ECCO 2018 Vic Reiner Lecture 3 Exercises

(1) We want to prove this LEMMA from lecture ...

LEMMA: Let $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ be a square matrix of variables, viewed as a linear map $V \xrightarrow{A} V$ where $V = \mathbb{C}(a_{ij})^n$.

Then one has an identity of power series in $\mathbb{C}[[a_{ij}]]$

$$\sum_{d \geq 0} \mathrm{Trace}\left(\mathrm{Sym}^d V \xrightarrow{\mathrm{Sym}^d A} \mathrm{Sym}^d V \right) = \frac{1}{\det(1_V - A)}$$

To prove it, start by extending the field $\mathbb{C}(a_{ij})$ of rational functions to any algebraically closed field $K \supset \mathbb{C}(a_{ij})$, and extend V to K^n.

Then one can triangularize A, that is, one can choose an ordered K-basis $(x_1, x_2, \dots, x_n)$ for K^n so that the linear map $K^n \xrightarrow{A} K^n$ has matrix of the form

$$\begin{array}{c|c} & \begin{matrix} x_1 & x_2 & \cdots & x_n \end{matrix} \\ \hline \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{matrix} & \begin{bmatrix} \lambda_1 & & * & \\ & \lambda_2 & & \\ & 0 & \ddots & \\ & & & \lambda_n \end{bmatrix} \end{array}.$$

(a) Show that the K-basis $\{ x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n} : j_1 + j_2 + \dots + j_n = d \}$ for $\mathrm{Sym}^d V$ can be ordered in such a way that $\mathrm{Sym}^d A$ acts triangularly.

(b) Explain why this implies $\mathrm{Sym}^d A$ acting on $\mathrm{Sym}^d V$ has trace $\displaystyle\sum_{j_1 + j_2 + \dots + j_n = d} \lambda_1^{j_1} \lambda_2^{j_2} \cdots \lambda_n^{j_n}$.

(c) Prove the LEMMA.

(d) Deduce the PROPOSITION that $\chi_{\mathrm{Sym}(\rho)}(g; q) = \dfrac{1}{\det(1_V - q \cdot \rho(g))}$

(e) Deduce Molien's theorem from the PROPOSITION.

② (a) Prove that the permutation representation $\mathfrak{S}_n \xrightarrow{\rho_{perm}} GL_n(\mathbb{C}) = GL(V)$ has the property that for any $\sigma \in \mathfrak{S}_n$

$$\det\left(1_V - q \cdot \rho_{perm}(\sigma)\right) = \prod_{\substack{\text{cycles } C \\ \text{of } \sigma}} \left(1 - q^{|C|}\right)$$

(b) Prove that for any irreducible character χ_λ of $\mathfrak{S}_n$, one has

$$\sum_{d \geq 0} \langle \chi_{\mathbb{C}[x_1,\dots,x_n]_d}, \chi_\lambda \rangle_{\mathfrak{S}_n} \cdot q^d = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_\lambda(\sigma)}\, p_\sigma(1, q, q^2, \dots)$$

where $p_\sigma(x_1, x_2, \dots) := \prod_{\substack{\text{cycles } C \\ \text{of } \sigma}} p_{|C|}(x_1, x_2, \dots)$

and $p_d(x_1, x_2, \dots) = x_1^d + x_2^d + \cdots$ (= the d^{th} power sum symmetric function)

REMARK: For those familiar with $\mathfrak{S}_n$-representations and the relation to symmetric functions, along with principal specializations of Schur functions $s_\lambda(x_1, x_2, \dots)$, as in Stanley's Enumerative Combinatorics Vol. 2 § 7.18, 7.21, the right side in (b) above equals

$$s_\lambda(1, q, q^2, \dots) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)} f^\lambda(q)$$

$$\text{where } f^\lambda(q) \overset{(*)}{=} q^{b(\lambda)} \frac{[n]!_q}{\prod_{\substack{\text{cells } x \text{ of} \\ \text{the Ferrers diagram} \\ \text{of } \lambda}} [h(x)]_q} \overset{(**)}{=} \sum_{\substack{\text{standard} \\ \text{Young tableaux } Q \\ \text{of shape } \lambda}} q^{maj(Q)}$$

See Stanley's COROLLARIES 7.21.3, 7.21.5 for the undefined terms here! Thus we have two interesting expressions for the fake degree polynomials $f^\psi(q)$ given by (*), (**) in the case where $G = \mathfrak{S}_n$.

③ Given a matrix $A = (a_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}} \in \mathbb{C}^{n\times n}$

and nonnegative integers $\underline{k} = (k_1,\dots,k_n) \in \mathbb{N}^n$,

define $\mathrm{per}_{\underline{k}}(A)$ as follows: letting $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be

two sets of variables related by $\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$,

then $\mathrm{per}_{\underline{k}}(A) :=$ coefficient of $x_1^{k_1}\cdots x_n^{k_n}$ in $y_1^{k_1}\cdots y_n^{k_n}$.

(a) Check that for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathrm{per}_{(1,1)}(A) = ad+bc$.

(b) Prove in general that $\mathrm{per}_{(1,1,\dots,1)}(A) = \sum_{\sigma \in \mathfrak{S}_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$

(= the <u>permanent</u> of A)

(c) Deduce from the LEMMA in Exercise ①,

<u>MacMahon's</u> $\overset{((9.6))}{}$ <u>Master Theorem</u>:

$$\sum_{\underline{k}\in\mathbb{N}^n} \mathrm{per}_{\underline{k}}(A)\, t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n} = \frac{1}{\det(I_n - TA)} \quad \text{where } T = \begin{bmatrix} t_1 & & 0 \\ & t_2 & \\ 0 & & \ddots\ t_n \end{bmatrix}$$

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④ (a) Fix $d \in \{1,2,3,\dots\}$. Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^d = 0$ if n is <u>odd</u>

(b) When $d=1$, show one still has $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^1 = 0$ if n is <u>even</u>.

(c) When $d=2$, show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^m \binom{2m}{m}$ if $n=2m$ is <u>even</u>.

(<u>HINT</u>: Interpret the left side as $\sum_{\substack{(A,B): \\ A,B\subset [n], \\ |A|+|B|=n}} (-1)^{|A|}$, and cancel it down to the terms where $A=B$)

(d) When $d=3$, <u>Dixon's identity</u> says $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = (-1)^m \binom{3m}{m,m,m}$ if $n=2m$ is <u>even</u>.

Deduce this from MacMahon's Master Theorem with $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$,

by showing the <u>left side</u> is $\mathrm{per}_{(2n,2n,2n)}(A)$ = coefficient of $x_1^{2n} x_2^{2n} x_3^{2n}$ in $(x_2-x_3)^{2n}(x_3-x_1)^{2n}(x_1-x_2)^{2n}$

while the <u>right side</u> is the coefficient of $t_1^{2n} t_2^{2n} t_3^{2n}$ in $\dfrac{1}{\det(I_3 - TA)} = \dfrac{1}{1+(t_1t_2+t_2t_3+t_1t_3)}$.