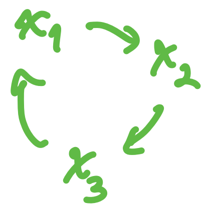
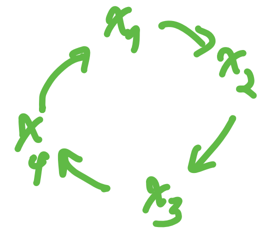
Invariant Theory of Cyclic Permutations: a test case





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1. Review invariant theory of finite groups
(ref: R. Stanley "Invariant theory of finite groups and their applications to combinatorics")
1979 Bull AMS

2. (tame) EXAMPLE: Symmetric group $\mathfrak{S}_n$
+ reflection groups

3. TEST CASE: Cyclic permutations C_n

4. Conjectures, Theorems and Questions

1. Review invariant theory of finite groups

$S = k[x_1, \ldots, x_n]$ polynomial ring over k a field

$\circlearrowright$

$GL_n(k)$ acts via linear substitutions of variables:

$$g \cdot f(\underline{x}) := f(g^{-1}\underline{x})$$

For a subgroup G, $S^G :=$ G-invariant subring

$$:= \{ f \in S : g(f) = f \quad \forall g \in G \}$$

MAIN QUESTIONS:

- Structure of S^G as a ring?
- Structure of S as an S^G-module?

EXAMPLES

symmetric group $\qquad$ cyclic group $\qquad$ alternating group

$$GL_3(k) > \mathfrak{S}_3 > C_3 \; (= \mathfrak{A}_3)$$

$\mathfrak{S}_3 = \{e, (12), (123), (13), (132), (23)\}$

$C_3 = \{e, (123), (132)\}$

$$\begin{tikzcd} x_1 \arrow[r] & x_2 \arrow[d] \\ & x_3 \arrow[ul] \end{tikzcd}$$

$$\begin{array}{ccccc} S^{\mathfrak{S}_3} & \hookrightarrow & S^{C_3} & \hookrightarrow & S \\ \| & & \| & & \| \\ k[e_1, e_2, e_3] & & k[e_1, e_2, e_3, \Delta] & & k[x_1, x_2, x_3] \end{array}$$

$e_1 = x_1 + x_2 + x_3$
$e_2 = x_1x_2 + x_1x_3 + x_2x_3$
$e_3 = x_1x_2x_3$

elementary symmetric polynomials

(algebraically independent)

$\Delta = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

$\Delta^2 = p(e_1, e_2, e_3)$
$= e_1^2e_2^2 - 4e_2^3 - 4e_1^3e_3 - 27e_3^2 + 18e_1e_2e_3$

a single degree 6 syzygy

GENERAL FACTS for G finite:

PROP (Easy): $S^G \hookrightarrow S = k[x_1,\ldots,x_n]$ is an integral ring extension

$\Longrightarrow$ $\begin{cases} S \text{ is a fin. gen'd } S^G\text{-module, and} \\ \text{Krull dim}(S^G) = n \;\; (= \text{Krull dim}(S)) \end{cases}$

THM: S^G is fin. gen'd as a k-algebra (1926)

(E. Noether) by elements of degree $\leq \#G$ if $\mathrm{char}(k)=0$ (1915)

THM: S is gen'd as S^G-module by elements

(Stanley 1977) of degree $\leq \#G$ if $\mathrm{char}(k)=0$

THM: If $\#G \in k^\times$, S^G is a Cohen-Macaulay ring, and

(Hochster-Eagon 1971) S is a Cohen-Macaulay S^G-module

Why are they C-M rings & modules?

One can split the S^G-module inclusion $S^G \underset{\pi_G}{\overset{\hookrightarrow}{\leftarrow}} S$

$$S = \underbrace{S^G}_{\mathrm{im}(\pi^G)} \oplus \ker(\pi^G)$$

where $\pi_G(f) := \frac{1}{\#G} \sum g(f)$

Reynolds operator (= averageing)

Any h.s.o.p. $\theta_1, \ldots, \theta_n$ for S^G
is also an h.s.o.p. for S,
is an S-regular sequence, ($S = k[x_1, \ldots, x_n]$ is a C-M ring)
so an S^G-regular sequence, via the splitting.

In fact, S splits further, always assuming $\#G \in k^\times$ ...

DEF'N: For each irreducible G-rep'n χ over k, let S^χ be the χ-isotypic component of $S = \bigoplus_{d=0}^{\infty} S_d$

e.g. for χ = trivial rep'n
$$\begin{array}{ccc} G & \longrightarrow & GL_1(k) = k^\times \\ g & \longmapsto & 1 \end{array}$$
, then $S^\chi = S^G$

Each S^χ is a C-M S^G-module, and one has an S^G-module decomposition
$$S = \bigoplus_{\substack{\text{irreducible} \\ G\text{-rep'ns } \chi}} S^\chi$$

COMPUTATIONAL AID:

All these rings and modules are $\mathbb{N}$-graded $M = \bigoplus_{d=0}^{\infty} M_d$

so one can ask for their Hilbert series

$$\mathrm{Hilb}(M,t) := \sum_{d=0}^{\infty} \dim_k(M_d) \cdot t^d$$

Molien's Theorem: Any G-irred. rep'n χ has

$$\mathrm{Hilb}(S^{\chi}, t) = \frac{\dim \chi}{\# G} \sum_{g \in G} \frac{\chi(g^{-1})}{\det(I - t \cdot g)}$$

character value – trace of g^{-1} acting in χ

EXAMPLE For $S = k[x_1, x_2, x_3]$,
where $S^{\mathfrak{S}_3} = k[e_1, e_2, e_3]$,

Molien's Thm. predicts

$$\mathrm{Hilb}(S^{\mathfrak{S}_3}, t) = \frac{1}{\#\mathfrak{S}_3} \sum_{w \in \mathfrak{S}_3} \frac{1}{\det(I_3 - t \cdot w)}$$

$$= \frac{1}{6}\Big[\underbrace{\frac{1}{(1-t)^3}}_{w=e} + 3\underbrace{\frac{1}{(1-t^2)(1-t)}}_{w=(12),(13),(23)} + 2\underbrace{\frac{1}{1-t^3}}_{w=(123),(132)}\Big]$$

$$= \frac{1}{(1-t^1)(1-t^2)(1-t^3)}$$

$$e_1 \qquad e_2 \qquad e_3$$

EXAMPLE Recall $C_3 = \{e, (123), (132)\} = \mathcal{C}l_3$ had

$$S^{C_3} = k[e_1, e_2, e_3, \Delta]$$ and one relation of degree 6.

with degrees $1, 2, 3, 3$

$$\Delta^2 = p(e_1, e_2, e_3)$$

Molien's Thm. predicts

$$\mathrm{Hilb}(S^{\mathfrak{S}_3}, t) = \frac{1}{\# C_3} \sum_{w \in C_3} \frac{1}{\det(I_3 - t \cdot w)}$$

$$= \frac{1}{3}\left[\underset{w=e}{\frac{1}{(1-t)^3}} + \underset{w=(123),\,(132)}{2\,\frac{1}{1-t^3}}\right]$$

relation $\Delta^2 = p(e_1, e_2, e_3)$

$$= \frac{1-t+t^2}{(1-t)^3(1+t+t^2)} = \frac{1-t^6}{\underset{e_1}{(1-t^1)}\,\underset{e_2}{(1-t^2)}\,\underset{e_3}{(1-t^3)}\,\underset{\Delta}{(1-t^3)}}$$

2. (Tame) EXAMPLE: Symmetric group $\mathfrak{S}_n$

$\mathfrak{S}_n$ has irred. rep'ns χ^λ indexed by partitions λ of n

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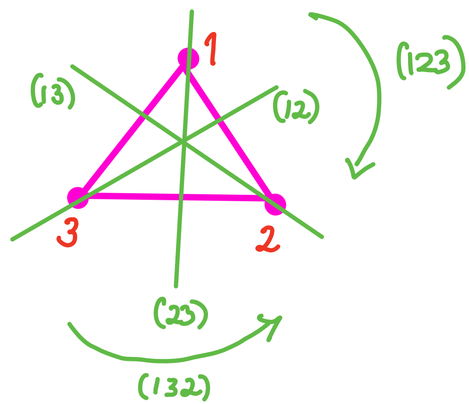
EXAMPLE $\mathfrak{S}_3$ has three irred. rep'ns

$\chi^{(3)}$ = trivial rep'n
$$\begin{aligned} \mathfrak{S}_3 &\longrightarrow GL_1(k) = k^\times \\ \sigma &\longmapsto 1 \quad \forall \sigma \in \mathfrak{S}_3 \end{aligned}$$

$\chi^{(1,1,1)}$ = sign rep'n
$$\begin{aligned} \mathfrak{S}_3 &\longrightarrow GL_1(k) = k^\times \\ \sigma &\longmapsto \operatorname{sgn}(\sigma) = \pm 1 \end{aligned}$$

$\chi^{(2,1)}$ = reflection rep'n $\mathfrak{S}_3 \longrightarrow GL_2(k)$

as linear symmetries of a regular 2-simplex



$$\underset{\substack{\shortparallel \\ k[x_1,x_2,x_3]}}{S} = S^{\chi^{(3)}} \oplus S^{\chi^{(2,1)}} \oplus S^{\chi^{(1,1,1)}}$$

where

Hilbert series:

$$S^{\chi^{(3)}} = S^{\mathfrak{S}_3} = k[e_1,e_2,e_3] = \text{free } S^{\mathfrak{S}_3}\text{-module with basis } \{1\}$$

$$\frac{1}{(1-t)(1-t^2)(1-t^3)}$$

$$S^{\chi^{(1,1,1)}} = \mathfrak{S}_3\text{-anti-invariants} = \Delta \cdot k[e_1,e_2,e_3]$$
$$= \text{free } S^{\mathfrak{S}_3}\text{-module with basis } \{\Delta\}$$

$$\frac{t^3}{(1-t)(1-t^2)(1-t^3)}$$

$$S^{\chi^{(2,1)}} = S^{\mathfrak{S}_3}\cdot\{x_1-x_2,\ x_2-x_3\} \oplus S^{\mathfrak{S}_3}\cdot\{x_1^2-x_2^2,\ x_2^2-x_3^2\}$$
$$= \text{free } S^{\mathfrak{S}_3}\text{-module with basis in red shown}$$

$$2\cdot\frac{t^1+t^2}{(1-t)(1-t^2)(1-t^3)}$$

These simplest behaviors are hallmarks of reflection groups...

THEOREM (Shephard-Todd 1955, Chevalley 1955, T.A. Springer 1977)

For finite subgroups $G < GL_n(k) \circlearrowright S = k[x_1, \ldots, x_n]$ with $\#G \in k^\times$, the following are equivalent:

- S^G is a polynomial subalgebra $S^G = k[f_1, f_2, \ldots, f_n]$
- S (and each S^χ) are free S^G-modules
- G is generated by (pseudo-) reflections $r \in G$

(r has fixed space on k^n a hyperplane $\Leftrightarrow$ r diagonalizes to $\begin{bmatrix} \zeta & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$ for some $\zeta \in k^\times$

We know tons about $\mathrm{Hilb}(S^\chi, t)$ for reflection groups G.

3. TEST CASE: Cyclic permutations C_n

Ring structure for S^{C_n} gets out of control quickly...

$n=2$: $k[x_1,x_2]^{C_2} = k[x_1,x_2]^{\mathfrak{S}_2} = k[e_1,e_2]$ a polynomial ring

degrees: 1, 2

$n=3$: $k[x_1,x_2,x_3]^{C_3} = k[e_1,e_2,e_3,\Delta]$

$\cong k[a,b,c,d]/(d^2-p(a,b,c))$ a hypersurface quotient

degrees: 1, 2, 3, 3   6

$n=4$: $k[x_1,x_2,x_3,x_4]^{C_4} = k[e_1,e_2,e_3,e_4,f_2,f_3,f_4]$

$\cong k[a,b,c,d,e,f,g]/(\text{six relations})$

degrees: 1 2 3 4 2 3 4   6 6 6 7 7 8

not even a complete intersection

SUPPRESSED STORY:

One can say a bit about S^{C_n} as a free module over $S^{\mathfrak{S}_n} = k[e_1, e_2, \ldots, e_n]$

- know how to predict degrees of basis elements via some reflection group theory, because n-cycle $(1\,2 \cdots n)$ is a regular element of $\mathfrak{S}_n$ (in a sense defined by Springer 1974)
- finding the basis elements is harder, but a 2-step method of Garsia-Stanton 1984 was conjectured (R.-White 2012) to apply at least for $n = p$ a prime;
  - STEP 1 confirmed by the REU 2021 students, (Garg-Lu-Ren-Sun)
  - STEP 2 still missing!

(old)

IDEA: C_n is abelian, so take advantage of $\mathbb{N}^n$-grading on S by changing variables $x_1, \ldots, x_n$ to an eigenbasis $y_0, y_1, \ldots, y_{n-1}$:

$$C_n = \langle (1\,2\,3 \cdots n) \rangle = \left\langle \begin{array}{c|cccc} & x_1 & x_2 & \cdots & x_n \\ \hline x_1 & 0 & 0 & \cdots & 1 \\ x_2 & 1 & 0 & & \vdots \\ \vdots & & 1 & \ddots & \\ x_n & 0 & & 1 & 0 \end{array} \right\rangle \circlearrowright S = k[x_1, x_2, \ldots, x_n]$$

$$\cong \left\langle \begin{array}{c|cccc} & y_0 & y_1 & \cdots & y_{n-1} \\ \hline y_0 & 1 & & & 0 \\ y_1 & & \zeta & & \\ \vdots & & & \zeta^2 & \\ y_{n-1} & 0 & & \ddots & \zeta^{n-1} \end{array} \right\rangle \circlearrowright S = k[y_0, y_1, \ldots, y_{n-1}]$$

where $\zeta = e^{\frac{2\pi i}{n}} \in k$.

Then $S^{C_n} = k\text{-span of } \left\{ y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} : 0 \cdot a_0 + 1 \cdot a_1 + 2 \cdot a_2 + \cdots + (n-1) a_n \equiv 0 \bmod n \right\}$

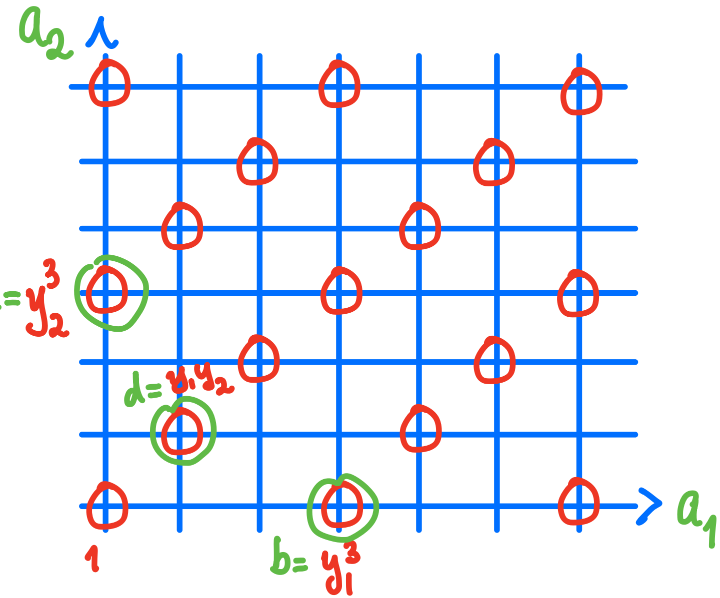
EXAMPLE

$$C_3 = \left\langle \begin{array}{c|ccc} & y_0 & y_1 & y_2 \\ \hline y_0 & 1 & 0 & 0 \\ y_1 & 0 & \zeta & 0 \\ y_2 & 0 & 0 & \zeta^2 \end{array} \right\rangle \curvearrowright S = \mathbb{C}[y_0, y_1, y_2] \qquad \zeta = e^{\frac{2\pi i}{3}}$$

$$S^{C_3} = \mathbb{C}[y_0, y_1^3, y_2^3, y_1y_2] = \mathbb{C}\text{-span}\left\{ y_0^{a_0} y_1^{a_1} y_2^{a_2} : 0\cdot a_0 + 1\cdot a_1 + 2\cdot a_2 \equiv 0 \bmod 3 \right\}$$

$$\cong \mathbb{C}[a, b, c, d] / (d^3 - bc)$$

$\mathbb{N}$-degrees: $1 \quad 3 \quad 3 \quad 2 \qquad 6$



$$\mathrm{Hilb}(S^{C_3}; y_0, y_1, y_2) = \frac{1 - (y_1y_2)^3}{(1-y_0)(1-y_1^3)(1-y_2^3)(1-y_1y_2)}$$

$$= \frac{1 + y_1y_2 + y_1^2y_2^2}{(1-y_0)(1-y_1^3)(1-y_2^3)}$$

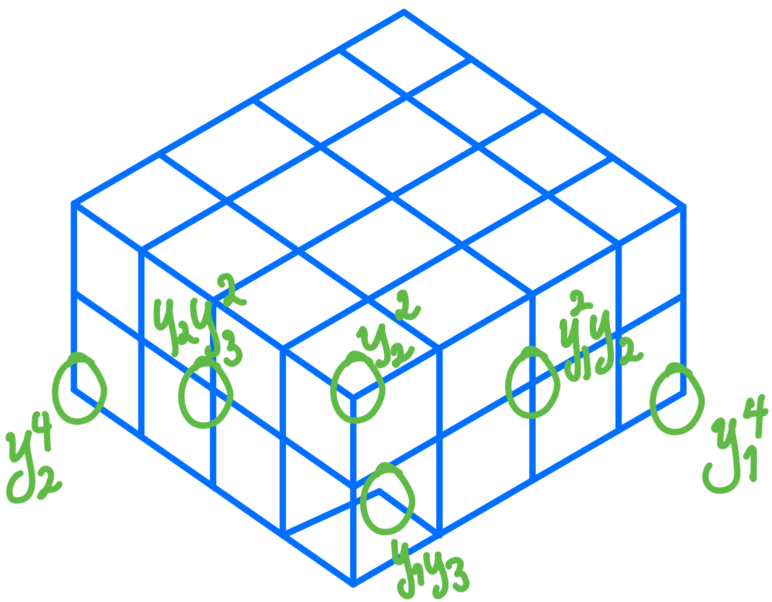
an affine semigroup ring

EXAMPLE $C_4 = \left\langle \begin{array}{c|cccc} & y_0 & y_1 & y_2 & y_3 \\ \hline y_0 & 1 & 0 & 0 & 0 \\ y_1 & 0 & i & 0 & 0 \\ y_2 & 0 & 0 & -1 & 0 \\ y_3 & 0 & 0 & 0 & -i \end{array} \right\rangle \circlearrowright S = \mathbb{C}[y_0, y_1, y_2, y_3]$

$$S^{C_4} = \mathbb{C}[y_0, y_1^4, y_2^2, y_3^4, y_1y_3, y_1^2y_2, y_2y_3^2]$$

$$\cong \mathbb{C}[a, b, c, d, e, f, g] \,/\, (e^4 - bd,\ f^2 - bc,\ g^2 - cd,\ fg - ce^2,\ e^2f - bg,\ e^2g - df)$$

$\mathbb{N}$-degrees: 1 4 2 4 2 3 3   8 6 6 6 7 7



a more complicated affine semigroup ring

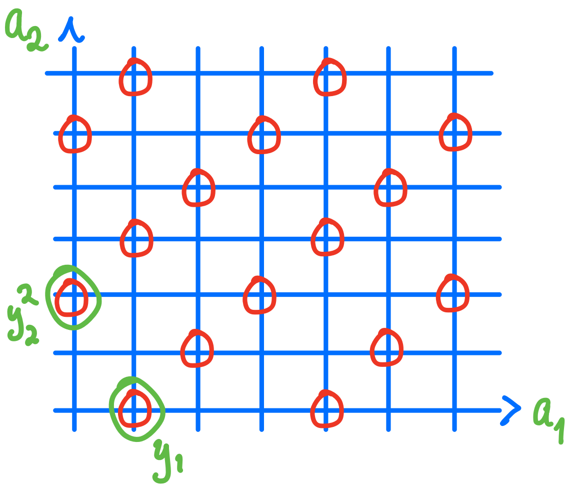
What about S and S^{χ} as S^{C_n}-modules?

Irreducible C_n-rep'ns are $\chi^{(d)} : C_n \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$

$d = 0, 1, \ldots, n-1$ $\quad \{1, c, c^2, \ldots, c^{n-1}\}$ $\quad c \longmapsto \zeta^d$ $\quad \zeta = e^{2\pi i/n}$

and $S^{\chi^{(d)}} = \mathbb{C}\text{-span of } \{ y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} : \sum_{i=0}^{n-1} i a_i \equiv d \bmod n \}$

EXAMPLE C_3 has $S^{\chi^{(1)}} = S^{C_3} \cdot y_1 + S^{C_3} \cdot y_2^2$



with a 2-periodic resolution

by work of Eisenbud 1980

on maximal C-M modules

over hypersurface rings

and matrix factorizations.

$$0 \longleftarrow S^{\mathcal{X}^{(1)}} \longleftarrow \begin{matrix} R(-1) \\ \oplus \\ R(-2) \end{matrix} \xleftarrow{\begin{bmatrix} -c & d^2 \\ d & -b \end{bmatrix}} \begin{matrix} R(-4) \\ \oplus \\ R(-5) \end{matrix} \xleftarrow{\begin{bmatrix} b & d^2 \\ d & c \end{bmatrix}} \begin{matrix} R(-7) \\ \oplus \\ R(-8) \end{matrix} \xleftarrow{\begin{bmatrix} -c & d^2 \\ d & -b \end{bmatrix}} \cdots$$

$$y_1 \longleftarrow\!\!\shortmid\; e_1$$

$$y_2^2 \longleftarrow\!\!\shortmid\; e_2$$

where $R = S^{C_3} = \mathbb{C}[y_0, y_1^3, y_2^3, y_1 y_2]$
$\cong \mathbb{C}[a,b,c,d]/(d^3 - bc)$

$\Rightarrow$ Poincaré series calculation

$$\mathrm{Poin}_{S^{C_3}}\left(S^{\mathcal{X}^{(1)}}; y_0, y_1, y_2, t\right) := \sum_{\underline{a} \in \mathbb{N}^3} \sum_{i \geq 0} t^i \cdot y_0^{a_0} y_1^{a_1} y_2^{a_2} \cdot \beta_{i,\underline{a}}$$

where $\beta_{i,\underline{a}} = \dim_k \mathrm{Tor}_i^{S^{C_3}}\left(S^{\mathcal{X}^{(1)}}, k\right)_{\underline{a}}$

$$= \frac{y_1 + y_2^2 + t\left(y_1 y_2^3 + y_1^3 y_2^2\right)}{1 - t^2 y_1^3 y_2^3}$$

$\mathbb{N}^3$-graded

$y_0 = y_1 = y_2 = y$

$\mathbb{N}$-graded

$$\mathrm{Poin}_{S^{C_3}}\left(S^{\mathcal{X}^{(1)}}; y, t\right) = \frac{y + y^2 + t\left(y^4 + y^5\right)}{1 - t^2 y^6}$$

What about C_n for $n \geq 4$?

Even minimal k-algebra generators for S^{C_n} are not completely understood:

$$\left\{ \begin{array}{c} \text{min. generators} \\ y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} \\ \text{for } S^{C_n} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{multi-subsets} \\ \{\bar{0}^{a_0}, \bar{1}^{a_1}, \cdots (\overline{n-1})^{a_{n-1}}\} \subset \mathbb{Z}/n\mathbb{Z} \\ \text{summing to } \bar{0}, \text{ with} \\ \text{no subsums } \bar{0} \end{array} \right\}$$

- EASY: They live in degrees $0, 1, \ldots, n$
- HARDER: Simple description in degrees $\geq n/2$
  (CONJ of Elashvili 1994, THM of P. Yuan 2007)
- NOT KNOWN in degrees $< n/2$

Similarly, one has

$$\left\{ \begin{array}{c} \text{min. generators} \\ y_0^{a_0} y_1^{a_1} \cdots y_{n-1}^{a_{n-1}} \\ \text{for } S^{X^{(d)}} \\ \text{as } S^{C_n}\text{-module} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{multi-subsets} \\ \{\bar{0}^{a_0}, \bar{1}^{a_1}, \ldots, (\overline{n-1})^{a_{n-1}}\} \subset \mathbb{Z}/n\mathbb{Z} \\ \text{summing to } \bar{d}, \text{ with} \\ \text{no subsums } \bar{0} \end{array} \right\}$$

- **EASY:** They live in degrees $0, 1, \ldots, n-1$
- **THEOREM** (REU 2021 Garg-Lu-Ren-Sun) Simple description in degrees $\geq n/2$, (using the previous results!)
- **NOT KNOWN** in degrees $< n/2$

Resolutions of $S^{X^{(d)}}$ over S^{C_n} ?

total:	2	2	2
0:	.	.	.
1:	1	.	.
2:	1	.	.
3:	.	1	.
4:	.	1	.
5:	.	.	1
6:	.	.	1

$S^{X^{(1)}}$ over S^{C_3}

$n=3$

(2-periodic from before)

	0	1	2	3	4	5	6
total:	3	8	24	72	216	648	1944
1:	1	.	.	.	.	.	.
2:	1	.	.	.	.	.	.
3:	1	1	.	.	.	.	.
4:	.	3	.	.	.	.	.
5:	.	3	2	.	.	.	.
6:	.	1	7	.	.	.	.
7:	.	.	9	4	.	.	.
8:	.	.	5	16	.	.	.
9:	.	.	1	25	8	.	.
10:	.	.	.	19	36	.	.
11:	.	.	.	7	66	16	.
12:	.	.	.	1	63	80	.
13:	.	.	.	.	33	168	32
14:	.	.	.	.	9	192	176
15:	.	.	.	.	1	129	416
16:	.	.	.	.	.	51	552
17:	.	.	.	.	.	11	450
18:	.	.	.	.	.	1	231
19:	.	.	.	.	.	.	73
20:	.	.	.	.	.	.	13
21:	.	.	.	.	.	.	1

$S^{X^{(1)}}$ over S^{C_4}

	0	1	2	3	4	5	6
total:	3	8	24	72	216	648	1944
1:	1	.	.	.	.	.	.
2:	2	.	.	.	.	.	.
3:	.	2	.	.	.	.	.
4:	.	4	.	.	.	.	.
5:	.	2	4	.	.	.	.
6:	.	.	10	.	.	.	.
7:	.	.	8	8	.	.	.
8:	.	.	2	24	.	.	.
9:	.	.	.	26	16	.	.
10:	.	.	.	12	56	.	.
11:	.	.	.	2	76	32	.
12:	.	.	.	.	50	128	.
13:	.	.	.	.	16	208	64
14:	.	.	.	.	2	176	288
15:	.	.	.	.	.	82	544
16:	.	.	.	.	.	20	560
17:	.	.	.	.	.	2	340
18:	.	.	.	.	.	.	122
19:	.	.	.	.	.	.	24
20:							2

$S^{X^{(2)}}$ over S^{C_4}

$n=4$

total:	6	54	534	5286	52326
1:	1	.	.	.	.
2:	2	.	.	.	.
3:	2	2	.	.	.
4:	1	8	.	.	.
5:	.	15	6	.	.
6:	.	16	32	.	.
7:	.	10	82	18	.
8:	.	3	130	120	.
9:	.	.	137	390	54
10:	.	.	96	806	432
11:	.	.	42	1162	1698
12:	.	.	9	1210	4306
13:	.	.	.	911	7798
14:	.	.	.	480	10566
15:	.	.	.	162	10922
16:	.	.	.	27	8618
17:	.	.	.	.	5097
18:	.	.	.	.	2160
19:	.	.	.	.	594

$S^{\chi^{(1)}}$ over S^{C_5}

$n=5$

total:	8	102	1390	18950
1:	1	.	.	.
2:	2	.	.	.
3:	3	2	.	.
4:	1	12	.	.
5:	1	26	9	.
6:	.	27	67	.
7:	.	19	192	42
8:	.	12	299	361
9:	.	3	311	1285
10:	.	1	254	2617
11:	.	.	157	3618
12:	.	.	69	3815
13:	.	.	26	3209
14:	.	.	5	2137
15:	.	.	1	1150
16:	.	.	.	501
17:	.	.	.	163
18:	.	.	.	44
19:	.	.	.	7
20:				1

$S^{\chi^{(1)}}$ over S^{C_6}

$n=6$

4. Conjectures, Theorems and Questions

CONJECTURE (G-L-R-S) REU 2021 For $S^{\chi^{(d)}}$ $d=1,2,\dots,n-1$

$$\beta_{ij} \neq 0 \implies 3i+1 \leq j \leq ni+n-1$$

$:= \dim_k \mathrm{Tor}_i^{S^{C_n}}(S^{\chi^{(d)}}, k)_j$

THEOREM: (G-L-R-S) REU 2021 ($\Leftarrow$) holds in the above conjecture for $S^{\chi^{(1)}}$

CONJECTURE holds for $n=2$: trivially

$n=3$: from 2-periodic resolution

$n=4$: from this next ...

THEOREM: (G-L-R-S) REU 2021 Have an explicit minimal free resolution of $S^{\chi^{(1)}}, S^{\chi^{(2)}}, S^{\chi^{(3)}}$ as S^{C_4}-modules.

MFR structure is 2-recursive in the sense that $\partial^{(i)}$ for $S^{\chi^{(d)}}$ is block upper-triangular with blocks coming from $\partial^{(i-1)}, \partial^{(i-2)}$ for $S^{\chi^{(d')}}$

COROLLARY

$$\mathrm{Poin}_{S^{C_4}}(S^{\chi^{(1)}}; y, t) = \frac{y+y^2+y^3+t(y^5+y^6+y^7-y^4)-t^2y^8}{(1+ty^4)(1-t(2y^3+y^4))}$$

$$\mathrm{Poin}_{S^{C_4}}(S^{\chi^{(2)}}; y, t) = \frac{y+2y^2-ty^5}{1-t(2y^3+y^4)}$$

(actually 1-recursive)

Conjecture would also follow for $n=5$ if one could show...

CONJECTURE (G-L-R-S REU 2021)

$$\mathrm{Poin}_{S^{C_5}}\left(S^{\chi^{(d)}};y,t\right)=\frac{y+2y^2+2y^3+y^4-t\left(y^4+2y^5+2y^6+y^7\right)}{1-t\left(3y^3+4y^4+3y^5\right)+t^2y^8}$$

for $d=1,2,3,4$

Denominator is again quadratic in t, suggesting 2-recursive MFR.

QUESTION: Does the MFR for $S^{\chi^{(d)}}$ as S^{C_n} always have a 2-recursive structure?

- Should the $n=4$ explicit MFR for S^X over S^{C_4} generalize to all

cyclic groups $C_m = \left\langle \begin{array}{c|c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} & \begin{bmatrix} \zeta^a & 0 & 0 \\ 0 & \zeta^b & 0 \\ 0 & 0 & \zeta^c \end{bmatrix} \end{array} \right\rangle \circlearrowright S = \mathbb{C}[y_1, y_2, y_3]$

$\zeta = e^{2\pi i/m}$

Harris & Wehlau 2013 at least present/resolve the ring S^{C_m} when the cyclic group $C_m \hookrightarrow SL_3(\mathbb{C})$, i.e. $a+b+c \equiv \text{mod } m$

- Should we expect 2-recursive structure for the MFR of S^X over S^G when G is abelian?

Thanks for your attention!