(1)

# ECCO 2018

## q-counting and representation theory

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Lec 1. q-counting quotients of Boolean algebras

2. rep. theory & reflection groups

3. Molien's theorem & coinvariant algebras

4. Cyclic sieving phenomena & Springer's theorem

Start with some important posets
(= partially ordered sets)

the Boolean algebra $2^{[n]}$ where $[n] := \{1, 2, \ldots, n\}$

consisting of all subsets $S \subseteq [n]$,

partially ordered by inclusion: $S \leq T$ means $S \subseteq T$

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The Hasse diagram depicts the graph with

vertices = poset elements

edges $\{S, T\}$ whenever $S \lessdot T$ "S is covered by T", meaning $S \leq T$ and $\nexists\, U$ with $S \lneq U \lneq T$

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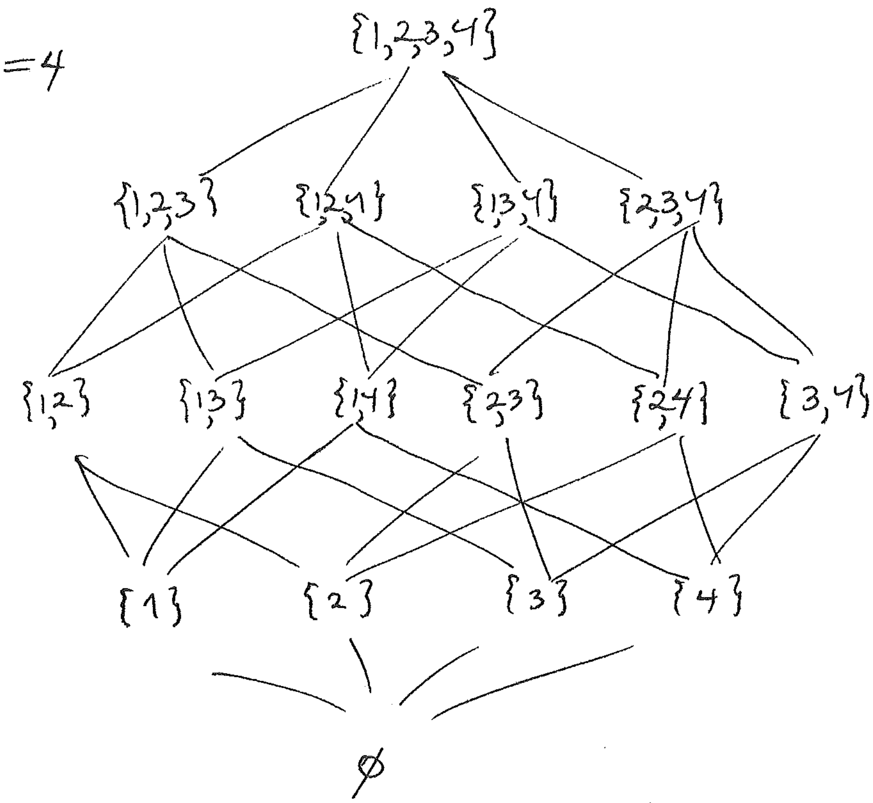
e.g. $n = 4$

$2^{[4]} =$




| rank | rank sizes |
| --- | --- |
| 4 | $r_4 = \binom{4}{4} = 1$ |
| 3 | $r_3 = \binom{4}{3} = 4$ |
| 2 | $r_2 = \binom{4}{2} = 6$ |
| 1 | $r_1 = \binom{4}{1} = 4$ |
| 0 | $r_0 = \binom{4}{0} = 1$ |

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Want to generalize these 4 properties of the rank sizes

$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ :

Symmetry: $\binom{n}{k} = \binom{n}{n-k}$

Alternating sum: $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots \pm \binom{n}{n} = 0$

Rank generating function: $\binom{n}{0} + \binom{n}{1}q + \binom{n}{2}q^2 + \ldots + \binom{n}{n}q^n = (1+q)^n$

Unimodality: $\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \ldots \leq \binom{n}{\lfloor n/2 \rfloor}$

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We'll generalize them by considering a

permutation subgroup $G \subseteq \mathfrak{S}_n :=$ symmetric group permuting $\{1,2,\ldots,n\}$

and the orbit poset $2^{[n]}/G$ whose elements are

$G$-orbits $\mathcal{O}$ of subsets, with $\mathcal{O}_1 \leq \mathcal{O}_2$ if $\exists\, S_1 \in \mathcal{O}_1$
$S_2 \in \mathcal{O}_2$
having $S_1 \subseteq S_2$

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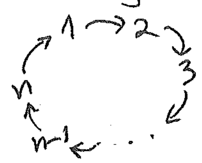
# Three important examples

## ① (Black-white) Necklaces

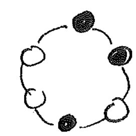
$$G = \langle \underbrace{(1,2,\dots,n)}_{n\text{-cycle}} \rangle \subset \mathfrak{S}_n$$

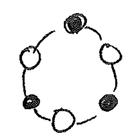
$G \cong \mathbb{Z}/n\mathbb{Z}$



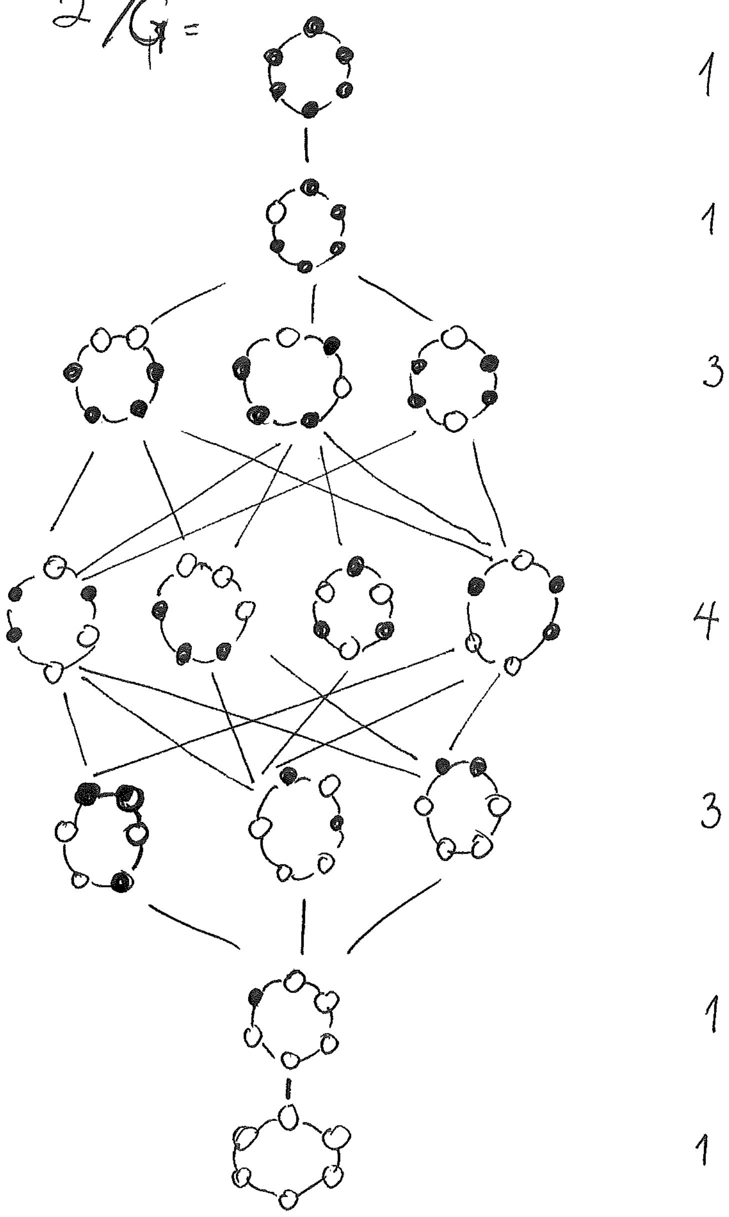

has $G$-orbits $\mathcal{O}$ in bijection with black-white necklaces having $n$ beads

e.g. $n=6$

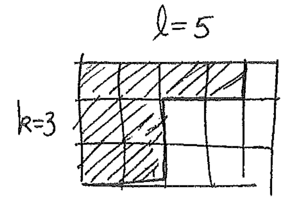
$\mathcal{O}_1 = \{\{1,2,4\},\{2,3,5\},\dots\} \longleftrightarrow$ 

$\mathcal{O}_2 = \{\{1,3,5\},\{2,4,6\}\} \longleftrightarrow$ 

$2^{[6]}/G =$



## ② Ferrers diagrams inside a $k \times \ell$ rectangle




$G = \mathfrak{S}_k[\mathfrak{S}_\ell]$ = wreath product containing $\mathfrak{S}_\ell \times \mathfrak{S}_\ell \times \ldots \times \mathfrak{S}_\ell$ that permutes within rows arbitrarily

$\cap$

$\mathfrak{S}_{k\ell}$

but also $\mathfrak{S}_k$ that wholesale swaps rows

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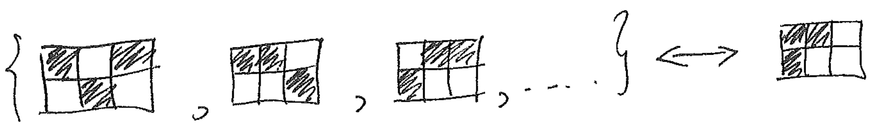
e.g. $G = \mathfrak{S}_2[\mathfrak{S}_3] \subset \mathfrak{S}_6$

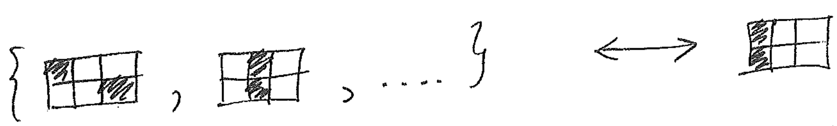
| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |

$k=2$
$\ell=3$

contains $\mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5,6\}}$

but also $\mathfrak{S}_2 = \langle (14)(25)(36) \rangle$

$\mathcal{O}_1 =$ 

$\mathcal{O}_2 =$ 

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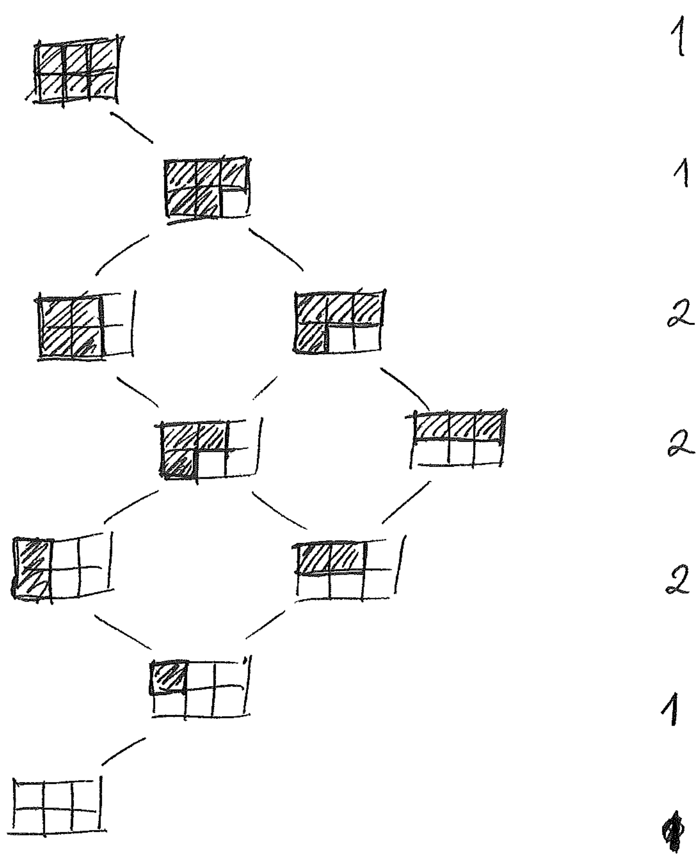
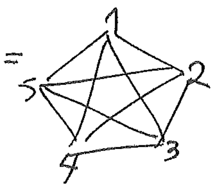
$2^{[6]}/G =$



(6)

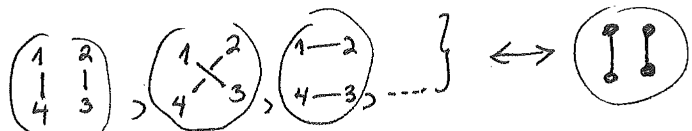
## ③ Unlabeled (simple) graphs on $v$ vertices

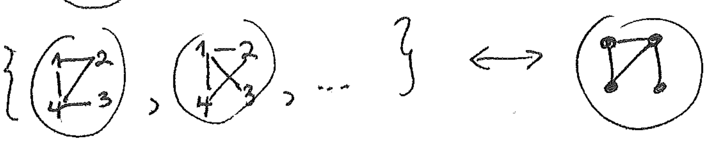
$$G = \mathfrak{S}_v \subset \mathfrak{S}_{\binom{[v]}{2}} \quad \text{where } \binom{[v]}{2} = \text{edges of complete graph } K_v \text{ on vertices } [v]$$

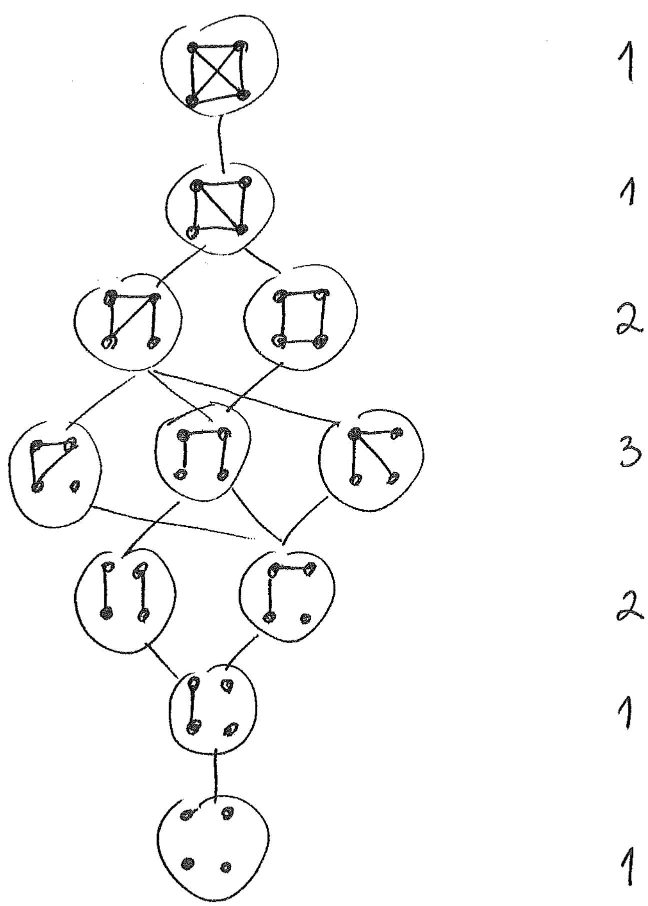
e.g. $K_5 =$ 


$G$-orbits $\mathcal{O}$ $\longleftrightarrow$ isomorphism classes of simple graphs

e.g. $n=4$

$\mathcal{O}_1 = \{$ 


$\mathcal{O}_2 = \{$ 


$n=4$, $\binom{n}{2}=6$ $\quad 2^{[6]}/G =$ 

(7)

Given $G$ a subgroup of $\mathfrak{S}_n$, let $r_0, r_1, \ldots, r_n$ be the rank sizes of the orbit poset $2^{[n]}/G$, that is, $r_k = \left| \underbrace{\binom{[n]}{k}/G}_{G\text{-orbits of } k\text{-element subsets}} \right|$, we will show....

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- (easy) PROPOSITION (Symmetry): $r_k = r_{n-k}$

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- (deBruijn 1959) THEOREM (Alternating sum):

$$r_0 - r_1 + r_2 - \cdots \pm r_n = \#\text{ of self-complementary } G\text{-orbits } \mathcal{O}$$

$S \in \mathcal{O} \Leftrightarrow [n] \setminus S \in \mathcal{O}$

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- (Redfield 1927, Polya 1937) THEOREM (Generating function):

$$r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} (1 + q^{|C|})$$

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- (Stanley 1982) THEOREM (Unimodality):

$$r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_{\lfloor \frac{n}{2} \rfloor}$$

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... aided by some (multi-)linear algebra

(8) Check the alternating sums in the three examples:

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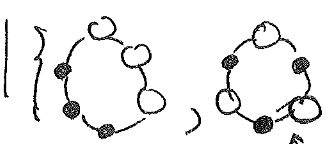
① Necklaces for $n=6$

$$1 - 1 + 3 - 4 + 3 - 1 + 1 = 2$$

$= \left| \left\{ \right. \right.$  $\left. \left. \right\} \right|$

2 self-complementary necklaces

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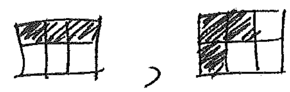
② Ferrers diagrams inside $k=2$  $l=3$

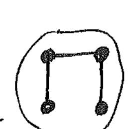
$$1 - 1 + 2 - 2 + 2 - 1 + 1 = 2$$

$= \left| \left\{ \right. \right.$  $\left. \left. \right\} \right|$

self-complementary Ferrers diagrams in 

③ Unlabeled simple graphs on 4 vertices

$$1 - 1 + 2 - 3 + 2 - 1 + 1 = 1$$

$= \left| \left\{ \right. \right.$  $\left. \left. \right\} \right|$

self-complementary graphs

(9)

Check the rank-generating function in the <u>necklace</u> example:

$$G = \langle c \rangle \text{ where } c = (1,2,3,4,5,6) \in \mathfrak{S}_6$$

$$= \{e, c, c^2, c^3, c^4, c^5\}$$

$$= \{e\} \cup \{c, c^5\} \cup \{c^2, c^4\} \cup \{c^3\}$$

$$e = (1)(2)(3)(4)(5)(6) \rightsquigarrow (1+q)^6$$

$$c = (123456) \rightsquigarrow (1+q^6)$$

$$c^5 = (165432) \rightsquigarrow (1+q^6)$$

$$c^2 = (135)(246) \rightsquigarrow (1+q^3)^2$$

$$c^4 = (153)(264) \rightsquigarrow (1+q^3)^2$$

$$c^3 = (14)(25)(36) \rightsquigarrow (1+q^2)^3$$

$$
\begin{array}{rccccccc}
(1+q)^6 = & 1 & + 6q & + 15q^2 & + 20q^3 & + 15q^4 & + 6q^5 & + q^6 \\
2(1+q^6) = & 2 & & & & & & + 2q^6 \\
2(1+q^3)^2 = & 2 & & & + 4q^3 & & & + 2q^6 \\
(1+q^2)^3 = & 1 & & + 3q^2 & & + 3q^4 & & + q^6 \\
\hline
\text{TOTAL} & 6 & + 6q & + 18q^2 & + 24q^3 & + 18q^4 & + 6q^5 & + 6q^6
\end{array}
$$

divide by $|G| = 6$

$$\frac{1}{|G|}(\text{TOTAL}) = 1 + q + 3q^2 + 4q^3 + 3q^4 + q^5 + q^6 \quad \checkmark$$

IDEA: Linearize and treat

- cardinalities as dimensions

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- generating functions as graded dimensions
  or Hilbert series

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- prove equalities via isomorphisms
  inequalities via injections or surjections

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- many identities come from
  equality of traces for conjugate group
  elements $g$, $hgh^{-1}$ in a group $G$

acting in a representation on $V$:

Given a homomorphism

$$G \xrightarrow{\rho} GL(V)$$

$$\mathrm{Trace}_V\left(\rho(hgh^{-1})\right) = \mathrm{Trace}_V\left(\rho(h)\rho(g)\rho(h)^{-1}\right) = \mathrm{Trace}_V(\rho(g))$$

$$\left[\begin{array}{l}\text{since } \mathrm{Tr}(AB) = \mathrm{Tr}(BA) \\ \qquad \text{implies } \mathrm{Tr}(PAP^{-1}) = \mathrm{Tr}(P^{-1} \cdot PA) = \mathrm{Tr}(A)\end{array}\right]$$

(11)

Start with $V = \mathbb{C}^2$ having $\mathbb{C}$-basis $\{\underset{\text{black}}{b}, \underset{\text{white}}{w}\}$

Then elements $T \in GL(V) = GL_2(\mathbb{C})$ act on $V$

e.g.

$$t = \begin{array}{c|cc} & b & w \\ \hline b & 0 & 1 \\ w & 1 & 0 \end{array} \quad \text{acts via } \begin{array}{l} t(b)=w \\ t(w)=b \end{array}$$

$$s = \begin{array}{c|cc} & b & w \\ \hline b & -1 & 0 \\ w & 0 & +1 \end{array} \quad \text{acts via } \begin{array}{l} s(b)=-b \\ s(w)=+w \end{array}$$

The $n^{th}$ tensor power $T^n(V) = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ factors}} = V^{\otimes n}$

has actions of

- $GL(V)$ diagonally: $T(v_1 \otimes \cdots \otimes v_n) = T(v_1) \otimes \cdots \otimes T(v_n)$ (and expand multilinearly)

- $\mathfrak{S}_n$ positionally: $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$

and the two actions commute:

$$\sigma T(v_1 \otimes \cdots \otimes v_n) = T\sigma(v_1 \otimes \cdots \otimes v_n)$$

$$= T(v_{\sigma^{-1}(1)}) \otimes \cdots \otimes T(v_{\sigma^{-1}(n)})$$

(12)

$V^{\otimes n}$ has a natural $\mathbb{C}$-basis indexed by <u>subsets</u> $S \in 2^{[n]}$

$$\{e_S\}_{S \in 2^{[n]}}$$

decomposable tensor $v_1 \otimes \cdots \otimes v_n$ having $\begin{cases} b \text{ in positions } S \\ w \text{ in positions } [n] \setminus S \end{cases}$

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e.g. $n=4$

$$e_{\{2\}} = \overset{1}{w} \otimes \overset{2}{b} \otimes \overset{3}{w} \otimes \overset{4}{w} \longleftrightarrow wbww$$

$$e_{\{1,4\}} = b \otimes w \otimes w \otimes b \longleftrightarrow bwwb$$

---

For a permutation group $G \subset \mathfrak{S}_n$,

the <u>$G$-fixed subspace</u> $(V^{\otimes n})^G$ has a natural $\mathbb{C}$-basis

indexed by <u>$G$-orbits</u> $\mathcal{O} \in 2^{[n]}/G$

$$\{e_{\mathcal{O}}\}_{\mathcal{O} \in 2^{[n]}/G} \quad \text{where } e_{\mathcal{O}} := \sum_{S \in \mathcal{O}}' e_S$$

---

e.g. $n=4$ $\quad G = \langle (1,2,3,4) \rangle \cong \mathbb{Z}/4\mathbb{Z}$

$e$  $= wbwb + bwbw$

$e$  $= wbbb + bwbb + bbwb + bbbw$

(13) Both $V^{\otimes n}$ and $(V^{\otimes n})^G$ are graded $\mathbb{C}$-vector spaces:

$$V^{\otimes n} = \bigoplus_{k=0}^{n} \underbrace{\left(V^{\otimes n}\right)_k}_{\substack{\mathbb{C}\text{-span of} \\ \{e_S\}_{S \in \binom{[n]}{k}}}} \qquad \left(V^{\otimes n}\right)^G = \bigoplus_{k=0}^{n} \underbrace{\left(V^{\otimes n}\right)^G_k}_{\substack{\mathbb{C}\text{-span of} \\ \{e_{\mathcal{O}}\}_{\mathcal{O} \in \binom{[n]}{k}/G}}}$$

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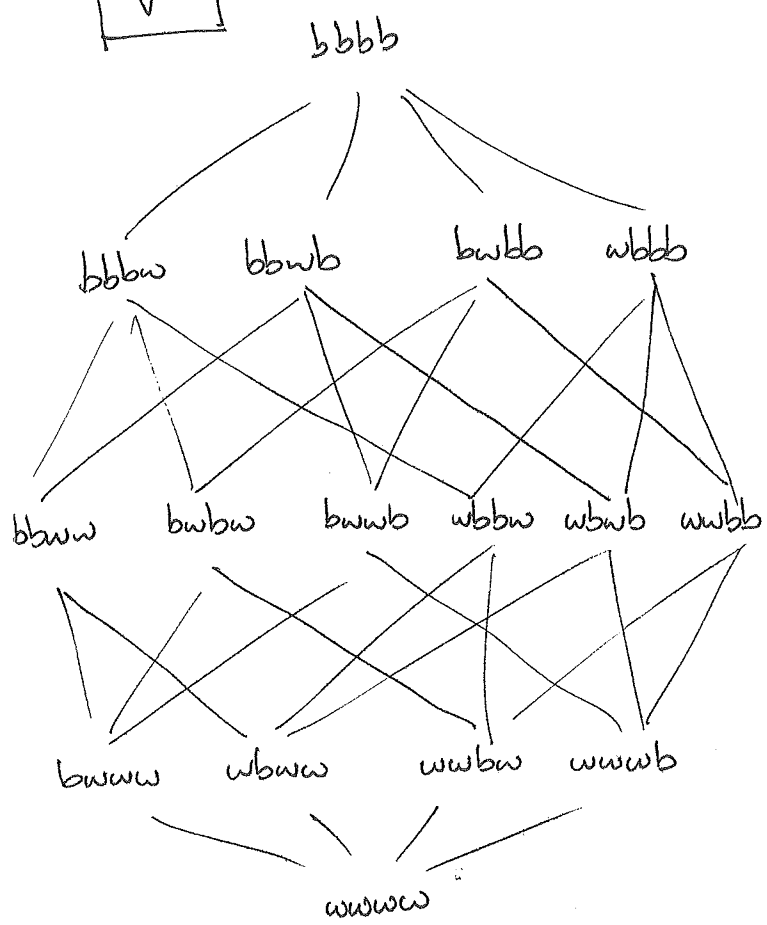
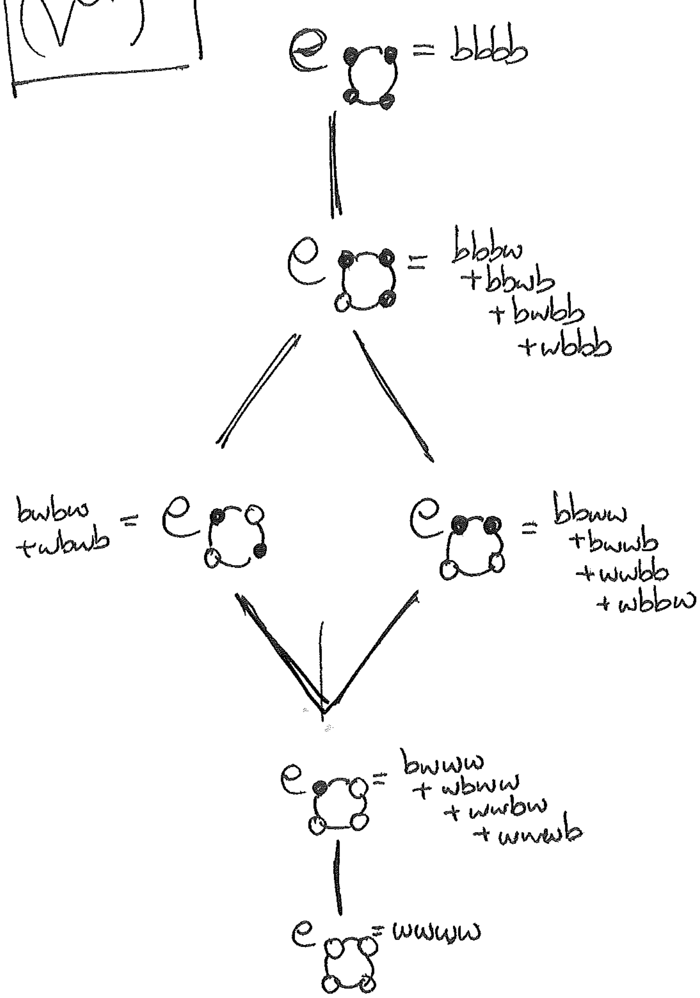
e.g. $n=4$ $\quad G = \langle (1,2,3,4) \rangle \cong \mathbb{Z}/4\mathbb{Z}$

$\boxed{V^{\otimes 4}}$




$\boxed{(V^{\otimes 4})^G}$



(14)

Since the rank sizes $r_0, r_1, \ldots, r_n$ of the orbit poset $2^{[n]}/G$ can now be reinterpreted as dimensions

$$r_k = \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_k \quad \left(= \left| \binom{[n]}{k}/G \right| \right)$$

we can now give a (silly) proof of the easy ...

---

PROPOSITION (Symmetry) $\quad r_k = r_{n-k}$

proof: Recall $t = \begin{array}{c|cc} & b & w \\ \hline b & 0 & 1 \\ w & 1 & 0 \end{array} \in GL(V)$ swaps $b$ and $w$,

and so it permutes the $\mathbb{C}$-basis $\{e_S\}_{S \in 2^{[n]}}$ for $V^{\otimes n}$

by swapping $\quad e_S \xleftrightarrow{t} e_{[n]\setminus S}$

(e.g. $t(\underset{\substack{\uparrow \\ e_{\{1,3\}}}}{bwbww}) = \underset{\substack{\uparrow \\ e_{\{2,4,5\}}}}{wbwbb}$ )

and giving a $\mathbb{C}$-linear isomorphism $\left(V^{\otimes n}\right)_k \xrightarrow[\sim]{t} \left(V^{\otimes n}\right)_{n-k}$.

But since $t \in GL(V)$ commutes with the action of $\mathfrak{S}_n$,

and hence with the action of $G \subset \mathfrak{S}_n$,

this same map $t$ restricts to a $\mathbb{C}$-linear isomorphism

$$\underbrace{\left(V^{\otimes n}\right)^G_k}_{\text{dimension } r_k} \xrightarrow[\sim]{t} \underbrace{\left(V^{\otimes n}\right)^G_{n-k}}_{\text{dimension } r_{n-k}} \qquad \blacksquare$$

(15)

For a (less silly) proof of the alternating sum result, start by reinterpreting the rank generating function.

PROPOSITION: $s(q) := \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix} \in GL(V)$

acts on $(V^{\otimes n})^G$ with trace $r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n$.

In particular, $s = s(-1) = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$ acts on $(V^{\otimes n})^G$

with trace $r_0 - r_1 + r_2 - \cdots \pm r_n$.

proof: Note $s(q)$ scales the basis element $e_S$ for $V^{\otimes n}$ by $q^{|S|}$:

$$s(q)(e_S) = q^{|S|} e_S, \quad \text{e.g. } \overset{n=5}{s(q)(e_{\{1,3\}})}$$

$$= s(q)(b \otimes w \otimes b \otimes w \otimes w)$$

$$= qb \otimes w \otimes qb \otimes w \otimes w$$

$$= q^2 \underbrace{b \otimes w \otimes b \otimes w \otimes w}_{e_{\{1,3\}}}$$

Hence $s(q)$ scales all of $(V^{\otimes n})_k$ by $q^k$,

so $s(q)$ scales $(V^{\otimes n})^G_k$ by $q^k$,

and hence its trace on $(V^{\otimes n})^G = \bigoplus_{k=0}^{n} (V^{\otimes n})^G_k$

will be $\sum_{k=0}^{n} q^k \cdot \underbrace{\dim_{\mathbb{C}} (V^{\otimes n})^G_k}_{r_k}$ ∎

(16)

Now we can prove

(de Bruijn 1959)
THEOREM (Alternating sum): $r_0 - r_1 + r_2 + \ldots \pm r_n = \#$ self-complementary $G$-orbits

Proof: Note that $s = \begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix}$, and $t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are conjugate with $GL(V)$,

since $t$ is diagonalizable with eigenvalues $-1, +1$

eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvector $\begin{bmatrix} +1 \\ +1 \end{bmatrix}$.

Hence in the representation of $GL(V)$ on $(V^{\otimes n})^G$, they must act with the same trace, which is $r_0 - r_1 + r_2 \ldots \pm r_n$ for $s$, and hence also for $t$.

Thus it remains to show that $t$ acts with trace on $(V^{\otimes n})^G$ equal to the # of self-complementary $G$-orbits:

- We saw $t$ permutes the $\mathbb{C}$-basis $\{e_S\}_{S \in 2^{[n]}}$ for $V^{\otimes n}$ by swapping $e_S \xleftrightarrow{t} e_{[n] \setminus S}$
- This means $t$ also permutes the $\mathbb{C}$-basis $\{e_{\mathcal{O}}\}_{\mathcal{O} \in 2^{[n]}/G}$ for $(V^{\otimes n})^G$ by $\begin{cases} \text{fixing } e_{\mathcal{O}} \text{ if } \mathcal{O} \text{ is self-complementary} \\ \text{swapping } e_{\mathcal{O}} \xleftrightarrow{t} e_{\mathcal{O}'} \text{ if } S \in \mathcal{O} \text{ but } [n] \setminus S \in \mathcal{O}' \neq \mathcal{O} \end{cases}$

---

e.g. $t(e$  $) = t(bwbw + wbwb) = wbwb + bwbw = e$ 

$t(e$  $) = t(bwww + wbww + wwbw + wwwb) = wbbb + bwbb + bbwb + bbbw = e$ 

---

Hence trace of $t$ counts these fixed points ∎

Let's sketch the remaining proofs, with missing details in the EXERCISES.

(Redfield–Polya) (1927, 1937)
<u>THEOREM</u> (Generating function)

$$r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} (1 + q^{|C|})$$

<u>proof:</u>

$$r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \sum_{k=0}^{n} \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_k \cdot q^k$$

$$\stackrel{?}{=} \sum_{k=0}^{n} \left( \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}_{(V^{\otimes n})_k}(\sigma) \right) \cdot q^k$$

See <u>EXERCISE</u> 1:
For a representation $G \xrightarrow{\rho} GL(\mathcal{U})$ of a finite group $G$,
$\dim_{\mathbb{C}}(\mathcal{U}^G) = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}(\rho(\sigma))$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{k=0}^{n} q^k \cdot \mathrm{Trace}_{(V^{\otimes n})_k}(\sigma)$$

EX<u>ERC</u>ISE 2: These are equal

$$= \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} (1 + q^{|C|})$$

∎

Lastly ...

(Stanley 1982)
THEOREM (Unimodality): $r_0 \le r_1 \le \dots \le r_{\lfloor \frac{n}{2} \rfloor}$

proof: Since we want to show for $k < \frac{n}{2}$ that

$$\begin{array}{ccc} r_k & \le & r_{k+1} \\ \| & & \| \\ \dim_{\mathbb{C}}(V^{\otimes n})^G_k & & \dim_{\mathbb{C}}(V^{\otimes n})^G_{k+1} \end{array}$$

let's try to find an injective $\mathbb{C}$-linear map

$$(V^{\otimes n})^G_k \hookrightarrow (V^{\otimes n})^G_{k+1} .$$

We could do this for all permutation groups $G \subseteq \mathfrak{S}_n$ at once if we could find an injective $\mathbb{C}$-linear map

$$(V^{\otimes n})_k \xhookrightarrow{U_k} (V^{\otimes n})_{k+1}$$

that was also commuting with the $\mathfrak{S}_n$-action on $V^{\otimes n}$.

OBVIOUS CANDIDATE:

$$U_k(e_S) = \sum_{\substack{T \in \binom{[n]}{k+1}: \\ S \subset T}} e_T$$

e.g. $n=5$

$$U_2(e_{\{1,3\}}) = e_{\{1,2,3\}} + e_{\{1,3,4\}} + e_{\{1,4,5\}}$$

$$U_2(bwbww) = bbbww + bwbbw + bwbwb$$

EXERCISE 3:
$U_k$ ~~does commute~~ commute with the $\mathfrak{S}_n$-action, and is injective for $k < \frac{n}{2}$

... completing the proof ∎