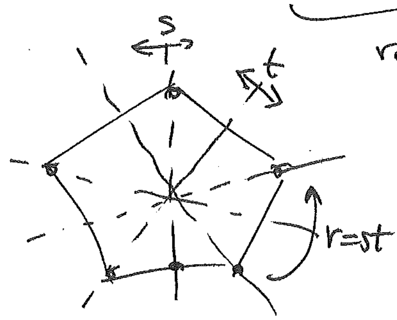
(1)

ECCO 2018 Vic Reiner Lecture 2 Exercises

① Recall $G = I_2(m)$ = linear symmetries of a regular m-sided polygon

(a) Prove that $G = \underbrace{\{e, r, r^2, \ldots, r^{m-1}\}}_{\text{rotations}} \sqcup \underbrace{\{s, sr, sr^2, \ldots, sr^{m-1}\}}_{\text{reflections}}$



(b) Prove the presentation for G as

$$G \cong \langle s, r \mid s^2 = r^m = e,\ srs = r^{-1} \rangle$$

(c) Prove the (Coxeter) presentation for G as

$$G \cong \langle s, t \mid s^2 = t^2 = e = (st)^m \rangle$$

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② (a) Prove that every 1-dimensional irreducible representation for $G = I_2(m)$ sends s, t to values in $\{\pm 1\}$, and they exactly are

$$\{\underset{\substack{s \mapsto +1 \\ t \mapsto +1}}{\mathbb{1}},\ \underset{\substack{s \mapsto -1 \\ t \mapsto -1}}{\det}\} \text{ if } m \text{ is odd}, \quad \{\mathbb{1},\ \det,\ \underset{\substack{s \mapsto -1 \\ t \mapsto +1}}{\rho_s},\ \underset{\substack{s \mapsto +1 \\ t \mapsto -1}}{\rho_t}\} \text{ if } m \text{ is even}.$$

(b) Prove that there is a representation for each $j \in \mathbb{Z}$ uniquely defined by

$$G = I_2(m) \xrightarrow{\rho^{(j)}} GL_2(\mathbb{C})$$
$$s \xmapsto{\rho^{(j)}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
$$r \xmapsto{\rho^{(j)}} \begin{bmatrix} \zeta^{j} & 0 \\ 0 & \zeta^{-j} \end{bmatrix}$$

where $\zeta := e^{2\pi i/m}$

(c) Prove $\rho^{(j)} = \rho^{(j+m)} = \rho^{(m-j)}$

$\rho^{(0)} \cong \mathbb{1} \oplus \mathbb{1}$, $\rho^{(\frac{m}{2})} \cong \rho_s \oplus \rho_t$ for m even, $\rho^{(1)} \cong \rho_{\text{ref}}$

(d) Prove the $I_2(m)$-irreducibles are $\{\rho^{(j)}\}_{j=1,2,\ldots,\lfloor\frac{m-1}{2}\rfloor} \cup \begin{Bmatrix} \{\mathbb{1}, \det\} & m \text{ odd} \\ \{\mathbb{1}, \det, \rho_s, \rho_t\} & m \text{ even} \end{Bmatrix}$

3 Let $G \subset \mathfrak{S}_n$ be a permutation group, and $G \xrightarrow{\rho} GL_n(\mathbb{C})$ the associated permutation representation for G acting on $[n]$.

(a) Show that the permutation representation where G permutes the ordered pairs $(i,j) \in [n] \times [n]$ via $g(i,j) = (g(i), g(j))$ has character χ_ρ^2

(b) If G is doubly-transitive, meaning G acts transitively on the set of pairs $\{(i,j) : 1 \leq i \neq j \leq n\}$, then show $\langle \chi_\rho, \chi_\rho \rangle_G = 2$.

(c) If G is doubly-transitive, show $\rho = \mathbb{1} \oplus \rho'$ with ρ' irreducible.
(Hint: Explain why $\rho = \mathbb{1} \oplus \rho'$ for some representation ρ', then calculate $\langle \chi_{\rho'}, \chi_{\rho'} \rangle_G$.)

(d) Prove $G = \mathfrak{S}_n \xrightarrow{\rho_{perm}} GL_n(\mathbb{C})$ decomposes as $\rho_{perm} = \mathbb{1} \oplus \underbrace{\rho_{ref}}_{\text{= symmetries of regular } (n-1)\text{-simplex}}$

(e) Prove $G = \mathfrak{S}_n$ has ρ_{ref} irreducible.

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4 Prove that $G = \mathfrak{S}_4$ has the following list of (inequivalent) irreducibles

$$\{ \mathbb{1},\ sgn,\ \rho_{ref},\ sgn \otimes \rho_{ref},\ \rho_2 \}$$

(sends $\sigma \mapsto sgn(\sigma) \cdot$ (permutation matrix of σ))

where ρ_2 is the following composite: $\mathfrak{S}_4 \to \mathfrak{S}_4 / \underbrace{V_4}_{\substack{\text{Klein-four} \\ \text{subgroup} \\ \{e, (12)(34), (13)(24), (14)(23)\}}} \cong \mathfrak{S}_3 \xrightarrow{\rho_{ref}} O_2(\mathbb{R})$

Compute the irreducible character table for $\mathfrak{S}_4$ by giving their character values on $\{e, (ij), (ijk), (ij)(kl), (ijkl)\}$