Sandpiles for group representations

Vic Reiner
Univ. of Minnesota
(joint with Georgia Benkart
Carly Klivans)

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THE PLAN:

- Review (directed) graph Laplacians and sandpile groups
- Define McKay-Cartan matrices and sandpile groups for representations
- Examples, results

Γ = a directed graph with $\ell+1$ vertices

$$L(\Gamma) = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & \multicolumn{3}{c}{\begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix}} \\ 2 & & & \end{array}$$

outdegrees

has Laplacian

$$L(\Gamma) \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$$

$$L(\Gamma)_{ij} := \begin{cases} \mathrm{outdeg}_\Gamma(i) & \text{if } i=j \\ -\#\left(\begin{matrix}\text{arcs} \\ i \to j\end{matrix}\right) & \text{if } i \neq j \end{cases}$$

$$L(\Gamma) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array}$$

is always singular

since $\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a (right-) nullvector.

REMARK:

The (primitive) left-nullvector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

will also play a role later...

THEOREM

The principal minors of $L(\Gamma)$

$$L(\Gamma)^{i,i} = L(\Gamma) - \begin{Bmatrix} i^{th} \text{ row,} \\ i^{th} \text{ column} \end{Bmatrix}$$

have nonnegative determinant,

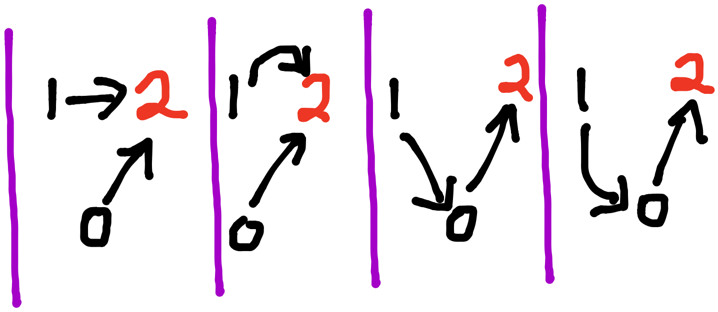
since

$$\det L(\Gamma)^{i,i} = \# \begin{pmatrix} \text{spanning trees inside } \Gamma \\ \text{directed toward vertex } i \\ \text{as root} \end{pmatrix}$$

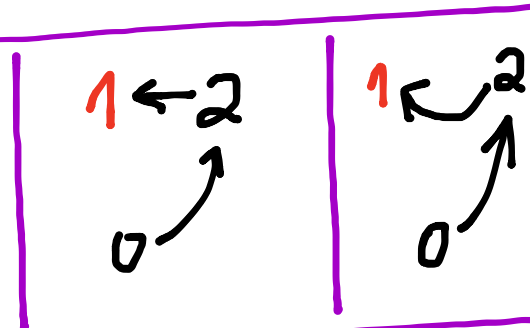
arborescences

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & -1 \\ 1 & -2 & 4 & -2 \\ 2 & 0 & -2 & 2 \end{array}$$

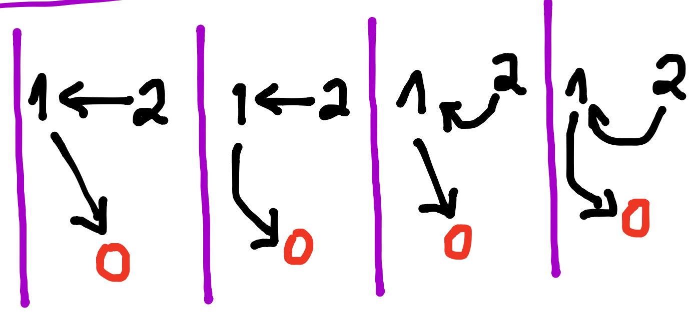
$$\det \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & -2 & 4 \end{array} = 4$$



$$\det \begin{array}{c|cc} & 0 & 2 \\ \hline 0 & 1 & -1 \\ 2 & 0 & 2 \end{array} = 2$$



$$\det \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 4 & -2 \\ 2 & -2 & 2 \end{array} = 4$$



This means $L(\Gamma) \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$
and each $L(\Gamma)^{i,i} \in \mathbb{Z}^{\ell\times\ell}$
have $\mathbb{Q}$-rank ℓ.

Q: What about their ranks when we reduce mod p, i.e. over $\mathbb{F}_p$?

Better yet, what about their Smith normal form over $\mathbb{Z}$?

RECALL: $A \in \mathbb{Z}^{\ell \times \ell}$ has $P, Q \in GL_\ell(\mathbb{Z})$

with $\underbrace{PAQ = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_\ell \end{bmatrix}}_{\text{Smith normal form of } A}$ with $d_i \mid d_{i+1}$

Equivalently,

$$\operatorname{coker}\left(\mathbb{Z}^\ell \xrightarrow{A^T} \mathbb{Z}^\ell\right)$$

$$= \mathbb{Z}^\ell / \operatorname{span}_{\mathbb{Z}}\{\text{rows of } A\}$$

$$\cong \mathbb{Z}/d_1\mathbb{Z} \times \mathbb{Z}/d_2\mathbb{Z} \times \cdots \times \mathbb{Z}/d_\ell\mathbb{Z}$$

DEFINITION

The Sandpile group

$$K(\Gamma,i) := \mathrm{coker}\left(\mathbb{Z}^{\ell} \xrightarrow{L(\Gamma)^{i,i}} \mathbb{Z}^{\ell}\right)$$

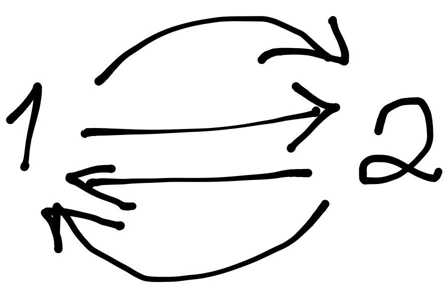
$L(\Gamma)^{i,i}$	Smith normal form	$K(\Gamma,i)$
$\begin{array}{c\|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & -2 & 4 \end{array}$	$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$	$\mathbb{Z}/4\mathbb{Z}$
$\begin{array}{c\|cc} & 0 & 2 \\ \hline 0 & 1 & -1 \\ 2 & 0 & 2 \end{array}$	$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$	$\mathbb{Z}/2\mathbb{Z}$
$\begin{array}{c\|cc} & 1 & 2 \\ \hline 1 & 4 & -2 \\ 2 & -2 & 2 \end{array}$	$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Why sandpile group?

Cosets of $\mathbb{Z}^\ell / \mathrm{span}_{\mathbb{Z}}\{\text{rows of } L(\Gamma)^{i,i}\}$

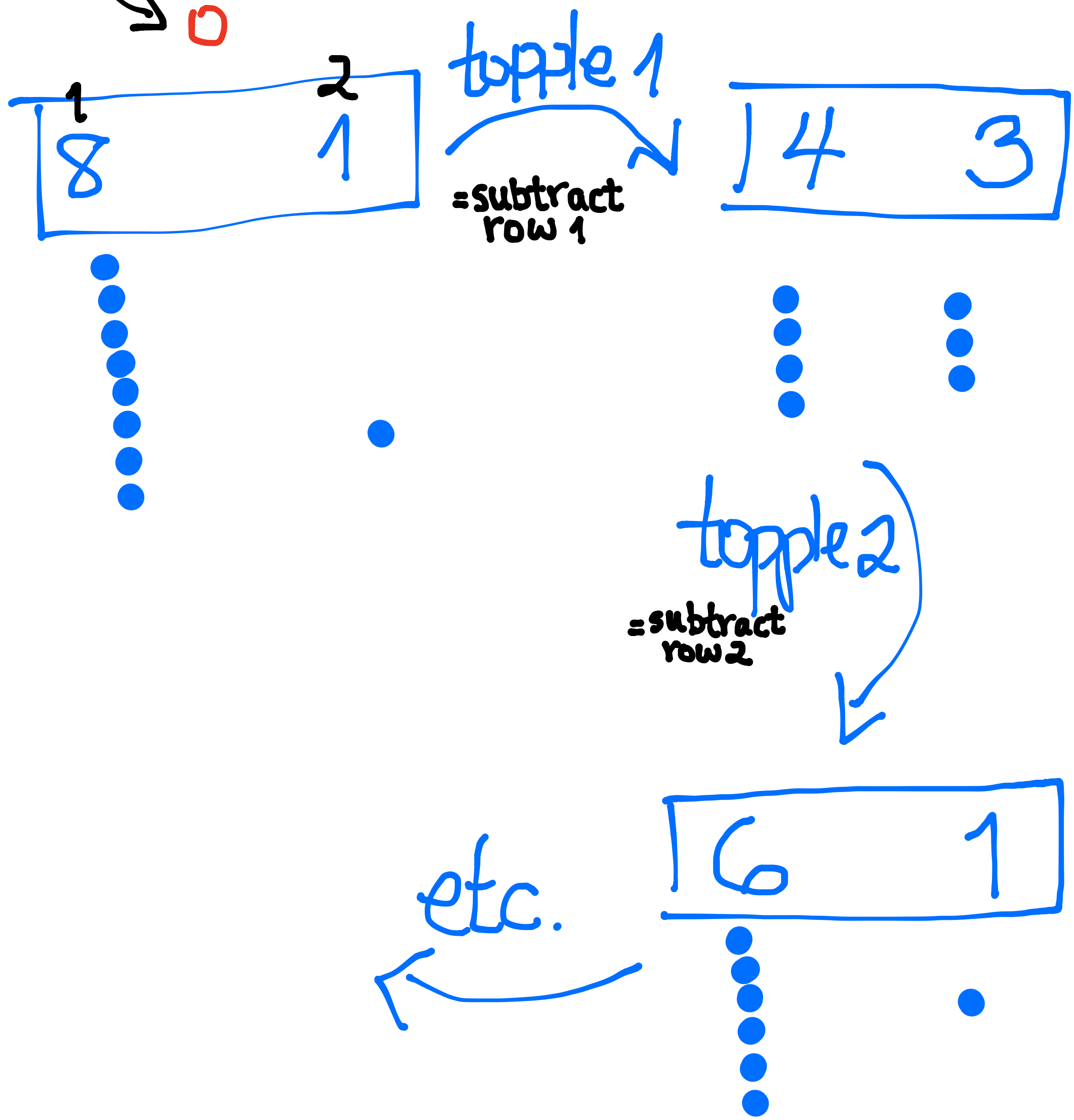
have interesting representatives in $\mathbb{N}^\ell$ coming from a game on Γ:

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 1 & 0 & -1 \\ 1 & -2 & 4 & -2 \\ 2 & 0 & -2 & 2 \end{array}$$



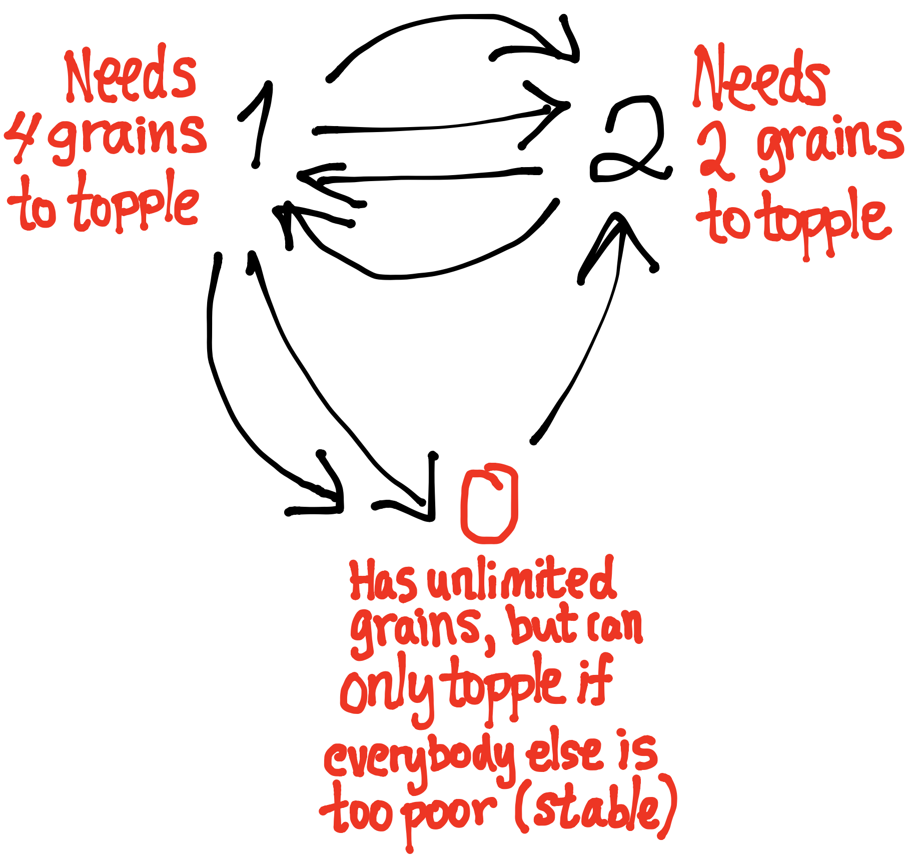
$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 4 & -2 \\ 2 & -2 & 2 \end{array}$$

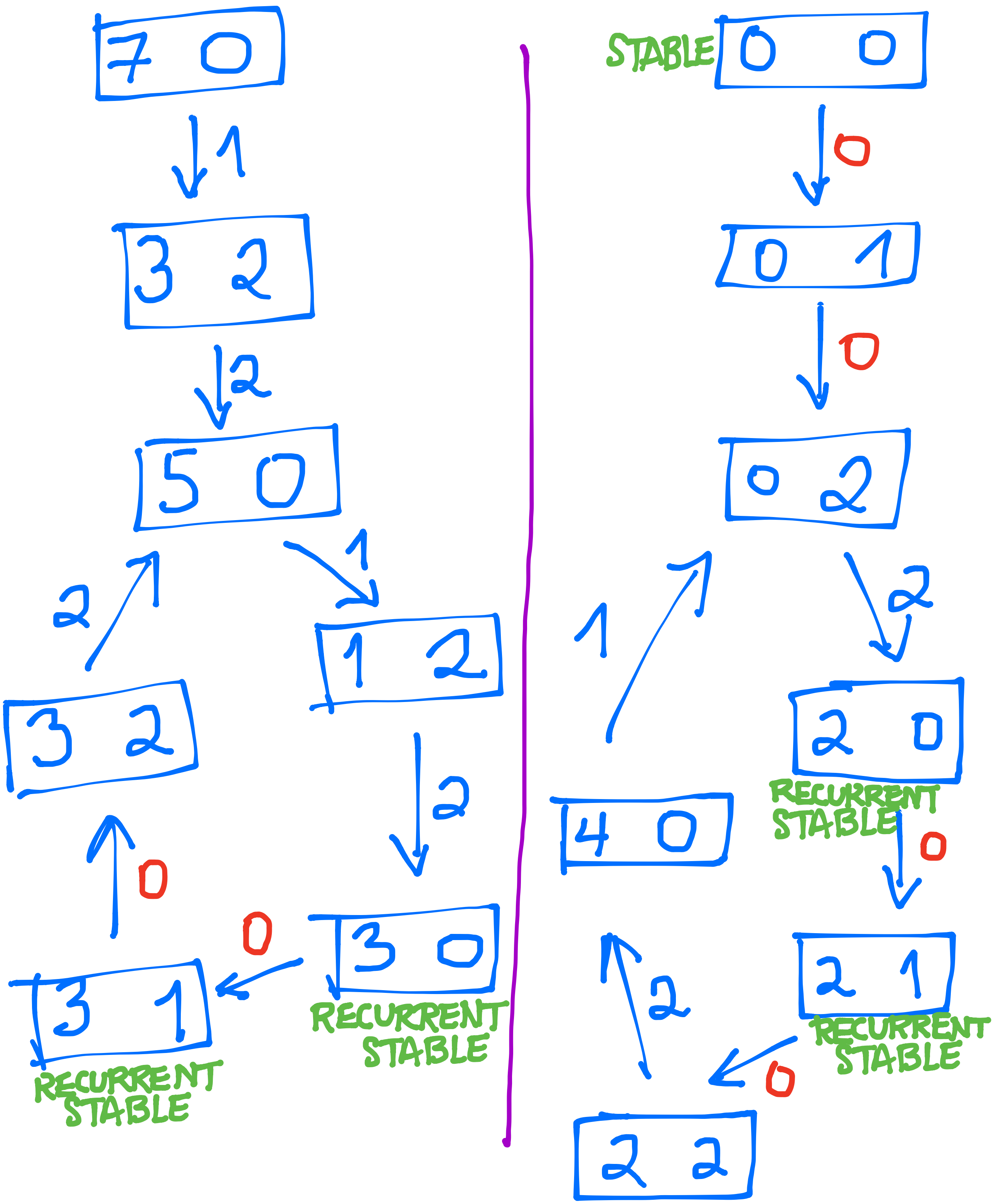
$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 4 & -2 \\ 2 & -2 & 2 \end{array} \quad \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$$



Toppling rules:

$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & -1 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array}$$





7 0
1
3 2
2
5 0
2
1
3 2
1 2
2
0
0
3 1
3 0
RECURRENT STABLE
RECURRENT STABLE
STABLE
0 0
0
0 1
0
0 2
1
2
2 0
RECURRENT STABLE
4 0
0
2
2 1
RECURRENT STABLE
0
2 2

THEOREM (Dhar, Gabrielov)

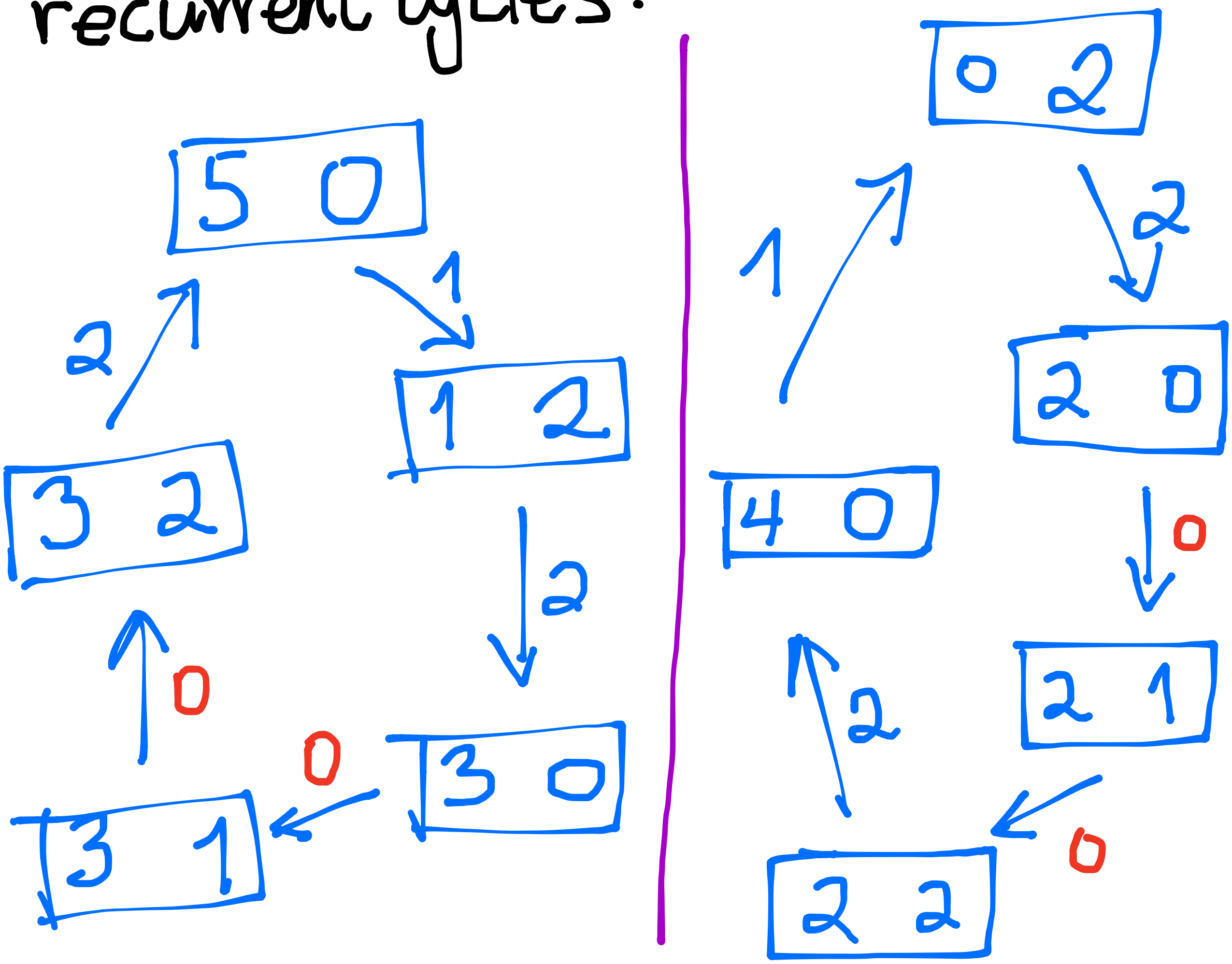
$K(\Gamma, i) = \mathrm{coker}\left(\mathbb{Z}^\ell \xrightarrow{L(\Gamma)^{ii}} \mathbb{Z}^\ell\right)$

has coset representatives given by the recurrent (stable) configurations in $\mathbb{N}^\ell$

e.g. above

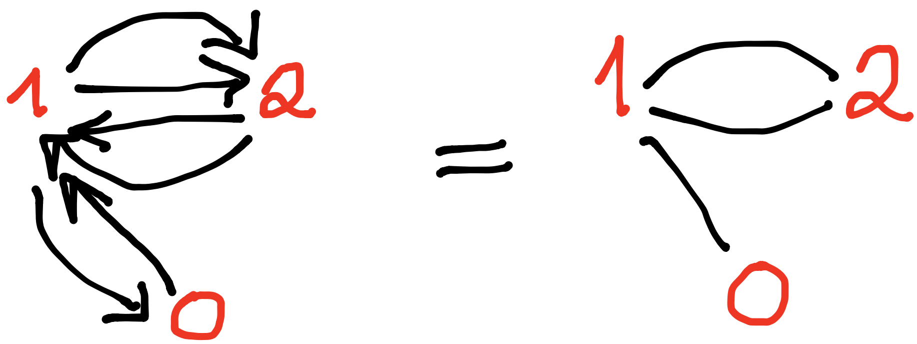
$$K(\Gamma, 0) = \left\{ \boxed{2 \quad 0},\ \boxed{2 \quad 1},\ \boxed{3 \quad 0},\ \boxed{3 \quad 1} \right\}$$

The left-nullvector $\begin{array}{c} 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ tells how many times each node topples during recurrent cycles:



... and its 0 entry tells how many times 0 should topple to preserve cosets.

When Γ is undirected several better things happen



- $$L(\Gamma) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array}$$

  is a symmetric matrix

  so $\mathbb{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is both

  left- and right-nullvector.

- Principal minors $L(\Gamma)^{i,i}$ have

  - determinant
  - Smith normal form
  - $K(\Gamma, i)$

  independent of i

  call this $K(\Gamma)$

- $L(\Gamma)^{i,i}$ is positive definite

  $L(\Gamma)$ is positive semidefinite

  eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_\ell$

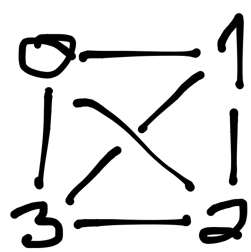
- $$\# K(\Gamma) = \det L(\Gamma)^{i,i} = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell + 1}$$

This leads to many families of undirected graphs where we

- know $L(\Gamma)$'s eigenvalues,
- guess, prove structure of $K(\Gamma)$.

EXAMPLE

K_n = complete graph



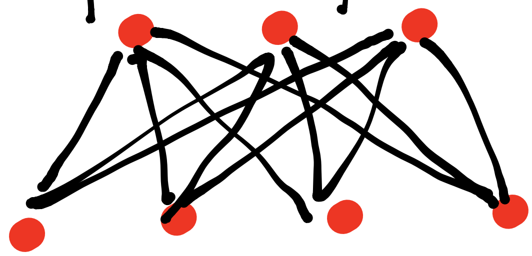
has $L(K_n)$ eigenvalues $(\underbrace{n, n, \ldots, n}_{n-1 \text{ times}})$

$$\#K(K_n) = n^{n-2}$$

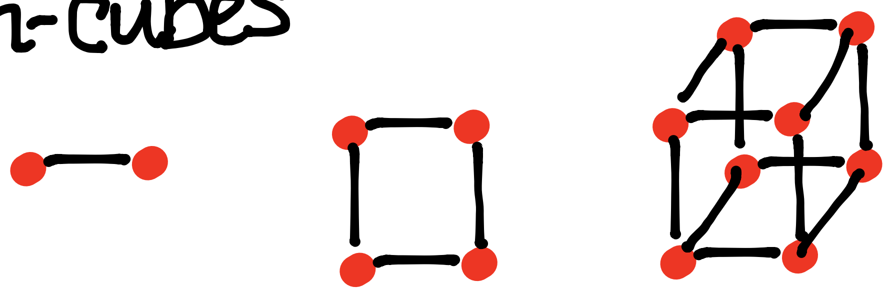
$$K(K_n) = (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

OTHER EXAMPLES

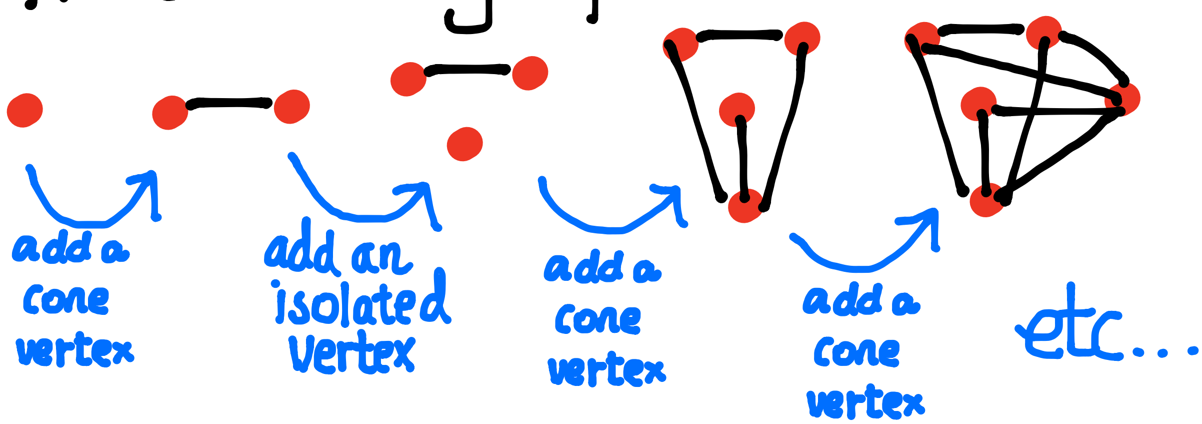
Complete bipartite, multipartite graphs



n-cubes



Threshold graphs



McKay-Cartan matrices

Fix G a finite group

with irreducible representations

or characters

$$\{\chi_0, \chi_1, \ldots, \chi_\ell\}$$

$\chi_0 = \mathbb{1}_G$ trivial G-rep

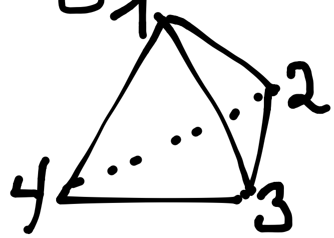
and irreducible degrees

$$\{d_0, d_1, \ldots, d_\ell\}$$

$d_0 = 1$

EXAMPLE

$G = A_4$ = rotational symmetries of



CHARACTER TABLE

	e	(123)	(132)	$(12)(34)$
$1\!\!1_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

$$\omega = e^{2\pi i/3}$$

DEFINITION: Given a faithful representation $G \xhookrightarrow{\gamma} GL_d(\mathbb{C})$

- **McKay matrix** $M = (m_{ij})$

$$\chi_i \otimes \chi_\gamma = \sum_{j=0}^{\ell+1} m_{ij} \chi_j$$

- **Extended McKay-Cartan matrix**

$$\tilde{C} := dI - M \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$$

- **McKay-Cartan matrix**

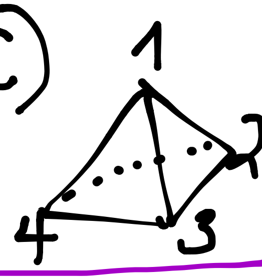
$$C := \tilde{C} - \{\chi_0 \text{ row}, \chi_0 \text{ column}\}$$

- **Sandpile group**

$$K(\gamma) := \operatorname{coker}(C^T)$$
$$= \mathbb{Z}^\ell / \operatorname{span}_{\mathbb{Z}}\left\{\begin{matrix}\text{rows of} \\ C\end{matrix}\right\}$$

EXAMPLE

$G = \mathcal{A}_4 \xhookrightarrow{\gamma} SO_3(\mathbb{R}) \subseteq GL_3(\mathbb{C})$

via rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\chi_\gamma = \chi_3$	3	0	0	-1

$\chi_0 \otimes \chi_\gamma = \chi_1 \otimes \chi_\gamma = \chi_2 \otimes \chi_\gamma = \chi_3$

$\chi_3 \otimes \chi_\gamma = \chi_0 + \chi_1 + \chi_2 + 2\chi_3$

$\rightsquigarrow$

$$M = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$M = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

McKay

$$\tilde{C} = 3I - M = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \left[\begin{array}{c|ccc} 3 & 0 & 0 & -1 \\ \hline 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{array}\right] \end{array}$$

extended McKay-Cartan

$$C = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{normal form}]{\text{Smith}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

McKay-Cartan

$$K(\gamma) = \operatorname{coker} C^T \cong \mathbb{Z}/3\mathbb{Z}$$

Sandpile group

Why $\operatorname{coker} C^T$ versus $\operatorname{coker} \tilde{C}^T$?

FACT: $\tilde{C}$ is singular, with left- and right-nullvector $\begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_\ell \end{bmatrix}$ $d_0 = 1$

e.g. $$\begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightsquigarrow \operatorname{coker} \tilde{C}^T \cong \mathbb{Z} \oplus \underbrace{\operatorname{coker} C^T}_{K(\gamma)}$$

Actually, we know all eigenvectors and eigenvalues of $\tilde{C}$:

$$\sum_{j=0}^{\ell} m_{ij} \chi_j = \chi_i \otimes \chi_\gamma$$

$$\rightsquigarrow \quad M \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix} = \chi_\gamma(g) \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$$

the g^{th} column of character table

$$\rightsquigarrow \quad \tilde{C} \cdot \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix} = \underbrace{\left(n - \chi_\gamma(g)\right)}_{\text{eigenvalues for } \tilde{C}} \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$$

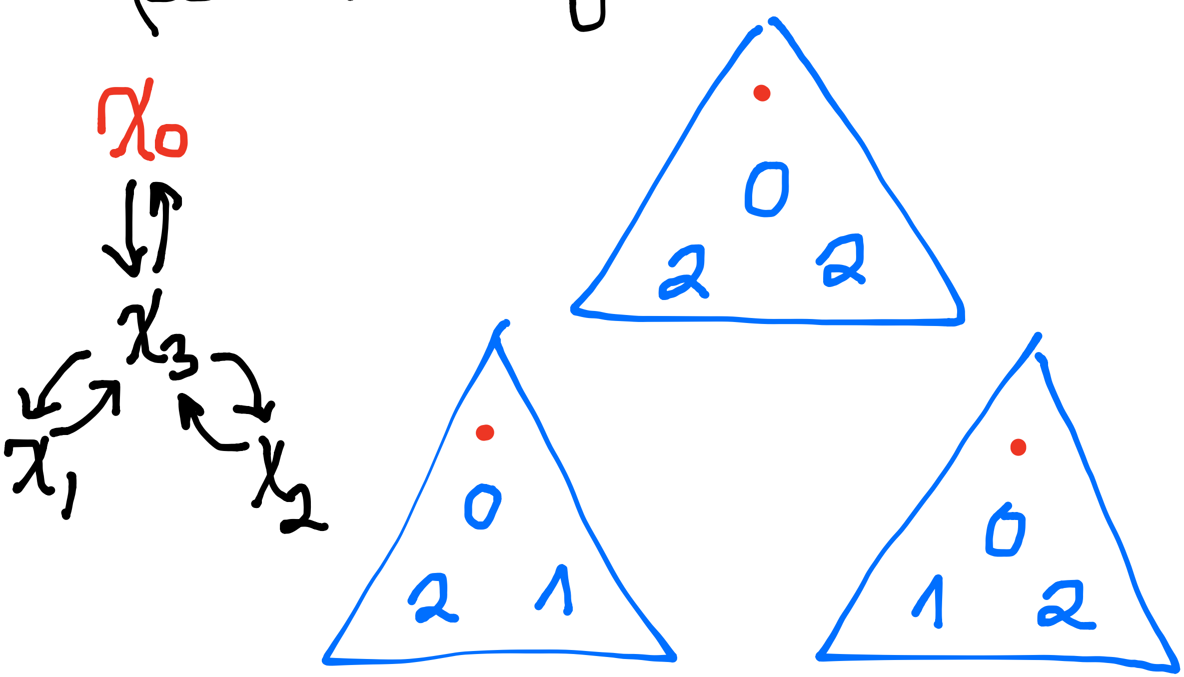
THEOREMS & EXAMPLES

THM: C is an avalanche-finite matrix.

COR: One can compute coset reps for

$$K(\gamma) = \mathbb{Z}^{\ell} / \mathrm{span}_{\mathbb{Z}}\{\text{rows of } C\}$$

via C-toppling, as the recurrent (stable) configurations in $\mathbb{N}^{\ell}$



THEOREM (C. Gaetz) If $G \xrightarrow{\gamma} GL_n(\mathbb{C})$ then

$$\#K(\gamma) = \frac{1}{\#G} \cdot \prod_{\substack{G\text{-conj. classes} \\ [g] \neq \{e\}}} (n - \chi_\gamma(g))$$

EXAMPLE $G = \mathfrak{A}_4 \xrightarrow{\gamma} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$

	e	(123)	(132)	$(12)(34)$
χ_γ	3	0	0	-1

$\rightsquigarrow$ $\#K(\gamma) = \frac{1}{12}(3-0)(3-0)(3-(-1))$

$= 3$

THEOREM If G is abelian then

$$K(\gamma) = \underbrace{K(\Gamma, \chi_0)}_{\substack{\text{usual digraph} \\ \text{sandpile group}}}$$

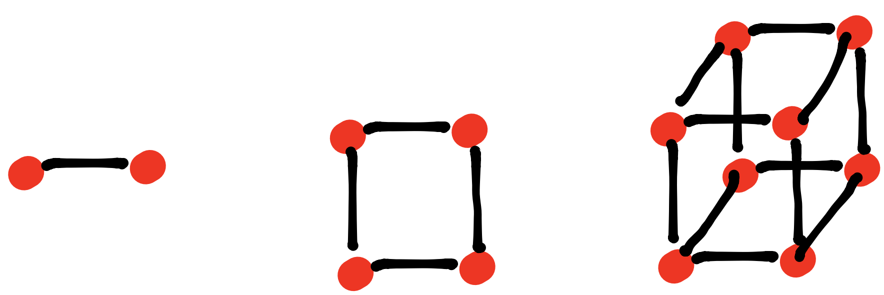
where Γ is the Cayley digraph of the group of G-characters $\{\chi_0, \chi_1, \ldots, \chi_\ell\}$ with respect to the generating multiset $\{\chi_{i_1}, \ldots, \chi_{i_n}\}$, where $G \xhookrightarrow{\gamma} GL_n(\mathbb{C})$ has $\chi_\gamma = \chi_{i_1} + \cdots + \chi_{i_n}$.

EXAMPLE

$$G = (\mathbb{Z}/2\mathbb{Z})^n \xhookrightarrow{\gamma} GL_n(\mathbb{C})$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \longmapsto \begin{bmatrix} (-1)^{\epsilon_1} & & & 0 \\ & (-1)^{\epsilon_2} & & \\ & & \ddots & \\ 0 & & & (-1)^{\epsilon_n} \end{bmatrix}$$

has $K(\gamma) = \underbrace{K(n\text{-cube})}_{\text{usual graph sandpile group}}$



EXAMPLE For any G, the regular representation

$$G \xhookrightarrow{\mathrm{reg}_G} GL_n(\mathbb{C})$$

where $n = \#G$ has

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{\#\left(\substack{G\text{-conjugacy}\\ \text{classes}}\right)-2}$$

$\rightsquigarrow$ G abelian

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

$\rightsquigarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

complete graph

A novel feature of $K(\gamma)$...

THEOREM There is a ring structure

$$\mathbb{Z} \oplus K(\gamma) = \operatorname{coker} \tilde{C}^T$$
$$\cong R_{\mathbb{Z}}(G)/J$$

where

- $R_{\mathbb{Z}}(G)$ is the ring of virtual G-characters with $\otimes$ as product
- J is the principal ideal in $R_{\mathbb{Z}}(G)$ generated by $n \cdot 1 - \chi_{\gamma}$

EXAMPLE $G = C_4$ has

$$R_{\mathbb{Z}}(G) \cong \mathbb{Z}^4$$

call this x call this y

$$= \mathbb{Z}\cdot\chi_0 + \mathbb{Z}\chi_1 + \mathbb{Z}\chi_2 + \mathbb{Z}\chi_3$$

$$\cong \mathbb{Z}[x,y] \big/ \left(x^3-1,\ xy-y,\ y^2-(1+x+x^2+2y)\right)$$

$$G = C_4 \xrightarrow{\gamma} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C}) \text{ has}$$

$$\mathbb{Z} \oplus K(\gamma) \cong R_{\mathbb{Z}}(G) \big/ (3-y)$$

$$\cong \mathbb{Z}[x] \big/ \left(x^3-1,\ 3(x-1),\ 2-x-x^2\right)$$

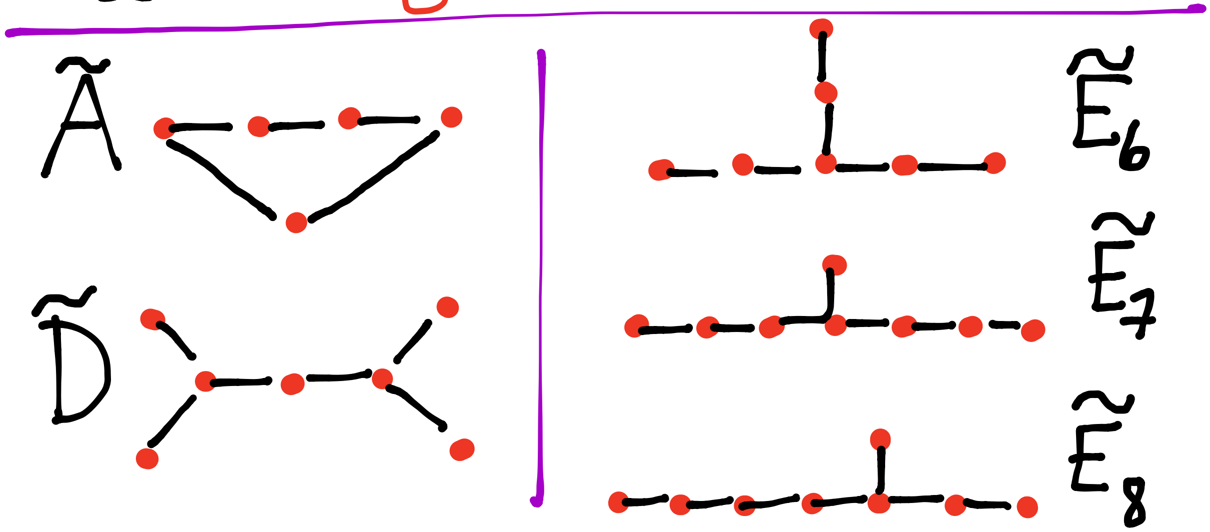
$$= \mathbb{Z}[x] \big/ \left(3(x-1),\ (x-1)^2\right)$$

$$\cong \mathbb{Z}[u] \big/ (3u, u^2)$$

$$= \mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})u$$

McKay's original setting

THEOREM (McKay 1980) If $G \overset{\gamma}{\hookrightarrow} SL_2(\mathbb{C})$ then $C, \tilde{C}$ are the Cartan and extended Cartan matrices for Φ a simply-laced finite root system, and the McKay digraph is the (bidirected) affine Dynkin diagram for Φ.



THEOREM

For $G \overset{\gamma}{\hookrightarrow} SL_2(\mathbb{C})$,

$$K(\gamma) \cong \underbrace{G^{ab}}_{\text{abelianization of } G} = G/[G,G]$$

$$\cong \text{weight lattice}(\Phi) \,/\, \text{root lattice}(\Phi)$$

$$\cong \pi_1\begin{pmatrix}\text{adjoint} \\ \text{compact Lie} \\ \text{group for } \Phi\end{pmatrix}$$

QUESTION:

Is there such a topological interpretation of $K(\gamma)$ more generally?

THANKS FOR

YOUR

ATTENTION !