Combinatorics of the Coincidental Reflection Groups

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July 5, 2019

FPSAC in Ljubljana

PSAs:

PSAs:

- Thanks FPSAC 2019 organizers!

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- Thanks FPSAC 2019 organizers!
- Come to OPAC 2020
  May 18-22, 2020
  Univ. of Minnesota-Minneapolis

- Reflection groups
  and their sources
- Who are the coincidentals?
  - They are hereditary.
  - They are product-oriented.
  - They have rootposets,
    that are doppelgängers!
  - They are dynamic.

• Reflection groups

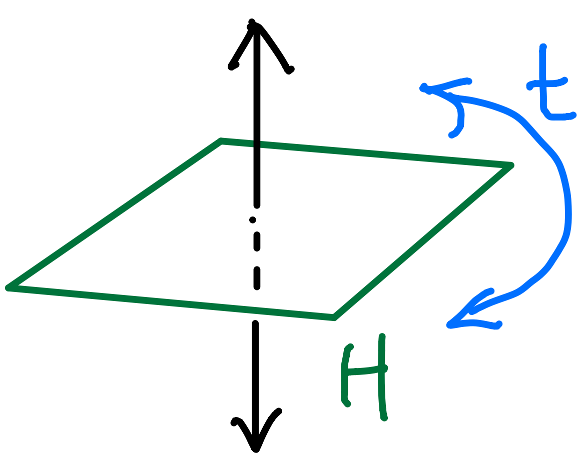
What's a reflection?

A real reflection $t \in GL(V)$
$V = \mathbb{R}^n$

has $H = V^t := \{v \in V : t(v) = v\}$

of dimension $n-1$,

and t negates the line $V^{\perp}$.



t diagonalizes to

$$\left[\begin{array}{c|ccc} -1 & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right]$$

A complex reflection (unitary, pseudo) $t \in GL(V)$, $V = \mathbb{C}^n$

has finite order, and again $H = V^t$ of dimension $n-1$.

So t diagonalizes to

$$\left[\begin{array}{c|ccc} \zeta & 0 & \cdots & 0 \\ \hline 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{array}\right]$$

ζ any root-of-unity

H is called the reflecting hyperplane for t.

A (complex) reflection group is a finite subgroup $W \subset GL(V)$, $V = \mathbb{C}^n$, generated by reflections.

Say that W is reducible if $V = V_1 \oplus V_2$ with

$$W \subset GL(V_1) \times GL(V_2) = \left\{ \begin{array}{c|cc} & V_1 & V_2 \\ \hline V_1 & * & 0 \\ V_2 & 0 & * \end{array} \right\}$$

and irreducible otherwise.

The irreducible reflection groups were classified by Shephard-Todd: (1955)

- One infinite family $G(de, e, n)$

$= \{$ $n \times n$ monomial matrices, nonzero entries $(de)^{th}$ roots-of-1 with product a d^{th} root-of-1 $\}$

$$\begin{bmatrix} 0 & 0 & 0 & \zeta^3 \\ \zeta^2 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & \zeta^5 & 0 \end{bmatrix}$$

- 34 exceptional groups!

(Very deceptive!)

Why did Shephard-Todd classify?
For the backward implication here:

THEOREM (Shephard-Todd 1955, Chevalley 1955)
When a finite subgroup $W \subset GL(V)$ acts on $S := \mathbb{C}[x_1, \ldots, x_n]$ via linear substitutions of variables, the W-invariant subalgebra

$$S^W := \{ f(\underline{x}) \in S : f(w\underline{x}) = f(\underline{x}) \}$$

is itself polynomial $S^W = \mathbb{C}[f_1, \ldots, f_n]$

$$\Longleftrightarrow$$

W is a reflection group

The $f_1, \ldots, f_n$ in $S^W = \mathbb{C}[f_1, \ldots, f_n]$ are not unique, but when chosen homogeneous, their multiset of degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ is unique.

This leads to much numerology, e.g.

THEOREM (Shephard–Todd 1955, Solomon 1963)

$$\#W = \prod_{i=1}^{n} d_i$$

$\uparrow q=1$

and more generally

$$\sum_{w \in W} q^{\dim(V^w)} = \prod_{i=1}^{n} (q + e_i)$$

where $(e_1, \ldots, e_n) := (d_1 - 1, \ldots, d_n - 1)$ are called the exponents of W.

There is similar numerology involving the co-exponents $(e_1^*, \dots, e_n^*)$, which are the roots of the characteristic polynomial

Möbius function

$$\chi(A_W, q) := \sum_{X \in \mathcal{L}(A_W)} \mu(V, X)\, q^{\dim(X)}$$

where $\mathcal{L}(A_W)$ is the poset of all intersections $X = H_{i_1} \cap \dots \cap H_{i_r}$ of reflecting hyperplanes.

THEOREM (Orlik-Solomon 1980)

For any reflection group W,

$$\chi(\mathcal{A}_W, q) = \sum_{w \in W} \det(w)\, q^{\dim(V^w)}$$
$$= \prod_{i=1}^{n} (q - e_i^*)$$

where the co-exponents $e_1^*, \ldots, e_n^*$ give the degrees appearing in any choice of homogeneous S^W-basis $\Theta_1, \ldots, \Theta_n$ for the (free) S^W-module $(S \otimes V)^W$,

i.e. $\Theta_i = \sum_{j=1}^{n} \underbrace{\Theta^{(j)}(x_1, \ldots, x_n)}_{\text{homogeneous of degree } e_i^*} \otimes y_j$

where $V = \mathbb{C}^n$ has $\mathbb{C}$-basis $y_1, \ldots, y_n$.

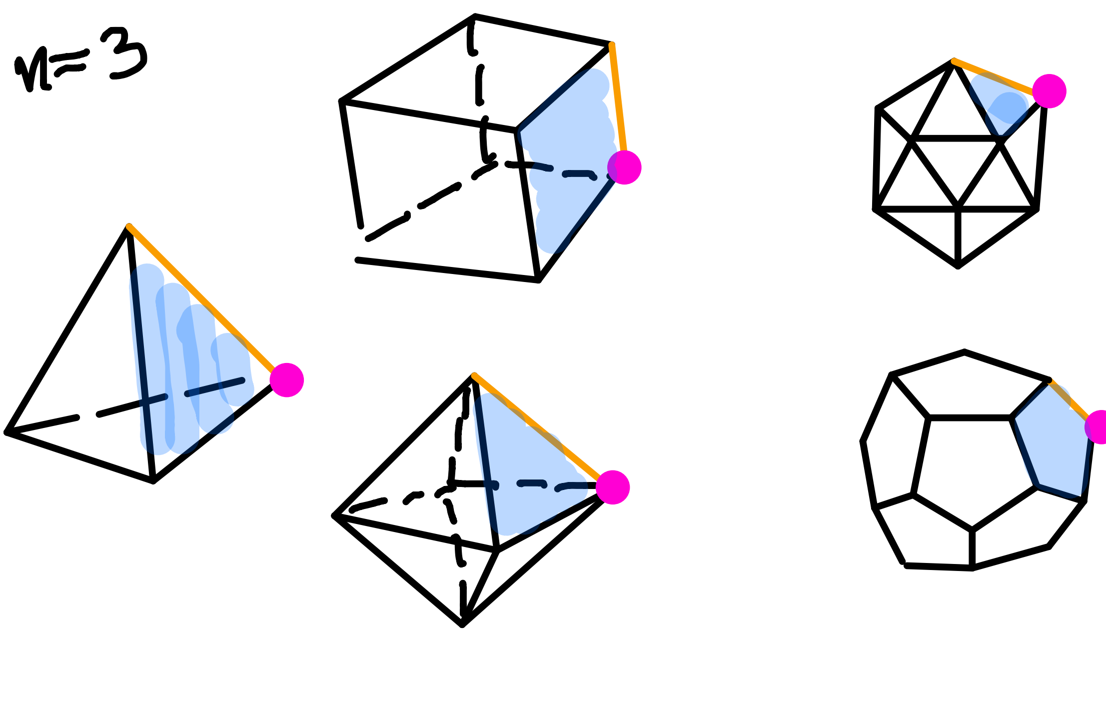
SOURCES of reflection groups

- **Regular polytopes** :=
  convex polytopes $P \subset \mathbb{R}^n = V$
  with linear symmetry group $W \subset GL(V)$
  transitive on maximal flags of faces

$$F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1}$$

F_0 = vertex, F_1 = edge, F_2 = polygon, F_{n-1} = facet

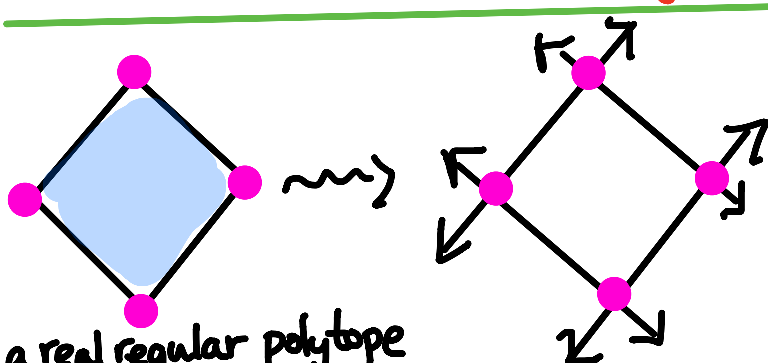
$n=3$



● Regular complex polytopes (!) (Shephard 1952)

:= arrangements of affine subspaces in $V = \mathbb{C}^n$ with linear symmetry group $W \subset GL(V)$ transitive on maximal flags $F_0 \subset F_1 \subset \ldots \subset F_{n-1}$, and maximal flags connected along ridges:

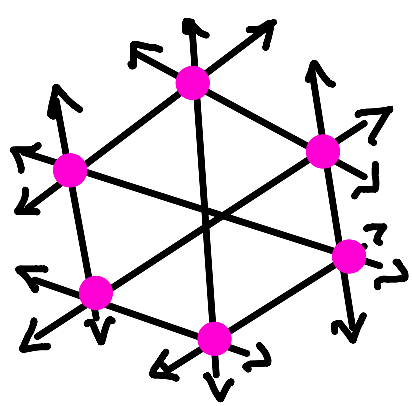
$$F_0 \subset F_1 \subset \ldots \subset F_{i-1} \overset{\subset F_i \subset}{\underset{\subset F_i' \subset}{}} F_{i+1} \subset \ldots \subset F_{n-1}$$



a real regular polytope

$\subset \mathbb{R}^2 \subset \mathbb{C}^2$

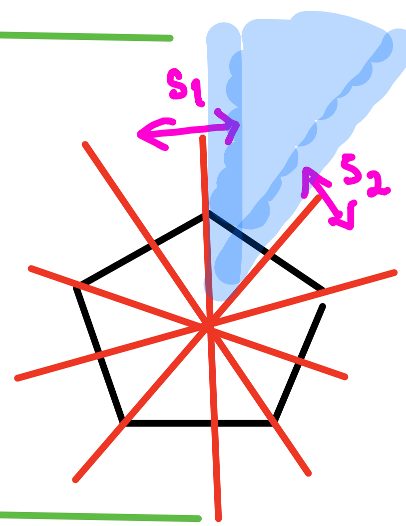
cartoon of a regular complex polytope



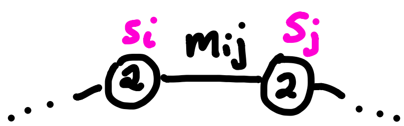
$\subset \mathbb{C}^2$

Finite Coxeter groups (= real reflection groups)

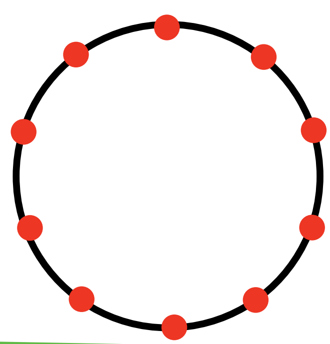
$$W = \Big\langle \underset{\{s_1, s_2, \ldots, s_n\}}{\underset{\parallel}{S}} \;\Big|\; \begin{matrix} s_i^2 = 1 \ \forall i \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij}\text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij}\text{ factors}} \end{matrix} \Big\rangle \text{ with } m_{ij} \in \{2,3,4,\ldots\}$$



$\rightsquigarrow$ Coxeter diagram



$\rightsquigarrow$ Coxeter complex = reflecting hyperplanes intersecting the unit sphere S^{n-1}



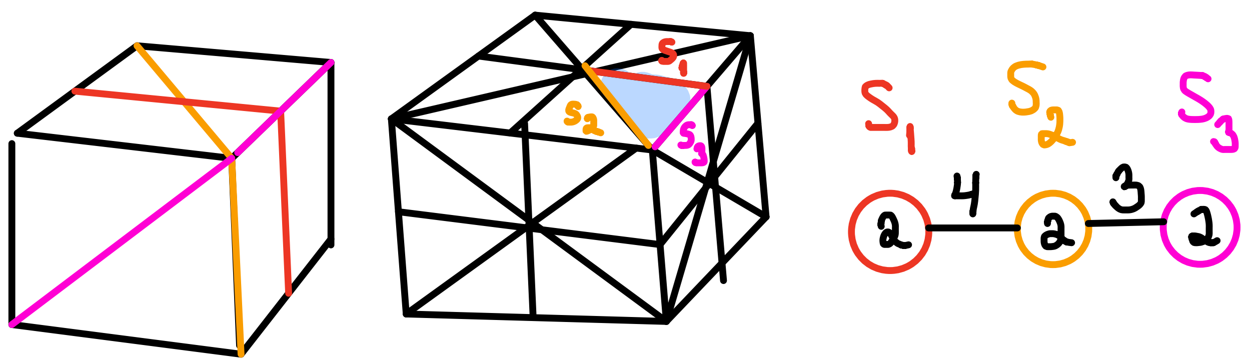
(Real W have $e_i^* = e_i$ Co-exponents = exponents)

For regular polytopes,

Coxeter complex

= barycentric subdivison of the boundary

= order complex of the poset of boundary faces of the polytope

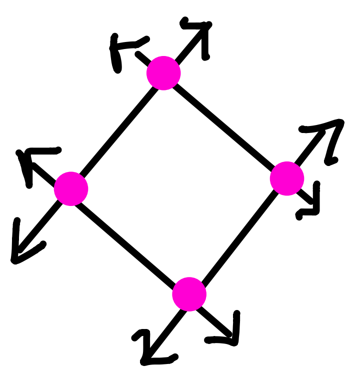


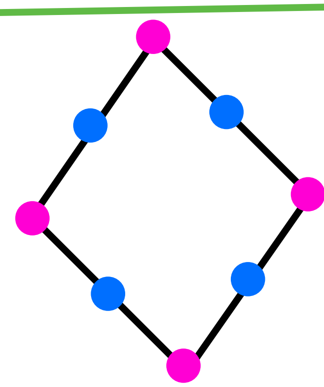
$$s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$$

$$s_2 s_3 s_2 = s_3 s_2 s_3$$

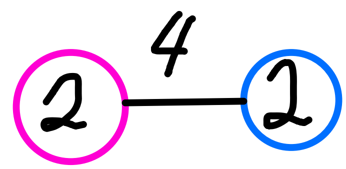
$$s_1 s_3 = s_3 s_1$$

Regular complex polytopes still have a Coxeter-Shephard-Milnor Fiber complex $\cong$ order complex of their poset of faces and a diagram presentation

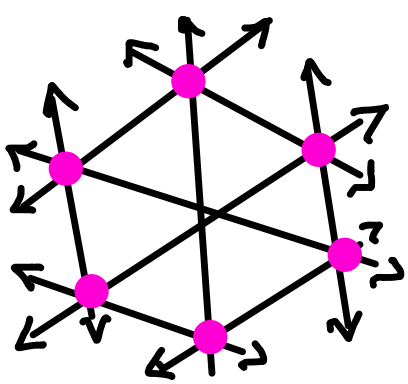


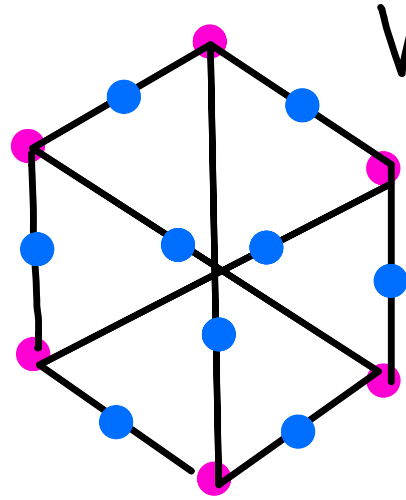


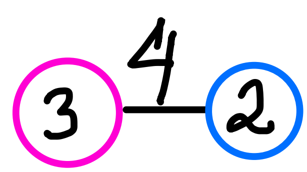
$W = B_2/C_2 = G(2,1,2)$



$W = G(3,1,2)$

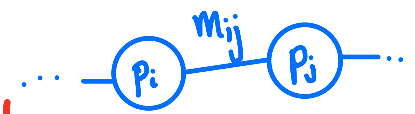






THEOREM (Koster 1975)

The finite groups W with diagram presentations



$$W \cong \Big\langle \underset{\{s_1,\dots,s_n\}}{\overset{\displaystyle S}{\shortparallel}} \;\Big|\; \begin{array}{c} s_i^{p_i} = 1 \\ \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ factors}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ factors}} \end{array} \Big\rangle$$

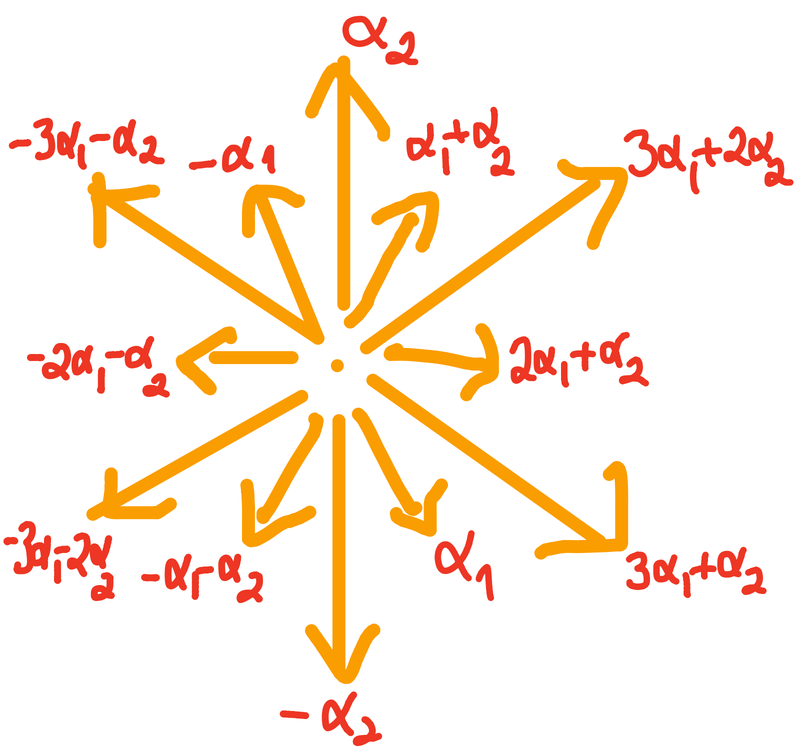
are exactly the

- real reflection groups, and
- symmetries of regular complex polytopes (= Shephard groups)

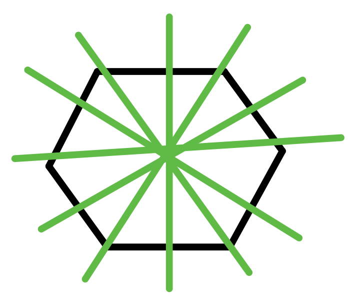
- **Lie groups & algebras** have root systems that give rise to Weyl groups W

$:=$ crystallographic, real reflection groups

preserving a full rank lattice $\mathbb{Z}^n < \mathbb{R}^n = V$

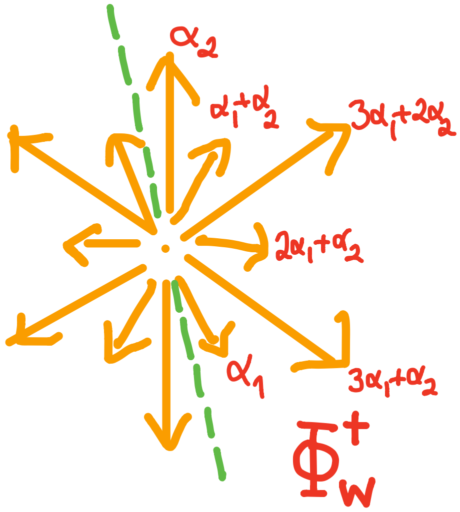


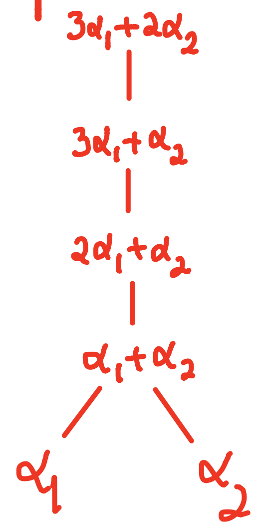
roots = normals to reflecting hyperplanes



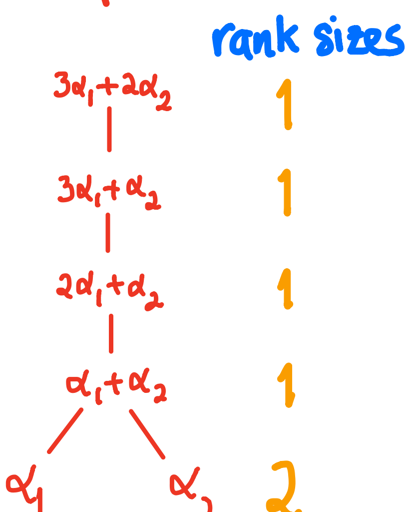
$$W = G_2 = I_2(6)$$

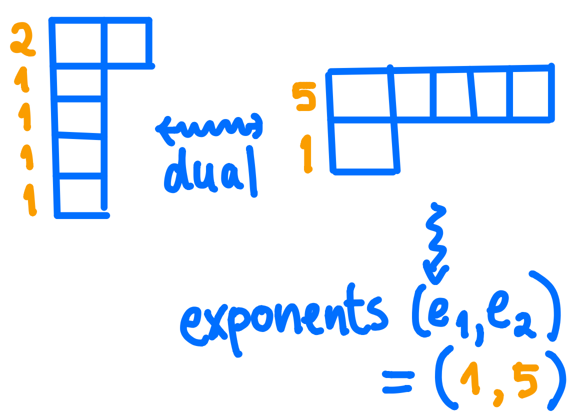
Weyl groups have a (positive) root poset Φ_W^+





… whose rank sizes re-express the exponents as their dual partition





Who are the coincidentals?

DEFINITION

$W \subset GL(V)$, $V = \mathbb{C}^n$

an irreducible reflection group is **coincidental** if

- it can be generated by n reflections (W is well-generated; equivalently $e_i^* + e_{n+1-i}^* = d_n$)
- its degrees $d_1 \leq d_2 \leq \ldots \leq d_n$ form an arithmetic sequence:

$$(d_1, d_1 + a, d_1 + 2a, \ldots, d_1 + (n-1)a)$$

Alex Miller (2015) gave 11 equivalent conditions!

So who are they really?

The Shephard groups except F_4, H_4.

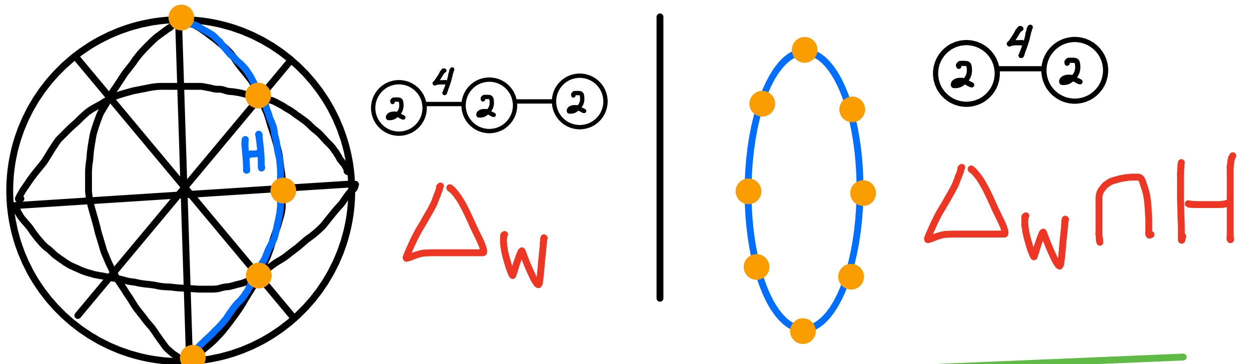
	REAL	COMPLEX
A_n	$\textcircled{2}-\textcircled{2}-\cdots-\textcircled{2}$	$\textcircled{3}-\textcircled{3}-\textcircled{3}$ $\textcircled{3}-\textcircled{3}-\textcircled{3}-\textcircled{3}$
B_n/C_n	$\textcircled{2}\overset{4}{-}\textcircled{2}-\cdots-\textcircled{2}$	$\textcircled{d}\overset{4}{-}\textcircled{2}-\cdots-\textcircled{2}$ $G(d,1,n)$ $\textcircled{2}\overset{4}{-}\textcircled{3}-\textcircled{3}$
H_3	$\textcircled{2}\overset{5}{-}\textcircled{2}-\textcircled{2}$	
$I_2(m)$	$\textcircled{2}\overset{m}{-}\textcircled{2}$	$\textcircled{p}\overset{m}{-}\textcircled{q}$ $\frac{1}{p}+\frac{1}{q}+\frac{2}{m}>1$ (with $p=q$ if m even) 12 of these

(Excluded: $G(de,e,n)$, $e,n \geq 2$, F_4, H_4, 13 more exceptionals)

What's so good about them?

They are hereditary.

For a finite real or Shephard group W with Milnor fiber complex Δ_W and any reflecting hyperplane H, call $\Delta_W \cap H$ a wall.



THEOREM (Abramenko 1994) A real reflection group W has every wall $\Delta_W \cap H \cong \Delta_{W'}$ $\Leftrightarrow$ W is coincidental

THEOREM (A. Miller 2017) A Shephard group W has every wall $\Delta_W \cap H$ containing some $\Delta_{W'}$ as a full-dim'l subcomplex $\Leftrightarrow$ W is coincidental

More generally, for any intersection $X = H_{i_1} \cap \ldots \cap H_{i_k}$ of reflecting hyperplanes consider the restriction arrangement in X

$$\mathcal{A}_W^X := \{ X \cap H : \underset{X \not\subseteq H}{H \text{ a reflecting hyperplane for } W} \}$$

Restriction arrangements share many properties with the original reflection arrangement $\mathcal{A}_W$, including integer factorizations

$$\chi(\mathcal{A}_W, q) = \prod_{i=1}^{\dim X} (q - e_i^X)$$

where $\{e_1^X, e_2^X, \ldots, e_{\dim X}^X\}$ are called the Orlik-Solomon (co-)exponents of X.

FACT (A. Miller 2015)

For real and Shephard groups W, the Orlik-Solomon exponents of X

$\{e_1^X, e_2^X, \ldots, e_{\dim X}^X\}$ depend only on $\dim X$

$\iff$ W is coincidental.

In fact, among well-generated groups W, one has that the Orlik-Solomon exponents

$$\{e_1^X, e_2^X, \ldots, e_{\dim X}^X\} = \{\underbrace{e_1^*, e_2^*, \ldots, e_{\dim X}^*}_{\text{the } \dim X \text{ smallest coexponents of } W}\} \quad \forall X$$

$\iff$ W is coincidental.

The coincidentals are product-oriented.

REFLECTION FACTORIZATIONS

In real reflection groups (W, S)
a Coxeter element is $c := s_1 s_2 \cdots s_n$
having order $h := d_n$ the Coxeter number

Recall a formula of Chapoton (2004) says

$$\#\left\{\begin{array}{c}\text{factorizations}\\ c = t_1 t_2 \cdots t_n\\ \text{with } t_i \text{ reflections}\end{array}\right\} = \frac{n!\, h^n}{|W|}$$

Deligne (1974) gave a recursion on the Coxeter diagram for (W,S) to compute the more general quantity

$$\#\left\{\begin{array}{l}\text{factorizations } c = a\,t_1 t_2 \cdots t_k\, b \\ t_i \text{ reflections, } \operatorname{codim}(V^a) + k + \operatorname{codim}(V^b) = n\end{array}\right\}$$

and Reading 2007 noted

THEOREM: This quantity almost has product formula

$$n(n-1)\cdots(n-k+1)\cdot\frac{h^k}{|W|}\cdot\prod_{i=k+1}^{n}(h+d_i)$$

which is correct $\Longleftrightarrow$ W is coincidental.

Explained by work of Douvropoulos (2017, 2019) why this follows from Orlik-Solomon exponents $\{e_1^X, e_2^X, \ldots, e_{\dim X}^X\}$ depending only on $\dim X$.

f-VECTORS AND h-VECTORS

Recall this product formula for real reflection groups W

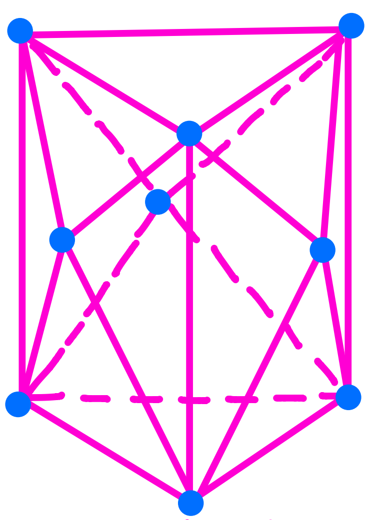
$$\mathrm{Cat}(W) := \prod_{i=1}^{n} \frac{h+d_i}{d_i}$$

W-Catalan number

$$W = A_{n-1} \rightsquigarrow \frac{(n+2)(n+3)\cdots(2n)}{2\cdot 3 \cdots n} = \frac{1}{n+1}\binom{2n}{n}$$

counts maximal faces in the W-cluster/W-Cambrian complexes

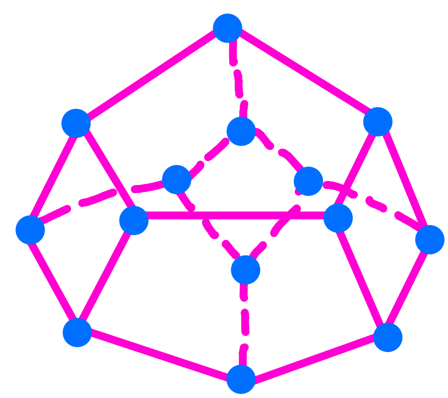
(Fomin-Zelevinsky 2001) (Reading 2004)



A_3-Cluster/Cambrian complex

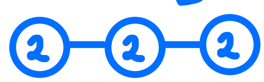
polar dual

$\longleftrightarrow$



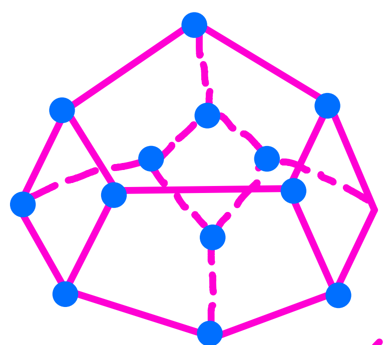
(simple) Associahedron or Stasheff polytope (1963)

$W = A_3$



What about other face numbers, i.e. the full f-vector $f=(f_0,f_1,f_2,\dots,f_n)$?

$W=A_3$ ②—②—②



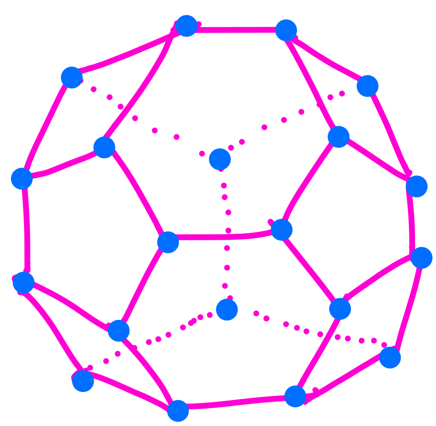
Associahedron/ Stasheff polytope

$$f=(f_0,f_1,f_2,f_3) = (14,21,9,1)$$

= Kirkman(1857)-Cayley(1890) or little Schröder(1870) numbers

$$\frac{1}{n}\binom{n}{k}\binom{n+k+1}{k}$$

$W=B_3$ ②—4—②—②



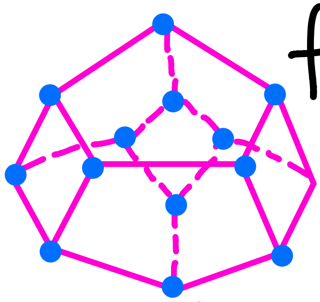
Cyclohedron/ Bott-Taubes polytope 1994

$$f=(20,30,12,1)$$

Computed by R.Simion(2003)

$$\binom{n}{k}\binom{n+k}{k}$$

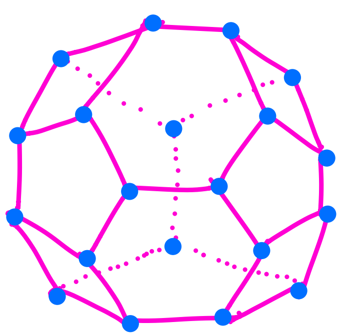
Alternatively, what about their
h-vectors $h=(h_0,h_1,\dots,h_n)$
defined by
$$\sum_{k=0}^{n} h_k t^k = \sum_{k=0}^{n} f_k (t-1)^k \quad ?$$



Associahedron

$f=(14,21,9,1) \rightsquigarrow h=(1,6,6,1)$
$=(f_0,f_1,f_2,f_3) \qquad =(h_0,h_1,h_2,h_3)$

Narayana (1955) numbers
$$\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$



Cyclohedron

$f=(20,30,21,1) \rightsquigarrow h=(1,9,9,1)$

Simion (2003)
$$\binom{n}{k}^2$$

Fomin & Reading 2005 gave a recursion on the Coxeter diagram of (W,S) to compute these f-vector and h-vectors. They then observed...

THEOREM: They almost have product formulas

$$f_k = \binom{n}{k} \prod_{i=1}^{n-k} \frac{h+d_i}{d_i}$$

$$h_k = \binom{n}{k} \prod_{i=1}^{k} \frac{d_{n+1-i}}{d_i}$$

which are

correct $\iff$ W is coincidental.

We understand this now (slightly) better.

THEOREM (Armstrong-R.-Rhoades 2012) The f-vector is

$$f_k = \left[\mathrm{Hilb}\left(\big(\underbrace{S}_{\text{grading tracked by } q} \otimes \overset{\text{grading tracked by } t}{\wedge} V^* \otimes \wedge^k V\big)^W, q, t \right) \right]_{\substack{t=-q^{h+1} \\ q=1}}$$

THEOREM (R-Shepler-Sommers)

Complex reflection groups W have this product formula

$$\mathrm{Hilb}\left(\left(S \otimes \wedge V^* \otimes \wedge^k V\right)^W, q, t \right)$$

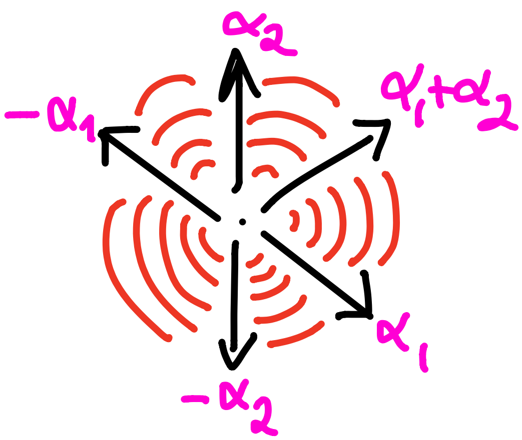
$$= \underbrace{\sigma_k}_{k^{\text{th}} \text{ elementary symmetric function}}\left(q^{e_1^*}, \ldots, q^{e_n^*}\right) \frac{\prod_{i=1}^{k}\left(1+q^{-e_i^*}t\right) \cdot \prod_{i=1}^{n-k}\left(1+q^{e_i}t\right)}{\prod_{i=1}^{n}\left(1-q^{d_i}\right)}$$

$\iff$ W is coincidental.

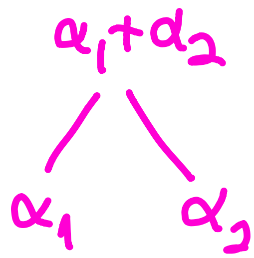
THEOREM (R.-Shepler-Sommers) For coincidental W the product formula for f_k converts to one for h_k via a hypergeometric transformation.

W-biCATALAN NUMBERS

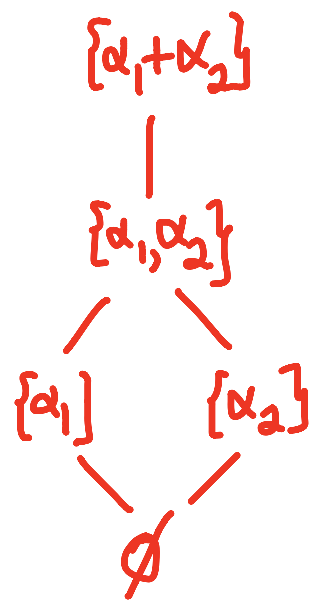
$Cat(W) = \prod_{i=1}^{n} \frac{h+d_i}{d_i}$ not only counts maximal W-cluster/Cambrian faces, but also antichains in the root poset Φ^+_W



5 maximal cones



Φ^+_W



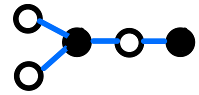
5 antichains in Φ^+_W

$W = A_2 =$ ②—②

$Cat(A_2) = \frac{1}{3+1}\binom{6}{3} = 5$

THEOREM (Barnard–Reading 2017)

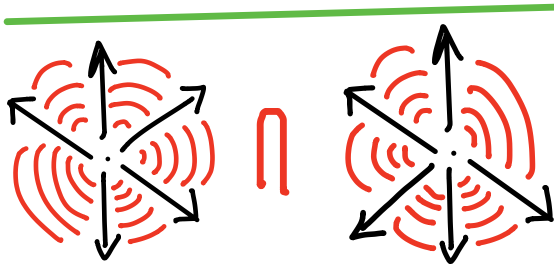
\#maximal cones in the common refinement of the Cambrian fan for a bipartite Coxeter element $c = \prod_{\text{black } s_i} s_i \cdot \prod_{\text{white } s_j} s_j$



and its opposite/negative fan

= \#antichains in doubled root poset

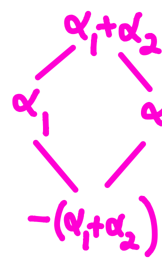
:= biCat(W) W-biCatalan number



=

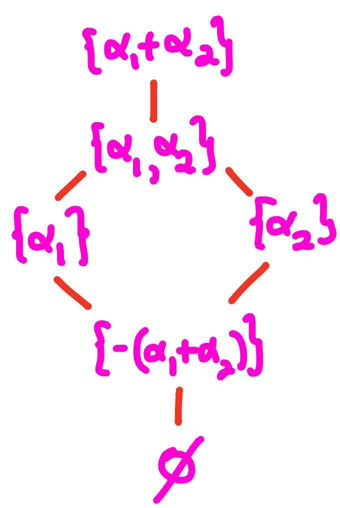
6 maximal cones

$\mathrm{biCat}(A_2) = 6$



doubled root poset

6 antichains



Barnard & Reading give formulas for $\mathrm{biCat}(W)$ for all real reflection groups W, and observe that the product formula

$$\prod_{i=1}^{n} \frac{h+e_i-1}{e_i}$$

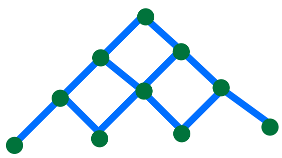
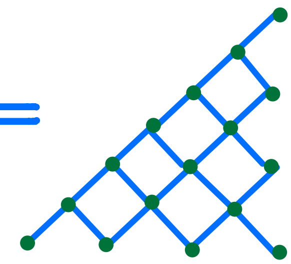
is almost $\mathrm{biCat}(W)$, and correct $\Longleftrightarrow$ W is coincidental.

E.g. A_2 has exponents $(e_1,e_2)=(1,2)$ and $h=d_2=e_2+1=3$, so

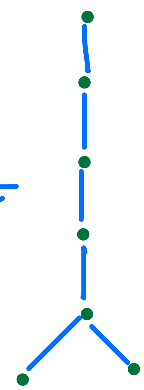
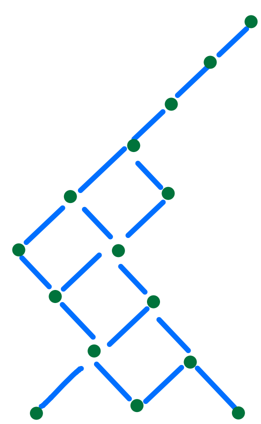
$$\prod_{i=1}^{n} \frac{h+e_i-1}{e_i} = \frac{(3+1-1)(3+2-1)}{1\cdot 2} = \frac{3\cdot 4}{2} = 6$$

$$= \mathrm{biCat}(A_2)$$

- The coincidentals have root posets!

$\Phi^+_{A_n} =$  $\Phi^+_{B_n/C_n} =$ 

Armstrong (2006) suggested these root posets Φ^+_W for the non-crystallographic real coincidental types:

$\Phi^+_{I_2(m)} :=$  $\Phi^+_{H_3} :=$ 

based on desired properties for

- exponents $\{e_i\}$ dual to rank sizes of Φ^+_W
- $Cat(W) = \prod_{i=1}^{n} \frac{h+d_i}{d_i} = \#$ antichains in Φ^+_W
- The M-triangle numerology of Chapoton (2004)

Cuntz & Stump (2012) showed that Armstrong's Φ^+_W for $W = I_2(m), H_3$ have many other desired/expected properties of Φ^+_W for crystallographic W, but there can be no such root poset $\Phi^+_{H_4}$!

not coincidental

THEOREM (N. Williams 2013) building on work of Haiman, Proctor, Purbhoo, Stanley

A real reflection group W has

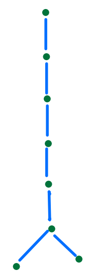
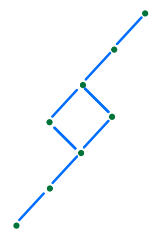
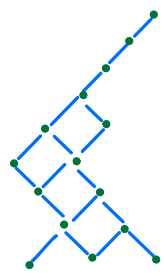
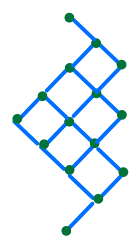
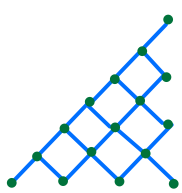
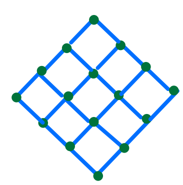
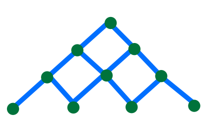
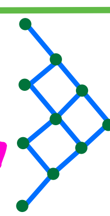
$$\#\left\{\begin{array}{c}\text{reduced } S\text{-words} \\ w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}\end{array}\right\} = \#\left\{\begin{array}{c}\text{linear} \\ \text{extensions} \\ \text{of } \Phi^+_W\end{array}\right\}$$

S-longest element $w_0 \in W$

when W is coincidental.

THEOREM (Hamaker-Patrias-Pechenik-Williams 2016)
Their root posets Φ_W^+ have (minuscule) doppelgängers!

= poset with same order polynomial $\Omega_P(m)$

coincidental W	Φ_W^+	minuscule doppelgänger
$I_2(m)$ (m even)		
H_3		
B_n		
A_n		

not quite doppel-gängers

- The coincidentals are dynamic!

(in the sense of dynamical algebraic combinatorics)

CLUSTERS & MULTI-CLUSTERS

THEOREM (S.-P. Eu & T.-S. Fu 2006)

For real reflection groups W, the product

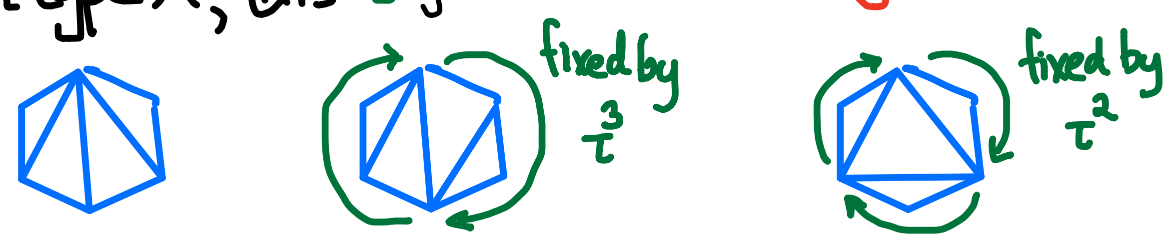
$$\mathrm{Cat}(W,q) := \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

W-q-Catalan

gives a cyclic sieving phenomenon (CSP) for the action of the deformed Coxeter element τ on (Fomin-Zelevinsky 2001) maximal faces of the W-cluster/Cambrian complex:

$$\#\left\{\begin{matrix}\text{maximal faces} \\ \text{fixed by } \tau^d\end{matrix}\right\} = \Big[\mathrm{Cat}(W,q)\Big]_{q=\left(e^{\frac{2\pi i}{h+2}}\right)^d}$$

In type A, this τ just rotates triangulations:



What about non-maximal faces?

The natural q-analogue

$$f_k(q) = \left[\mathrm{Hilb}\left(\left(S \otimes \Lambda V^* \otimes \Lambda^k V \right)^W, q, t \right) \right]_{t=-q^{h+1}}$$

seems* to give the analogous CSP

$\iff$ W is coincidental

$$\left(\iff f_k(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \prod_{i=1}^{n-k} \frac{[h+d_i]_q}{[d_i]_q} \right)$$

*We didn't fully check it works for H_3 nor that it fails for F_4, E_6, E_7, E_8, H_4

The E_n-F_n CSP result has a conjectural generalization by Ceballos-Labbé-Stump (2013) replacing...

- maximal faces in cluster complexes with maximal faces in multi-cluster complexes

- $\mathrm{Cat}(W)$ with

$$\mathrm{multiCat}(W,\ell,q) := \prod_{i=1}^{n} \prod_{j=0}^{\ell} \frac{[h+d_i+2j]_q}{[d_i+2j]_q}$$

- deformed Coxeter element τ with Auslander-Reiten translation τ, of order $h+2\ell$, so plugging in $q = \left(e^{\frac{2\pi i}{h+2\ell}}\right)^d$

...but only for W coincidental!

Ceballos-Labbé-Stump (2013)
S. Hopkins (2019)

P-PARTITIONS & ROWMOTION

The doppelgänger results of HPPW 2016 together with work of Proctor 1984 re-interpets

$$\mathrm{multiCat}(W,\ell,q) \left(:= \prod_{i=1}^{n} \prod_{j=0}^{\ell} \frac{[h+d_i+2j]_q}{[d_i+2j]_q} \right) = \sum_f q^{|f|}$$

where f runs through the set

$$\{P\text{-partitions } f : \Phi_W^+ \to \{0,1,2,\ldots,\ell\}$$

$$\alpha \le \beta \Rightarrow f(\alpha) \le f(\beta)\}$$

and $|f| := \sum_\alpha f(\alpha)$.

These P-partitions have another interesting action called PL-rowmotion ρ, of order $2h$ (not $h+2\ell$)

CONJECTURE (S. Hopkins 2019)
For coincidental W, $\mathrm{MultiCat}(W,\ell,q)$ has another CSP, for ρ on these P-partitions.

WHAT IS GOING ON WITH THE COINCIDENTALS ?!

Thanks for your attention!