Cyclic and Dihedral Sieving Phenomena

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Texas State University REU Seminar

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OUTLINE

1. Three combinatorial families, with counts:
   - subsets
   - triangulations
   - noncrossing partitions

2. q-counts and Cyclic Sieving Phenomena (CSP)

3. (q,t)-counts and Dihedral Sieving Phenomena (DSP)

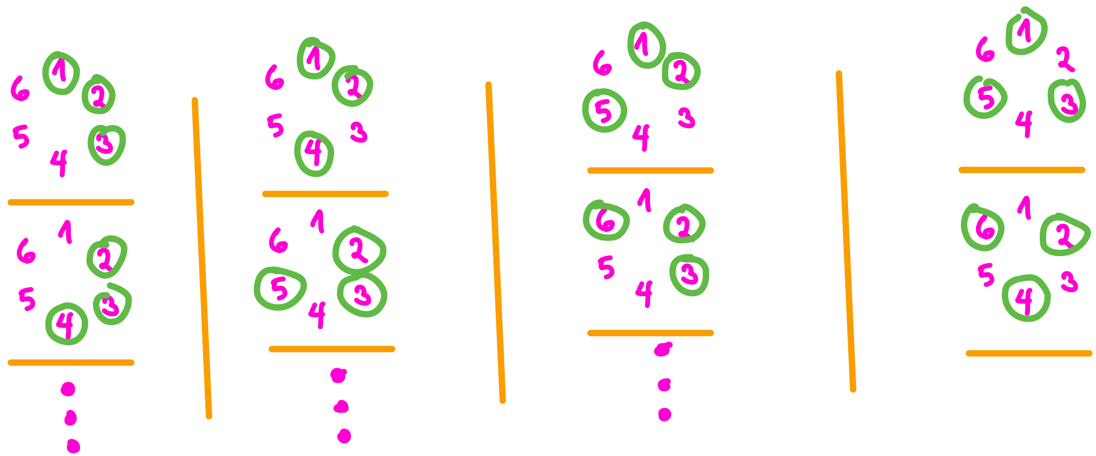
1. Three combinatorial families, with counts

$$\#\ k\text{-element subsets of } \{1,2,\dots,n\} = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

binomial coefficient

where $n! := n(n-1)\cdots 3\cdot 2\cdot 1$

EXAMPLE $n=6$, $k=3$: $\binom{6}{3} = \frac{6!}{3!\,3!} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} = 20$



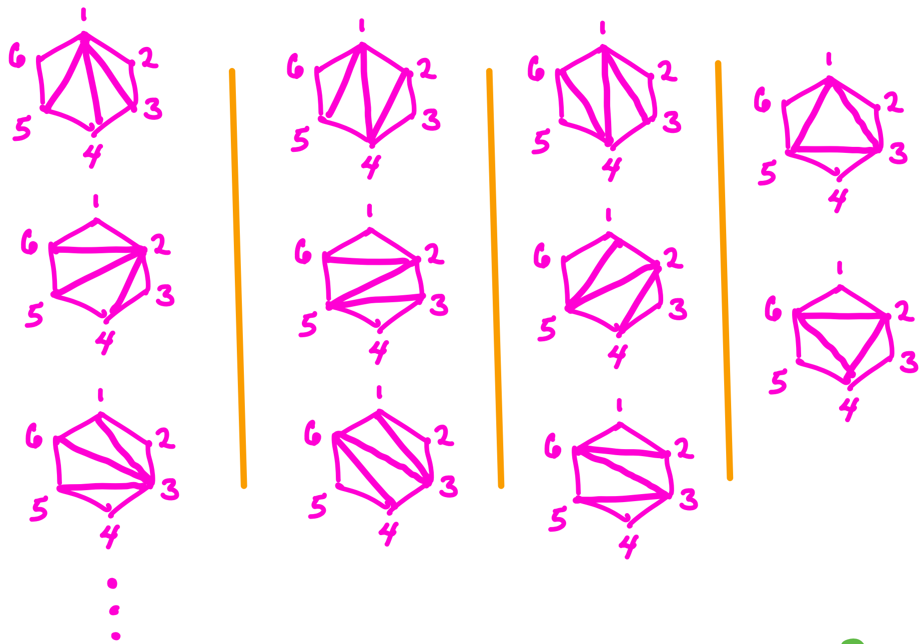
$6 + 6 + 6 + 2 = 20$

$$\#\text{triangulations of } (n+2)\text{-gon} = \frac{1}{n+1}\binom{2n}{n} =: C_n$$

Catalan number

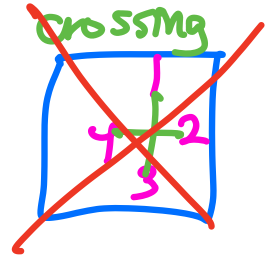
EXAMPLE $n=4$

$$C_4 = \frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{1}{5}\binom{8}{4} = \frac{8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2} = 14$$

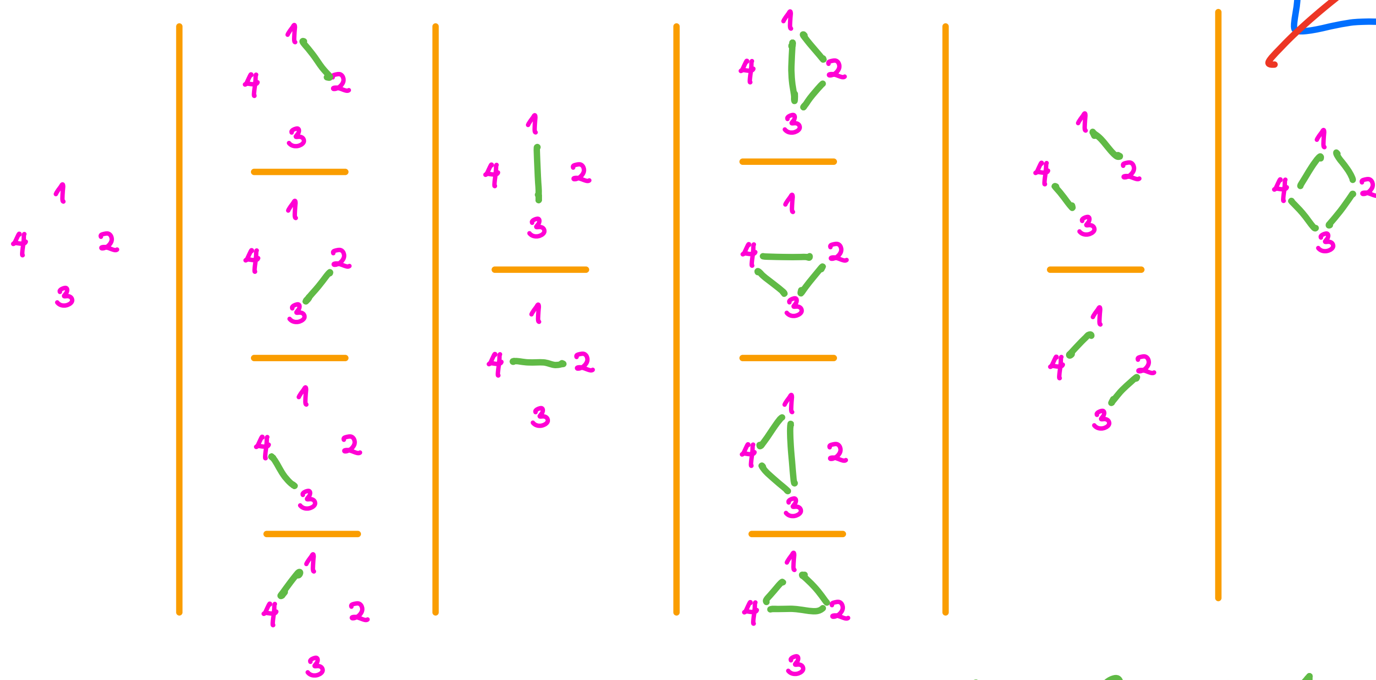


$$6 + 3 + 3 + 2 = 14$$

\# non-crossing (set) partitions of $\{1,2,\dots,n\}$ $= C_n = \frac{1}{n+1}\binom{2n}{n}$ (Kreweras 1972)



EXAMPLE $n=4$ $\quad C_4 = \frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{1}{5}\binom{8}{4} = 14$



$1 + 4 + 2 + 4 + 2 + 1 = 14$

2. q-counts and Cyclic Sieving Phenomena (CSP)

Suppose a finite set X is permuted by a cyclic group $C = \langle c \rangle = \{1, c, c^2, \ldots, c^{m-1}\}$ of order m.

DEFINITION:
Say that a polynomial $X(q)$ in variable q together with X and its C-action gives a CSP if every $c^d \in C$ has its fixed set

$$X^{c^d} := \{x \in X : c^d(x) = x\}$$

of size $\#X^{c^d} = \left[X(q)\right]_{q=(\zeta_m)^d}$ where $\zeta_m = e^{2\pi i/m} \in \mathbb{C}$

THEOREM (R.-Stanton-White 2004)

One has a CSP for $X = \{k\text{-element subsets of } \{1,2,\ldots,n\}\}$ with $C = \langle c \rangle$ in which $c =$

permutes cyclically mod n

and $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q := \dfrac{[n]!_q}{[k]!_q\,[n-k]!_q}$

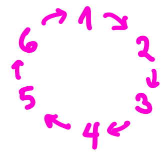
q-binomial coefficient

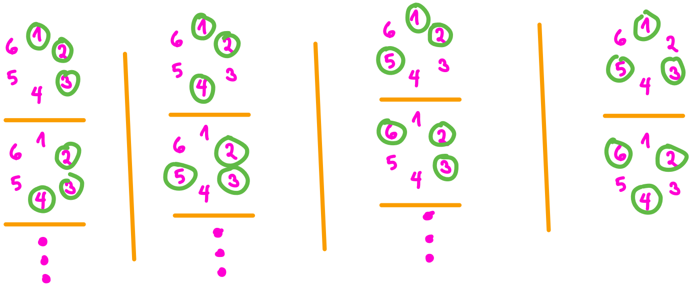
where $[n]!_q := [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

EXAMPLE

X = 3-element subsets of $\{1,2,3,4,5,6\}$

$C = \langle c \rangle$ permuting via 

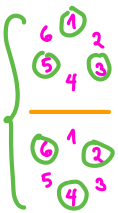


$$X(q) = \begin{bmatrix}6\\3\end{bmatrix}_q = \frac{[6]!_q}{[3]!_q\,[3]!_q} = \frac{[6]_q[5]_q[4]_q}{[3]_q[2]_q[1]_q}$$

$$= 1+q+2q^2+3q^3+3q^4+3q^5+3q^6+2q^7+q^8+q^9$$

$q=\zeta_6^0=1$ $\rightsquigarrow$ 20

$q=\zeta_6^3=-1$ $\rightsquigarrow$ 0

$q=\zeta_6^2,\ \zeta_6^4$ $\rightsquigarrow$ $2 = \#\Big\{$  $\Big\}$

$q=\zeta_6^1,\ \zeta_6^5$ $\rightsquigarrow$ 0

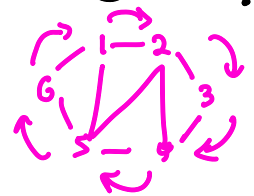
MacMahon's q-Catalan number (1915)

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

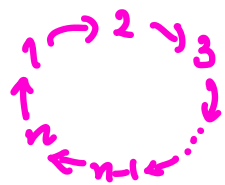
does double duty for us!

THEOREM (R.-Stanton-White 2004) One has a CSP for $X(q) = C_n(q)$ and both ...

(a) $X = \{$triangulations of $(n+2)$-gon$\}$ with $C = \langle c \rangle$ cyclically permuting vertices



(b) $X = \{$non crossing partitions of $\{1,2,\ldots,n\}\}$ with $C = \langle c \rangle$ cyclically permuting $c =$

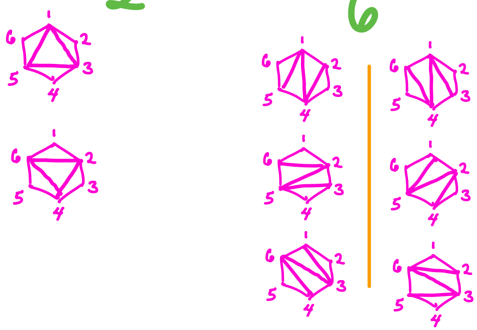


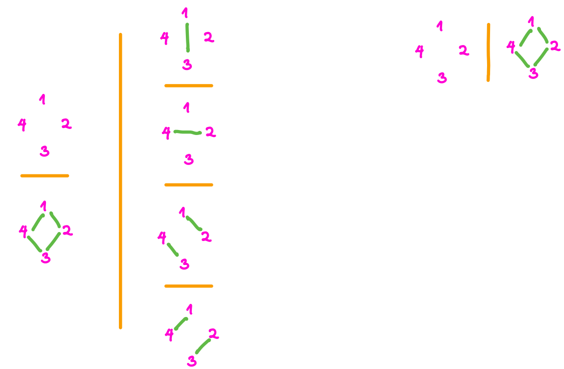
EXAMPLE $n=4$

$$X(q) = C_4(q) = \frac{1}{[5]_q}\begin{bmatrix} 2\cdot 4 \\ 4 \end{bmatrix}_q = \frac{[8]_q[7]_q[6]_q[5]_q}{[5]_q[4]_q[3]_q[2]_q}$$

$$= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$$

$q=\zeta_6^1, \zeta_6^5$	$q=\zeta_6^2, \zeta_6^4$	$q=\zeta_6^3=-1$	$q=1$	$q=\zeta_4^2=-1$	$q=\zeta_4^1, \zeta_4^3$
0	2	6	14	6	2





triangulations | non crossing set partitions

REMARKS

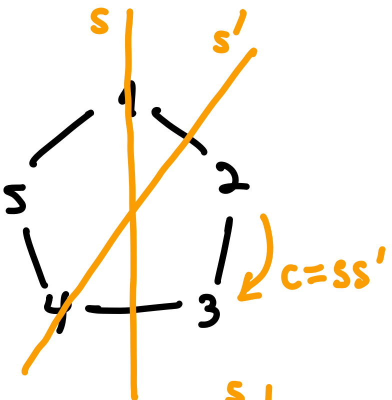
- $\left[X(q)\right]_{q=\zeta_m} \notin \mathbb{Z}$ for other values of m !
- Many generalizations exist.
- (proof quality)

  $\longleftarrow$ GOOD —— OK —— BRUTE FORCE ——

  subsets | noncrossing partitions | triangulations
- Bicyclic sieving phenomena with $C \times C'$ seem to show up too.

3. (q,t)-counts and Dihedral Sieving Phenomena (DSP)

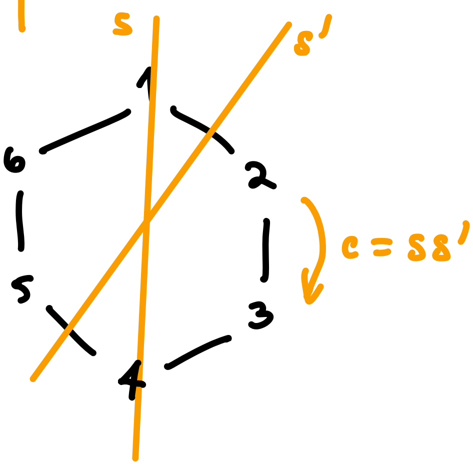
Often when X is permuted by $C = \langle c \rangle = \{1, c, c^2, \ldots, c^{m-1}\}$ it is also permuted by a dihedral group of order $2m$ (:= symmetries of regular m-gon)



$$I_2(m) \cong \langle s, s' \mid s^2 = (s')^2 = 1, \ (ss')^m = 1 \rangle$$

$$\cong \langle s, c \mid s^2 = c^m = 1, \ scs = c^{-1} \rangle$$

$$= \{ \underbrace{1, c, c^2, \ldots, c^{m-1}}_{\text{rotations}}, \underbrace{s, sc, \ldots, sc^{m-1}}_{\text{reflections}} \}$$

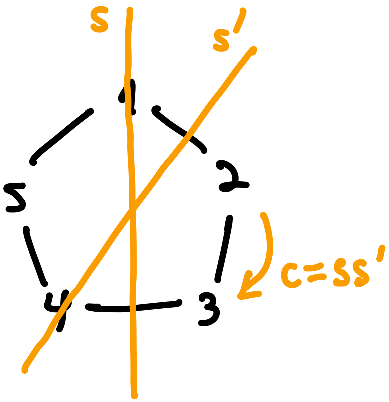


DEF'N: (Rao & Suk REU 2017) When X is permuted by dihedral group $I_2(m)$ with m odd (!), and $X(q,t)$ is a symmetric polynomial in q and t, say that $X(q,t)$ gives a DSP if

(i) $\#X^{c^d} = \left[X(q,t)\right]_{q=\zeta_m^d,\ t=\zeta_m^{-d}}$ $\left(\Leftrightarrow X(q,q^{-1}) \text{ gives a CSP}\right)$

and

(ii) $\#X^{s} = \left[X(q,t)\right]_{q=+1,\ t=-1}$

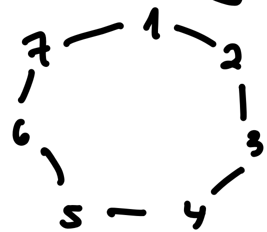


In other words, $\forall w \in I_2(m)$

$$\#X^{w} = \left[X(q,t)\right]_{\{q,t\} = \left\{\begin{array}{c}\text{eigenvalues}\\ \lambda_1, \lambda_2\\ \text{of } w \text{ on } \mathbb{R}^2\end{array}\right\}}$$

THEOREM (Rao-Suk REU 2017) For n odd,

$X = \{k\text{-subsets of } \{1,2,\ldots,n\}\}$ permuted dihedrally by $I_2(n)$ as in the vertices of the n-gon



one has a DSP using

$$X(q,t) = \left\{ {n \atop k} \right\} := \frac{\{n\}!}{\{k\}!\,\{n-k\}!}$$

where $\{n\}! := \{n\}\{n-1\}\cdots\{3\}\{2\}\{1\}$

and $\{n\} := q^{n-1} + q^{n-2}t + q^{n-3}t^2 + \ldots + qt^{n-2} + t^{n-1} = \dfrac{q^n - t^n}{q-t}$

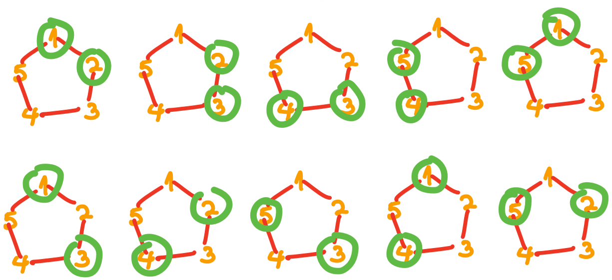
EXAMPLE $n=5$ $k=2$

$$\left\{ 5 \atop 2 \right\} = \frac{\{5\}!}{\{2\}!\,\{3\}!} = \frac{\{5\}\{4\}}{\{2\}\{1\}} = \frac{(q^4+q^3t+q^2t^2+qt^3+t^4)(q^3+q^2t+qt^2+t^3)}{(q+t)\ (1)}$$

$$= q^6+q^5t+2q^4t^2+2q^3t^3+2q^2t^4+qt^5+t^6$$

$q=\zeta_5^0=1$
$t=\zeta_5^0=1$

10

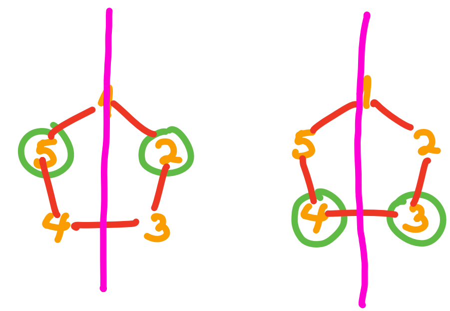


$q=\zeta_5^d$
$t=\zeta_5^{-d}$
$d=1,2,3,4$

0

$q=+1$
$t=-1$

2



THEOREM (Rao–Suk REU 2017) For n odd,

$X = \left\{\begin{matrix}\text{noncrossing partitions} \\ \text{of } \{1,2,\ldots,n\}\end{matrix}\right\}$ permuted dihedrally by $I_2(n)$

has a DSP with $X(q,t) = \frac{1}{\{n+1\}} \left\{\begin{matrix} 2n \\ n \end{matrix}\right\}$

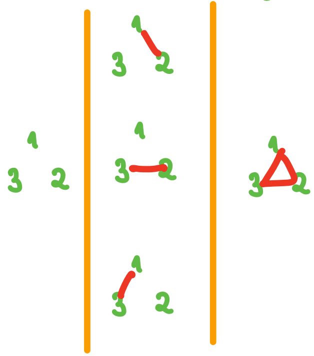
EXAMPLE $n=3$

$$X(q,t) = \frac{1}{\{4\}} \left\{\begin{matrix} 2\cdot 3 \\ 3 \end{matrix}\right\} = \frac{\{6\}\{5\}\{4\}}{\{4\}\{3\}\{2\}}$$

$$= q^6 + q^4t^2 + q^3t^3 + q^2t^4 + t^6$$

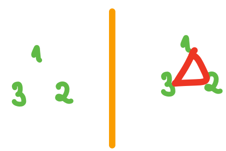
$q=\zeta_3^0=1$
$t=\zeta_3^0=1$

5



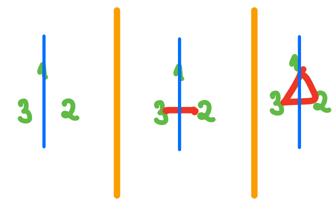
$q=\zeta_3^d$, $t=\zeta_3^{-d}$ $\quad d=1,2$

2



$q=+1$
$t=-1$

3



THEOREM (Rao-Suk REU 2017) For n odd,

$X = \left\{ \begin{array}{l} \text{triangulations} \\ \text{of an } (n+2)\text{-gon} \end{array} \right\}$ permuted dihedrally by $I_2(n+2)$

has a DSP with $X(q,t) = (qt)^{\binom{n}{2}} \mathrm{Cat}_n(q,t)$

Here $\mathrm{Cat}_n(q,t) :=$ Garsia & Haiman's (q,t)-Catalan #

$:=$ bigraded Hilbert series in q,t for the S_n-antisymmetric component of

$$\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n] / \left(\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n]^{S_n}_{+} \right)$$

$$= \sum_{\text{partitions } \mu \text{ of } n} \frac{t^{m} q^{m} (1-t)(1-q) \prod (1-q^{a'}t^{l'})}{\prod (q^{a}-t^{l+1})(t^{l}-q^{a+1})}$$

EXAMPLE $n=3$

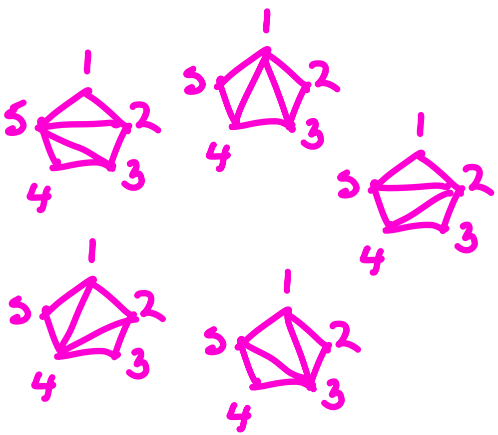
$$X(q,t) = (qt)^{\binom{3}{2}} \mathrm{Cat}_3(q,t)$$

$$= q^6t^3 + q^5t^4 + q^4t^4 + q^4t^5 + q^3t^6$$

$q=\zeta_5^0=1$

$t=\zeta_5^0=1$

5



$q=\zeta_5^d$ $\quad d=1,2,3,4$

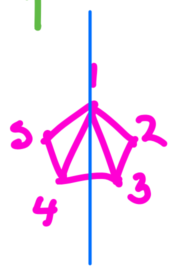
$t=\zeta_5^{-d}$

0

$q=+1$

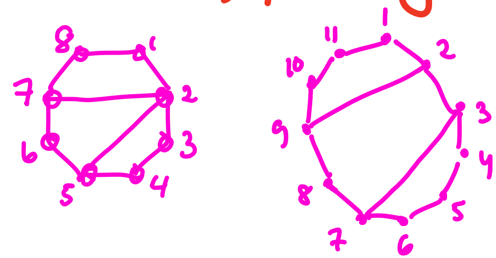
$t=-1$

1



REMARKS

- In REU 2019, Stier, Wellman & Xu generalized the triangulations DSP in two ways:
  - using quadrangulations, pentagonalizations, etc



and $X(q,t) =$ (q,t)-Fuss-Catalan #'s

  - using clusters in cluster algebras of finite type W

and $X(q,t) = (q,t)$-Catalan # of type W (Stump 2010)

- A satisfactory notion of DSP for actions of dihedral group $I_2(m)$ with m even (and convincing examples) still seems to be missing.

PROBLEM

Find such a notion!

Thanks for your attention!

REFERENCES

R., Stanton & White "The cyclic sieving phenomenon"
J. Comb. Thy. Ser A 108 (2004)

Rao & Suk "Dihedral sieving phenomena"
Disc. Math. 343 (2020)

Stier, Wellman & Xu "Dihedral sieving on cluster complexes"
arXiv: 2011.11885