Invariant theory for the free left-regular band and a q-analogue

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1. What is invariant theory?

2. What is a left-regular band (LRB)?

EXAMPLES

- free LRB $\mathcal{F}_n$
- q-analogue $\mathcal{F}_n^{(q)}$
- Tits face semigroup $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$

3. The invariant ring for the free LRB

4. The derangement representations

5. The whole ring for the free LRB

1. What is invariant theory?

Classically it asks, for a subgroup $G \subset GL_n(k)$ acting on $S = k[x_1, \dots, x_n]$ by linear substitutions $g(x_j) = \sum_j g_{ij} x_i$ ....

- Structure of the G-invariants $S^G := \{ f(x) \in S : f(g\underline{x}) = f(\underline{x}) \ \forall g \in G \}$ as a ring?
  Generators, relations?

- Structure of the whole ring S as an S^G-module and simultaneously as a G-representation?

Simplest answers for finite reflection groups $G \subset GL_n(\mathbb{C})$:

- $S^G = \mathbb{C}[f_1, f_2, \ldots, f_n]$ is also a polynomial algebra
  (n generators, 0 relations)

  e.g. $G = \mathfrak{S}_n$ permuting variables in $\mathbb{C}[x_1, \ldots, x_n]$
  has $\mathbb{C}[x_1, x_2, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

  elementary symmetric polynomials

- $S = \mathbb{C}[x_1, \ldots, x_n]$ is a free S^G-module

  $$S = \bigoplus_{\substack{G\text{-irreducible} \\ \text{characters } \chi}} S^{G,\chi}$$

  each χ-isotypic component is a free S^G-module, with $\chi(1)^2$ basis elements in known degrees.

2. What is a left-regular band (LRB)?

A monoid $\mathcal{M}$ (= semigroup with 1) in which

$$xyx = xy \quad \forall x, y \in \mathcal{M}$$

$\Downarrow$ $y=1$

[$x^2 = x \quad \forall x \in \mathcal{M}$ defines a band = idempotent monoid]

Studied by Bidigare, Bidigare-Hanlon-Rockmore,
Brown, Brown & Diaconis,
Saliola, Margolis-Saliola-Steinberg,
Aguiar-Mahajan, ...

EXAMPLE The free LRB $\mathcal{F}_n$ on letters $a_1, a_2, \ldots, a_n$

$\mathcal{F}_n = \{$ injective words on the letters $\}$ with multiplication

no repeated letters

$$a_1 a_2 \cdots a_\ell \cdot b_1 b_2 \cdots b_m = \underbrace{(a_1 a_2 \cdots a_\ell b_1 b_2 \cdots b_m)^{\wedge}}_{\text{concatenation}}$$

$^{\wedge}$ means remove $2^{nd}, 3^{rd}, \ldots$ occurrences of letters

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e.g. $n=3$ On letters $\{a,b,c\}$,

$\mathcal{F}_3 = \{1,$ a, b, c, ab, ac, ba, bc, ca, cb, abc, acb, bac, bca, cab, cba $\}$

with $1 = \binom{\text{empty}}{\text{word}}$

$a \cdot a = a$

$ac \cdot ac = ac$

$ac \cdot ab = acb$

$bac \cdot ab = bac$

$ab \cdot bac = abc$

EXAMPLE $\mathcal{F}_n^{(q)}$ = q-analogue of the free LRB $\mathcal{F}_n$

$= \{$ flags $(V_1, V_2, \ldots, V_\ell)$ of subspaces

$\underbrace{V_1}_{\text{line}} \subset \underbrace{V_2}_{\text{plane}} \subset \ldots \subset \underbrace{V_\ell}_{\ell\text{-subspace}}$ in $\mathbb{F}_q^n$ with $\dim V_i = i$ $\}$

with $(V_1, V_2, \ldots, V_\ell) \cdot (W_1, W_2, \ldots, W_m) :=$

$$\left(V_1, V_2, \ldots, V_\ell, V_\ell + W_1, V_\ell + W_2, \ldots, V_\ell + W_m\right)^\wedge$$

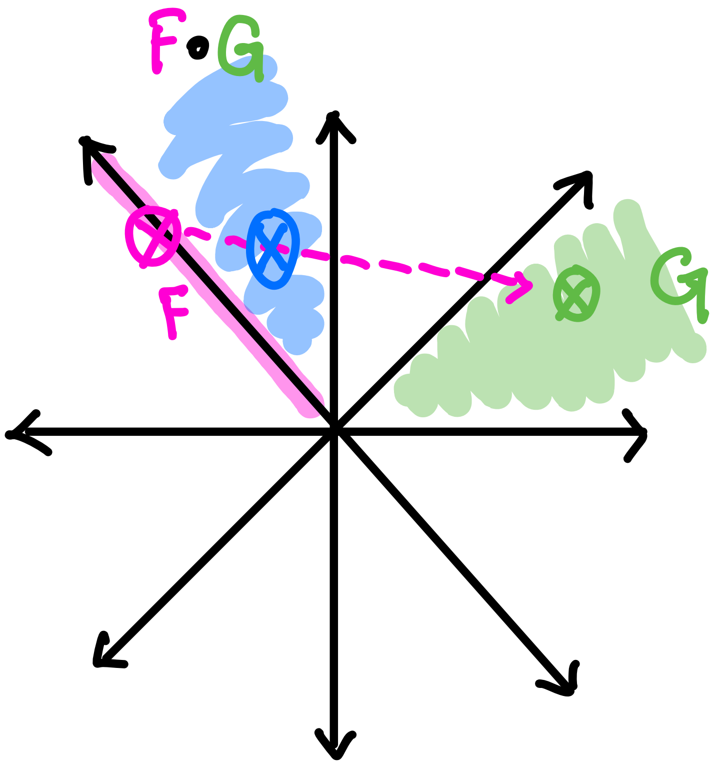
means remove any subspace that appears earlier in the list

MOTIVATING EXAMPLE

Tits's face semigroup of a (central) hyperplane arrangement $\mathcal{A} \subset \mathbb{R}^n$

$\mathcal{F}(\mathcal{A}) = \{$ faces F of $\mathcal{A} \}$ with $F \circ G$ = "face F perturbed toward face G"

chambers and all their subface cones



1 in $\mathcal{F}(\mathcal{A})$ is the 0-dimensional face at the origin $\{0\}$:

$\{0\} \circ G = G \quad \forall$ faces G

MOTIVATION:

Inside the monoid algebra $k\mathcal{M} := \left\{ \sum_{m \in \mathcal{M}} c_m m : c_m \in k \right\}$

one can model card-shuffling Markov chains

and use representation theory of $k\mathcal{M}$ to analyze

eigenvalues and mixing times.

EXAMPLE

Random-to-top shuffling

on $\mathfrak{S}_n = \left\{ \begin{array}{l} \text{permutations} \\ \text{of } a_1, \ldots, a_n \end{array} \right\}$

$$R2T(abc) = \frac{1}{3}(abc + bac + cab)$$

R2T on $\mathfrak{S}_n$ can be modeled inside $\mathbb{Q}\mathcal{F}_n$ ← free LRB

as left-multiplication by $\frac{1}{n} \cdot x$ where $x = a_1 + a_2 + \dots + a_n$.

e.g. n=3: $\frac{1}{3}(\overbrace{a+b+c}^{x}) \cdot (abc) = \frac{1}{3}(abc + bac + cab)$

Or inside $\mathbb{Q}\mathcal{F}(\mathcal{A}_n)$ as left-multiplication by $\frac{1}{n} \cdot x$ where

$x = 1|23 \cdots n + 2|134 \cdots n + \dots + n|12 \cdots n\text{-}1$

braid arrangement $\bigcup_{1 \le i < j \le n} \{x_i = x_j\}$

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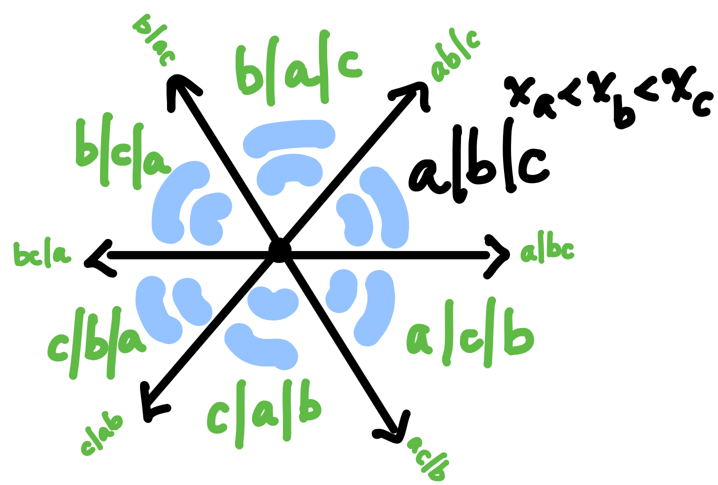
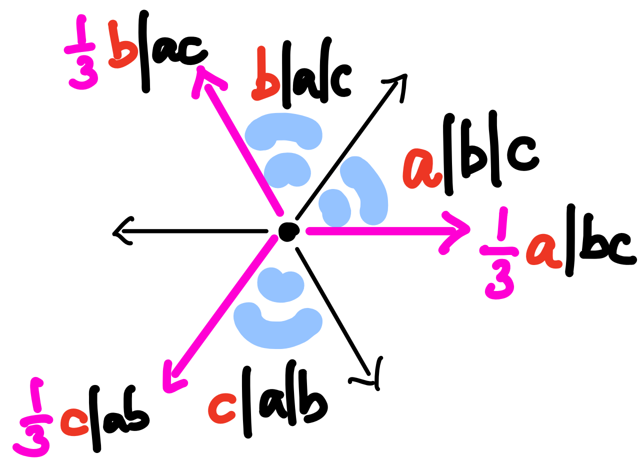
e.g. n=3: $\frac{1}{3}(\overbrace{a|bc + b|ac + c|ab}^{x}) \circ (a|b|c) = \frac{1}{3}(a|b|c + b|a|c + c|a|b)$

$\mathcal{A}_{\mathfrak{S}_3}$





THEOREM (Bidigare 1997) When $\mathfrak{S}_n$ acts on $k\mathcal{F}(\mathcal{A}_n)$ the $\mathfrak{S}_n$-invariant subalgebra

$$k\mathcal{F}(\mathcal{A}_n)^{\mathfrak{S}_n} \cong \underbrace{\mathrm{Sol}(\mathfrak{S}_n)^{opp}}$$

Solomon's descent algebra for $\mathfrak{S}_n$

(a non-semisimple algebra)

He applied this to R2T on $\mathfrak{S}_n$ and other symmetric random walks.

Further work on $k\mathcal{F}(\mathcal{A}_n)$ as $\mathfrak{S}_n$-rep and $k\mathcal{F}(\mathcal{A}_n)^{\mathfrak{S}_n}$-module by

Garsia & Reutenauer 1989

Uyemura-Reyes 2002

Commins 2022+ (ongoing thesis work)

3. The invariant ring for the free LRB

PROPOSITION (easy) The free LRB $\mathcal{F}_n$ has $\mathfrak{S}_n$-invariant subalgebra $(k\mathcal{F}_n)^{\mathfrak{S}_n}$ with k-basis of orbit sums

$$x_0 = 1$$
$$x_1 = a_1 + a_2 + \cdots + a_n$$
$$x_2 = a_1a_2 + a_2a_1 + a_1a_3 + \cdots + a_na_{n-1}$$
$$\vdots$$
$$x_n = a_1a_2\cdots a_n + \cdots + a_n\cdots a_2a_1$$

NOTE: $x_1 = x$ from before (having $R2T = \frac{1}{n}\cdot x$)

EXAMPLE $(k\mathcal{F}_3)^{\mathfrak{S}_3}$ has k-basis

$$x_0 = 1$$
$$x_1 = a+b+c$$
$$x_2 = ab+ba+ac+ca+bc+cb$$
$$x_3 = abc+acb+bac+bca+cab+cba$$

(easy)
PROPOSITION:
$x := x_1 = a_1 + a_2 + \ldots + a_n$ left-multiplies in this basis triangularly

$$x \cdot x_\ell = \ell \cdot x_\ell + x_{\ell+1}$$

(easy)
COROLLARY: The powers $\{1, x, x^2, \ldots, x^n\}$ expand unitriangularly in the orbit sum k-basis $\{1, x_1, x_2, \ldots, x_n\}$ for $(kF_n)^{\mathfrak{S}_n}$ with Stirling numbers $S(n,k)$ as coefficients: $x^m = \sum_k S(m,k)\, x_k$

EXAMPLE:

$x^0 = 1 = 1 \cdot x_0$

$x^1 = a+b+c = 1 \cdot x_1$

$x^2 = (a+b+c)^2 = 1 \cdot x_1 + 1 \cdot x_2$

$x^3 = (a+b+c)^3 = 1 \cdot x_1 + 3 \cdot x_2 + 1 \cdot x_3$

Stirling numbers $S(n,k)$

$n \backslash k$	0	1	2	3	4
0	1				
1		1			
2		1	1		
3		1	3	1	
4		1	6	7	1

$S(n,k) = \#$ of set partitions of $\{1,2,\ldots,n\} = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k$ with k blocks

$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

COROLLARY: $x = a_1 + a_2 + \dots + a_n$ generates $(kF_n)^{\mathfrak{S}_n}$,
(Brauner - Commins - R. 2022)
and one has a ring isomorphism

$$k[X]/(X(X-1)(X-2)\cdots(X-n)) \longrightarrow (kF_n)^{\mathfrak{S}_n}$$

sending $X \longmapsto x$

In particular, when $n! \in k^\times$, the invariant ring $(kF_n)^{\mathfrak{S}_n}$ is commutative and semisimple, and x acts with eigenvalues $0, 1, \dots, n$ in finite dimensional $(kF_n)^{\mathfrak{S}_n}$-modules.

CONCLUSION: To describe kF_n as $(kF_n)^{\mathfrak{S}_n}$-module and $\mathfrak{S}_n$-rep, only need to describe $\mathfrak{S}_n$-rep on each eigenspace $\ker(x - m)$ on kF_n
$m = 0, 1, 2, \dots, n$

... and same story for the q-analogue $\mathcal{F}_n^{(q)}$ with the action of $GL_n(\mathbb{F}_q)$:

- $x \rightsquigarrow x^{(q)} = \sum_{\substack{\text{lines } L \\ \text{in } \mathbb{F}_q^n}} (L) = (L_1) + (L_2) + \ldots + (L_{[n]_q})$

  where $[n]_q := 1 + q + q^2 + \ldots + q^{n-1}$

- Stirling numbers $S(n,k) \rightsquigarrow$ q-Stirling numbers (Milne 1982)

- $\left(k\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)} \cong k[X] / (X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q))$

- $\left(k\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)}$ is commutative, semisimple, and $x^{(q)}$ acts with eigenvalues $[0]_q, [1]_q, \ldots, [n]_q$ on modules.

4. The derangement representation of $\mathfrak{S}_n$

Recall the derangement numbers

$$d_n := n!\left(\frac{1}{1!}-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\dots+\frac{(-1)^n}{n!}\right)$$

count permutations $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_n)$ in $\mathfrak{S}_n$

- with no fixed points $\sigma_i = i$ (derangements)

– OR –

- having even first ascent position i with $\sigma_i < \sigma_{i+1}$ (desarrangements)

n	d_n
0	1
1	0
2	1
3	2
4	9

But d_n are also dimensions of an $\mathfrak{S}_n$-rep D_n whose associated symmetric function d_n was introduced by Désarménien & Wachs 1993.

(EQUIVALENT) DEFINITIONS:

- $d_n = \sum_{\substack{\text{standard} \\ \text{Young tableaux } Q \\ \text{whose 1st ascent } i \text{ is even}}} s_{\lambda(Q)}$ ← explicit $\mathfrak{S}_n$ irreducible decomposition of D_n

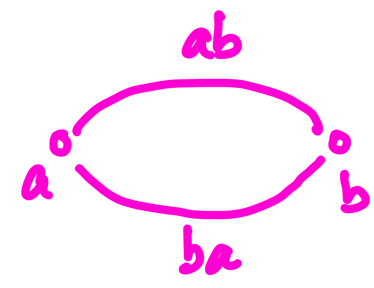
- $d_n = h_{1^n} - e_1 \cdot h_{1^{n-1}} + e_2 h_{1^{n-2}} + e_3 h_{1^{n-3}} - \dots + (-1)^n e_n$

- $h_{1^n} = d_n + h_1 d_{n-1} + h_2 d_{n-2} + \dots + h_{n-1} d_1 + h_n$

- $D_n \cong \ker(R2T : k\mathfrak{S}_n \longrightarrow k\mathfrak{S}_n)$

- $D_n \cong \mathrm{sgn}_{\mathfrak{S}_n} \otimes \left(\begin{array}{l} \text{top homology of the} \\ \text{(cell) complex } K_n \text{ of} \\ \text{injective words on } a_1, a_2, \dots, a_n \end{array} \right)$

K_2



n	standard Young tableaux Q with 1st ascent i even	symmetric function d_n	derangement number d_n
0	$\emptyset$	1	1
1	–	0	0
2	$\begin{smallmatrix}1\\2\end{smallmatrix}$	$s_{\ydiagram{1,1}}$	1
3	$\begin{smallmatrix}1&3\\2\end{smallmatrix}$	$s_{\ydiagram{2,1}}$	2
4	$\begin{smallmatrix}1\\2\\3\\4\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2\\4\end{smallmatrix}$ $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$ $\begin{smallmatrix}1&3&4\\2\end{smallmatrix}$	$s_{\ydiagram{1,1,1,1}} + s_{\ydiagram{2,1,1}} + s_{\ydiagram{2,2}} + s_{\ydiagram{3,1}}$	9

5. The whole ring for the free LRB

Filter $\mathcal{F}_n$ by word length:

$\mathcal{F}_{\geq \ell}$ = k-span of injective words of length $\geq \ell$

$$\mathcal{F}_n = \mathcal{F}_{\geq 0} \supset \mathcal{F}_{\geq 1} \supset \mathcal{F}_{\geq 2} \supset \cdots \supset \mathcal{F}_{\geq n-1} \supset \mathcal{F}_n$$

Semisimplicity of $(k\mathcal{F}_n)^{\mathfrak{S}_n}$ and of $k\mathfrak{S}_n$

$\Rightarrow$ sufficient to describe the $\mathfrak{S}_n$-rep on

- each x-eigenspace $\ker(x-m)$ for $m=0,1,\ldots,n$
- acting on each filtration factor $\mathcal{F}_{\geq \ell}/\mathcal{F}_{\geq \ell+1}$

THEOREM (Brauner-Commins-R. 2022) In kF_n, the x-eigenspace $\ker(x-m)$ for $m=0,1,\ldots,n$ when x acts on $F_{\geq \ell}/F_{\geq \ell+1}$ for $\ell=0,1,\ldots,n$ carries $\mathfrak{S}_n$-rep with symmetric function

$$h_{n-\ell}\cdot h_m \cdot d_{\ell-m}$$

derangement rep

(that is, the induction $\left(\mathbb{1}_{\mathfrak{S}_{n-\ell}} \otimes \mathbb{1}_{\mathfrak{S}_m} \otimes D_{\ell-m}\right)\Big\uparrow_{\mathfrak{S}_{n-\ell}\times\mathfrak{S}_m\times\mathfrak{S}_{\ell-m}}^{\mathfrak{S}_n}$)

... and same for q-analogue $kF_n^{(q)}$

- $\mathfrak{S}_n$-irreducibles $\rightsquigarrow$ $GL_n(\mathbb{F}_q)$ unipotent irreducibles
- induction $\mathfrak{S}_a\times\mathfrak{S}_b$ to $\mathfrak{S}_{a+b}$ $\rightsquigarrow$ parabolic induction $GL_a\times GL_b$ to GL_{a+b}

Proof ideas:

- In bottom of filtration, $kF_{\geq n} \cong k\mathfrak{S}_n$ = regular rep, and can construct m-eigenvectors for x on $k\mathfrak{S}_n$ by inducing $(\mathbb{1}_{\mathfrak{S}_m} \otimes -)\uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}$ nullvectors for x on $k\mathfrak{S}_{n-m}$

- Then use $h_{1,n} = d_n + h_1 d_{n-1} + h_2 d_{n-2} + \cdots + h_{n-1} d_1 + h_n$ to show nullspace must carry d_n, m-eigenspace must carry $h_m d_{n-m}$.

- j-eigenspace for x on $F_{\geq \ell}/F_{\geq \ell+1}$ is $\mathfrak{S}_n$-isomorphic to

$$\left(\begin{matrix} j\text{-eigenspace for } x \\ \text{on } k\mathfrak{S}_\ell \end{matrix} \right) \otimes \mathbb{1}_{\mathfrak{S}_{n-\ell}} \Big\uparrow_{\mathfrak{S}_\ell \times \mathfrak{S}_{n-\ell}}^{\mathfrak{S}_n} \rightsquigarrow \left(h_m \cdot d_{\ell-m} \right) \cdot h_{n-\ell}$$

Thanks for
your attention,
and thank you
Ed, Theo and Yasu!