \{x_i = -x_j\}$

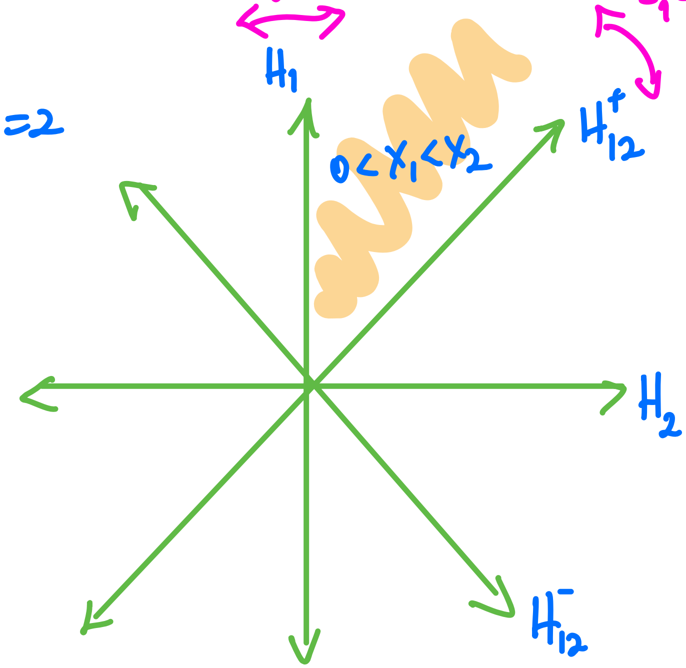
$H_i = \{x_i = 0\}$

$$S = \{ s_0, s_1, s_2, \ldots, s_{n-1} \}$$

$$s_0 = \begin{bmatrix} \boxed{-1} & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & \ddots \; +1 \end{bmatrix}, \quad s_1 = (1,2), \quad s_2 = (2,3), \quad s_{n-1} = (n-1,n)$$

$n=2$

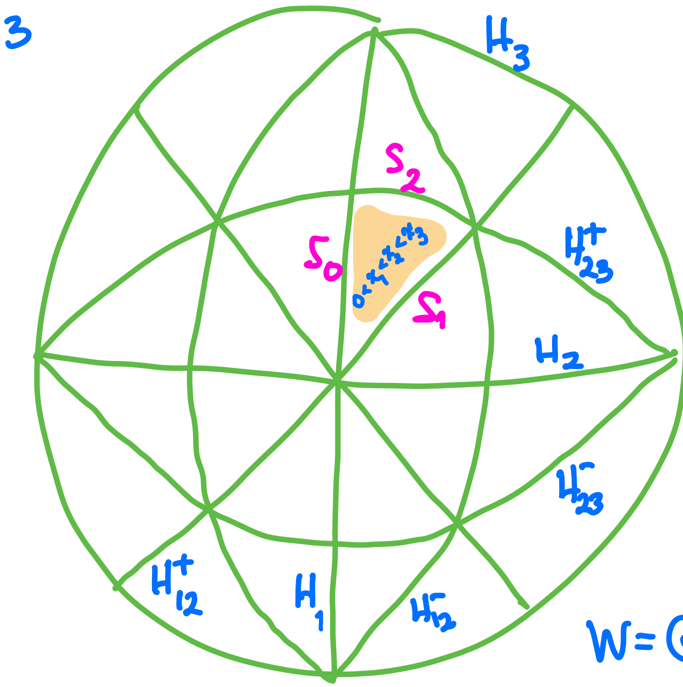
$s_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\qquad s_1 = (1,2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$W = \mathfrak{S}_2^{\pm}$

$S = \{s_0, s_1\}$

$n=3$



$W = \mathfrak{S}_3^{\pm}$

$S = \{s_0, s_1, s_2\}$

Why bother?

Thickening these reflection arrangement complements via $(-)\otimes_{\mathbb{R}} \mathbb{R}^d$ gives a linear subspace complement generalizing $\mathrm{Conf}_n \mathbb{R}^d$:

$$Y = \mathbb{R}^n \otimes \mathbb{R}^d - \bigcup_{\substack{\text{reflecting}\\ \text{hyperplanes}\\ H \text{ in } \mathcal{A}_W}} H \otimes_{\mathbb{R}} \mathbb{R}^d$$

$$\cong \begin{cases} \mathrm{Conf}_n(\mathbb{R}^d) & \text{if } W = \mathfrak{S}_n \\ \mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^d) = \{(p_1, p_2, \ldots, p_n) \in (\mathbb{R}^d)^n : p_i \neq \pm p_j,\ p_i \neq 0\} & \text{if } W = \mathfrak{S}_n^{\pm} \end{cases}$$

= Orbit configuration space for $\mathbb{Z}_2$-action
$\mathbb{R}^d \to \mathbb{R}^d$
$p \mapsto -p$

cohomology ring studied/presented by M. Xicotencatl 1997

This helps! Sundaram-Welker's 1997 equivariant Goresky-MacPherson formula specializes for any hyperplane arrangement $\mathcal{A} \subset \mathbb{R}^n$ with symmetries W

$$Y = \mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^d - \bigcup_{H \in \mathcal{A}} H \otimes_{\mathbb{R}} \mathbb{R}^d$$

to analyze $H^{\cdot} Y$ as a sum of contributions for each flat $X = H_{i_1} \cap \cdots \cap H_{i_r}$ compiled into contributions for each W-orbit of flats $[X] := \{$flats X' of form wX for $w \in W\}$

COROLLARY In the above setting,

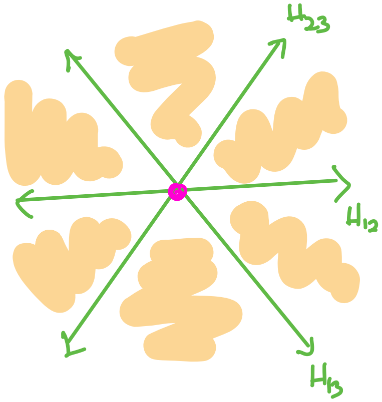
S. Brauner 2020

$$H^{k(d-1)} Y \cong \bigoplus_{\substack{\text{codimension } k \\ \text{flats } X}} \underbrace{\tilde{H}^{k-2} \Delta(X, \mathbb{R}^n)}_{\substack{\text{homology of the} \\ \text{poset of proper} \\ \text{flats containing } X}} \quad (\text{ignoring } W\text{-action})$$

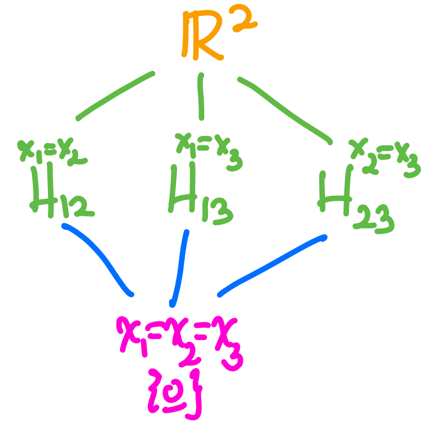
$$\big\uparrow_{N_X}^{W} \cong \bigoplus_{\substack{\text{codim } k \\ W\text{-orbits } [X] \\ \text{of flats}}} \mathrm{Ind}_{N_X}^{W} \tilde{H}^{k-2} \Delta(X, \mathbb{R}^n) \otimes \det_{\mathbb{R}^n/X}$$

$N_X :=$ setwise W-stabilizer of X

e.g. $W = \mathfrak{S}_3$



A_W has poset of flats



with W-orbits of flats

$$\begin{aligned}
\{0\} = [x_1=x_2=x_3] &\longleftrightarrow (3) \\
[H_{12}]=[H_{13}]=[H_{23}] &\longleftrightarrow (2,1) \\
[\mathbb{R}_2] &\longleftrightarrow (1,1,1)
\end{aligned}
\Bigg\}\ \text{partitions } \lambda \vdash n$$

$\Rightarrow$ $Y = \mathrm{Conf}_n(\mathbb{R}^d)$ has

$$H^0 Y \cong \mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_3} \tilde{H}^{-2}\Delta(\mathbb{R}^2, \mathbb{R}^2) \otimes \det_{\mathbb{R}^2/\mathbb{R}^2} \cong \chi^{(3)} \qquad = \mathrm{Lie}_{(1,1,1)}$$

$$H^1 Y \cong \mathrm{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_1}^{\mathfrak{S}_3} \tilde{H}^{-1}\Delta(H_{12}, \mathbb{R}^2) \otimes \det_{\mathbb{R}^2/H_{12}} \cong \chi^{(2,1)} + \chi^{(1,1,1)} \qquad = \mathrm{Lie}_{(2,1)}$$

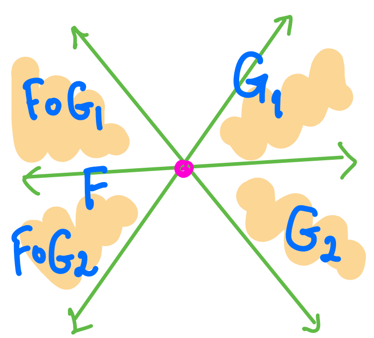
$$H^2 Y \cong \mathrm{Ind}_{\mathfrak{S}_3}^{\mathfrak{S}_3} \tilde{H}^{0}\Delta(\{0\}, \mathbb{R}^2) \otimes \det_{\mathbb{R}^2/\{0\}} \cong \chi^{(2,1)} \qquad = \mathrm{Lie}_{(3)}$$

On the other hand ...

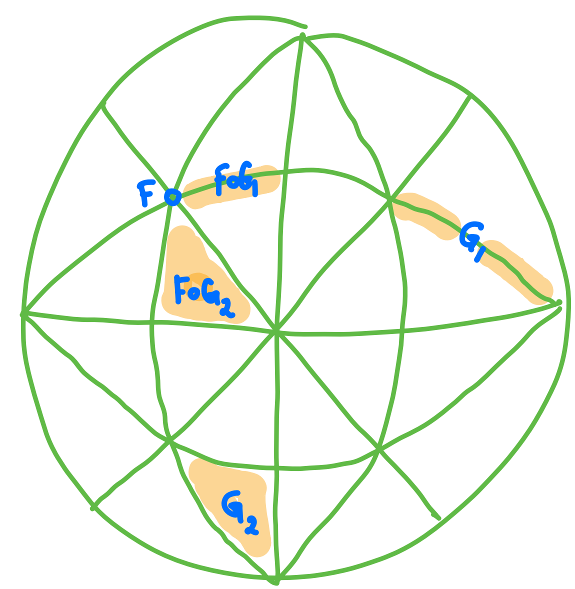
the flats X and W-orbits of flats $[X]$ appear naturally in work of

Bidigare 1998
Bidigare-Hanlon-Rockmore 1999
Brown-Diaconis 1998
Saliola 2008, 2009, 2012
Aguiar-Mahajan 2017

on the representation theory of Tits's face semigroup on the set of faces $\mathcal{F}$ of $\mathcal{A}$:
(= cones of all dimensions)



$F \circ G_1$ = "face F perturbed toward face G"



... that manages to explain why finite reflection groups W and S have a descent algebra:

$$\ell(w) := \min\{\ell : w = s_1 s_2 \cdots s_\ell \text{ with } s_i \in S\}$$

length function with respect to S

$$\mathrm{Des}(w) := \{s \in S : \ell(ws) < \ell(w)\}$$

(Mysterious) THEOREM: Solomon 1976

The $\mathbb{k}$-subspace $\left\{ \sum_{w \in W} c_w w : c_w \text{ depends only on } \mathrm{Des}(w) \right\} \subset \mathbb{k}W$ forms a subalgebra, called Solomon's descent algebra Sol_W.

Solomon's descent algebra $\mathrm{Sol}_n \subset \Bbbk\mathfrak{S}_n$

and higher Lie idempotents $\{e_\lambda\}_{\lambda \vdash n}$

generalizes to reflection groups W with conceptual understanding

BBHT idempotents in Sol_W

$\|$

flat orbit idempotents $\{e_{[X]}\}$ in $(\Bbbk\mathcal{F})^W \subset \mathrm{End}_W(\Bbbk\mathcal{C}) \cong \Bbbk W$

$\mathrm{Sol}_W \cong (\Bbbk\mathcal{F})^W$ ← Bidigare 1998

Bergeron-Bergeron-Howlett-Taylor 1992

group algebra

$\cap$

flat idempotents $\{e_X\}$ in $\boxed{\Bbbk\mathcal{F}}$ $\circlearrowright$ $\Bbbk\mathcal{C}$ = $\Bbbk$-subspace spanned by chambers

simplest rep theory!

In particular, the work of Saliola 2008–2012 shows ...

THEOREM: The primitive flat-orbit idempotents (= BBHT idempotents) $\{e_{[X]}\}$ in $\mathrm{Sol}_W \subset \mathbb{k}W$

decompose $\mathbb{k}W = \bigoplus_{\substack{\text{flat } W\text{-orbits} \\ [X]}} \mathbb{k}W \cdot e_{[X]}$ with W-representations

$$\mathbb{k}W \cdot e_{[X]} \cong \mathrm{Ind}_{N_X}^{W} \tilde{H}^{k-2}\Delta(X, \mathbb{R}^n) \otimes \det_{\mathbb{R}^n/X}$$

COROLLARY: Eulerian idempotents $\{e_k^W\}$ defined by $e_k^W := \sum_{\substack{\text{flat } W\text{-orbits} \\ [X] \text{ of dimension } k}} e_{[X]}$

Brauner 2020

have $H^{k(d-1)} Y \cong \mathbb{k}W \cdot e^W_{n-k}$

where $Y = \mathbb{R}^n \otimes \mathbb{R}^d - \bigcup_{H \in \mathcal{A}_W} H \otimes \mathbb{R}^d$ for d odd

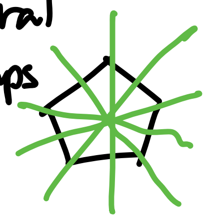
THEOREM: For the coincidental reflection groups

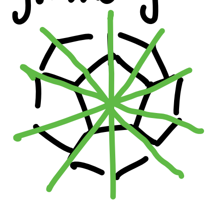
Brauner 2020

$W = \mathfrak{S}_n, \quad \mathfrak{S}_n^{\pm}, \quad I_2(m), \quad H_3$

$I_2(m)$ " dihedral groups

H_3 " symmetry of dodecahedron





the Eulerian idempotents $\{e_k^W\}$ are primitive idempotents for

Eulerian subalgebra $\mathcal{E}_W$ $\subset$ Solomon descent algebra Sol_W $\subset$ $\Bbbk W$

$:= \left\{ \sum_{w \in W} c_w \cdot w : c_w \text{ depends only on } \#\mathrm{Des}(w) \right\}$

generated by the analogue of Barr's element (and with nice generating functions for $\{e_k^W\}$).

$$B := \sum_{s \in S} \; \sum_{\substack{w \in W: \\ \mathrm{Des}(w) \subset \{s\}}} w$$

"Hidden" actions of $\mathfrak{S}_{n+1}$

THEOREM

S. Whitehouse
1994 for $k=n-1$
1997 for all k

The $\mathfrak{S}_n$-reps $\{\Bbbk\mathfrak{S}_n \cdot e_k\}_{k=0,1,\dots,n-1}$

are restrictions from $\mathfrak{S}_{n+1}$ to $\mathfrak{S}_n$ of

$\mathfrak{S}_{n+1}$-reps $\{\Bbbk\mathfrak{S}_{n+1} \cdot f_k\}_{k=0,1,\dots,n-1}$

where

$$f_k := e_k \cdot \frac{1}{n+1} \sum_{i=0}^{n} \underbrace{(1,2,\dots,n,n+1)}_{(n+1)\text{-cycle}}^{i} \in \Bbbk\mathfrak{S}_{n+1}$$

Whitehouse idempotents

$n=3$

Eulerian

$\Bbbk\mathfrak{S}_3 e_2$ | $\Bbbk\mathfrak{S}_3 e_1$ | $\Bbbk\mathfrak{S}_3 e_0$

$\chi^{(3)}$ | $\chi^{(2,1)} + \chi^{(1,1,1)}$ | $\chi^{(2,1)}$

$\Big\Uparrow$ restrict $\mathfrak{S}_4$ to $\mathfrak{S}_3$

$\Bbbk\mathfrak{S}_4 f_2$ | $\Bbbk\mathfrak{S}_4 f_1$ | $\Bbbk\mathfrak{S}_4 f_0$

$\chi^{(4)}$ | $\chi^{(2,1,1)}$ | $\chi^{(2,2)}$

Whitehouse

$n=4$

Eulerian

$\Bbbk\mathfrak{S}_4 e_3$ | $\Bbbk\mathfrak{S}_4 e_2$ | $\Bbbk\mathfrak{S}_4 e_1$ | $\Bbbk\mathfrak{S}_4 e_0$

$\chi^{(4)}$ | $\chi^{(3,1)} + \chi^{(2,1,1)}$ | $\chi^{(3,1)} + 2\chi^{(2,2)} + \chi^{(2,1,1)} + \chi^{(1,1,1,1)}$ | $\chi^{(3,1)} + \chi^{(2,1,1)}$

$\Big\Uparrow$ restrict $\mathfrak{S}_5$ to $\mathfrak{S}_4$

$\Bbbk\mathfrak{S}_5 f_3$ | $\Bbbk\mathfrak{S}_5 f_2$ | $\Bbbk\mathfrak{S}_5 f_1$ | $\Bbbk\mathfrak{S}_5 f_0$

$\chi^{(5)}$ | $\chi^{(3,1,1)}$ | $\chi^{(3,2)} + \chi^{(2,2,1)} + \chi^{(1,1,1,1,1)}$ | $\chi^{(3,1,1)}$

Whitehouse

Compare with ...

THEOREM (O. Mathieu 1996): $H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$ for $d = 3, 5, 7, \ldots$ odd, as $\mathfrak{S}_n$-rep lifts to an $\mathfrak{S}_{n+1}$-rep, on a certain "limit ring" with abstract presentation.

THEOREM (N. Early - R. 2019):

$$H^\bullet \mathrm{Conf}_n(\mathbb{R}^3) \cong H^\bullet \ \mathrm{Conf}_{n+1}(SU_2)/\Delta SU_2 \text{ as } \mathfrak{S}_{n+1}\text{-rep}$$

where $SU_2 = \left\{ \begin{bmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{bmatrix} : \begin{array}{c} \alpha, \beta \in \mathbb{C} \\ |\alpha|^2 + |\beta|^2 = 1 \end{array} \right\} = \mathbb{S}^3 \supset \mathbb{R}^3$

2×2 special unitary group

and the $\mathfrak{S}_n$-isomorphism $H^{2k} \mathrm{Conf}_n(\mathbb{R}^3) \cong \mathbb{k}\mathfrak{S}_n \cdot e_{n-k}$

lifts to an $\mathfrak{S}_{n+1}$-isomorphism $H^{2k} \mathrm{Conf}_{n+1}(SU_2)/SU_2 \cong \mathbb{k}\mathfrak{S}_{n+1} \cdot f_{n-k}$

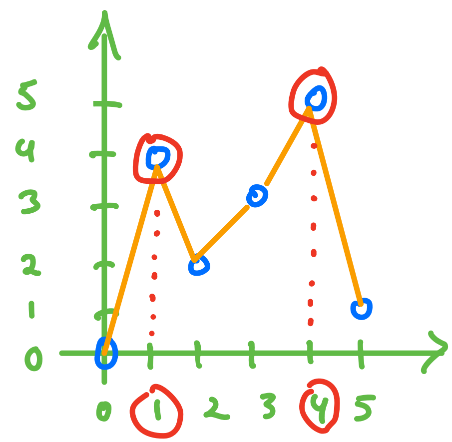
REMARK: R. Pagaria (2022) computed generating functions for the $\mathfrak{S}_{n+1}$-characters on $\{\mathbb{k}\mathfrak{S}_{n+1} \cdot f_k\}$ to prove a conjecture of Moseley-Proudfoot-Young (2017)

Peak idempotents

For $w \in \mathfrak{S}_n$, its **peak set** is $\mathrm{Peak}(w) := \left\{ i : \begin{array}{c} 1 \le i \le n-1, \\ w_{i-1} < w_i > w_{i+1} \end{array} \right\}$
$= (w_1, w_2, \ldots, w_n)$ (with convention $w_0 := 0$)

e.g. $w = \underline{4}\,2\,3\,\underline{5}\,1 \in \mathfrak{S}_5$ $\quad (w_1, w_2, w_3, w_4, w_5)$

has $\mathrm{Peak}(w) = \{1, 4\}$



NOTE: $\mathrm{Des}(w)$ determines $\mathrm{Peak}(w)$

$\mathrm{Des}(w) = \{ i : w_i > w_{i+1} \}$

THEOREM
K. Nyman 2003

The $\Bbbk$-subspace $\mathcal{P}_n := \left\{ \sum_{w \in \mathfrak{S}_n} c_w \cdot w \ : \ c_w \text{ depends only on } \mathrm{Peak}(w) \right\} \subset \Bbbk\mathfrak{S}_n$

$$\mathcal{P}_n \subset \mathrm{Sol}_n \subset \Bbbk\mathfrak{S}_n$$

forms a subalgebra

(peak subalgebra)

THEOREM
Aguiar - N. Bergeron - Nyman 2004

In fact, $\mathcal{P}_n$ is the image of $\mathrm{Sol}_{\mathfrak{S}_n^{\pm}}$

under the forgetting signs map $\mathfrak{S}_n^{\pm} \xrightarrow{\pi} \mathfrak{S}_n = \mathfrak{S}_n^{\pm} / \mathbb{Z}_2^n$:

$$\begin{tikzcd}
\Bbbk\mathfrak{S}_n^{\pm} \arrow[r, two heads, "\pi"] & \Bbbk\mathfrak{S}_n \\
\mathrm{Sol}_{\mathfrak{S}_n^{\pm}} \arrow[u, phantom, "\cup" description] \arrow[r, two heads, "\pi"] & \mathcal{P}_n \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

Type B (hyperoctahedral) Solomon descent algebra

Peak subalgebra

COROLLARY: Aguiar-Brauner-R. 2023+

The Type B Eulerian idempotents $\{e_k^{\mathfrak{S}_n^\pm}\}_{k=0,1,\ldots,n} \subset \mathrm{Sol}_{\mathfrak{S}_n^\pm}$

map under π to a complete family of orthogonal

peak idempotents $\{e_k^{\mathcal{P}_n}\}_{k=0,1,\ldots,n} \subset \mathcal{P}_n$,

Aguiar-Bergeron-Nyman 2004

and their $\mathfrak{S}_n$-rep decomposition $$\Bbbk\mathfrak{S}_n = \bigoplus_{k=0}^{n} \Bbbk\mathfrak{S}_n \cdot e_k^{\mathcal{P}_n}$$

describes the cohomological decomposition

$$H^{k} Z_n = H^{2(n-k)} Z_n$$

where $Z_n := \mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^3)/\mathbb{Z}_2^n = \mathrm{Conf}_n\left((\mathbb{R}^3 - \{0\}) / \underset{p \mapsto -p}{\mathbb{Z}_2} \right) = \mathrm{Conf}_n(\mathbb{RP}^2 \times (0,\infty))$

another configuration space!

Useful for...

COROLLARY

Aguiar - Brauner - R.
2023+

$$\Bbbk\mathfrak{S}_n \cdot e_k^{\mathcal{P}_n} \cong \bigoplus_{\substack{\lambda \vdash n: \\ \lambda \text{ has} \\ k \text{ odd parts}}} \mathrm{Lie}_\lambda$$

higher Lie reps

---

proof idea:

$$\Bbbk\mathfrak{S}_n \cdot e_k^{\mathcal{P}_n} \cong H^{2(n-k)} Z_n \cong H^{2(n-k)} \mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^3)/\mathbb{Z}_2^n$$

$$\cong \left(H^{2(n-k)} \mathrm{Conf}_n^{\mathbb{Z}_2}(\mathbb{R}^3) \right)^{\mathbb{Z}_2^n}$$

$$\cong \bigoplus_{\substack{\lambda \vdash n: \\ \lambda \text{ has} \\ k \text{ odd parts}}} \mathrm{Lie}_\lambda$$

with some algebra!

$\blacksquare$

Thanks

for your

attention!