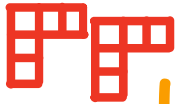
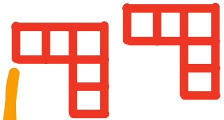


# Hooked



# on the work of Adin & Roichman

Vic Reiner

January 10, 2022

Reflections: On the occasion of Ron Adin's and Yuval Roichman's 60th birthdays

- Ron teaches me the beauty of colorful trees
- Ron & Yuval help me to appreciate Escher
- Ron & Yuval beat me (and Gerhard Röhrle, Götz Pfeiffer and Matt Douglass) to the punch!

- Ron teaches me the beauty of colorful trees

As a grad student, Ron spoke at an important conference (Stockholm 1989) on this:

COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 12 (3) (1992) 247–260

COUNTING COLORFUL MULTI-DIMENSIONAL TREES

RON M. ADIN

*Received 31 July, 1989*

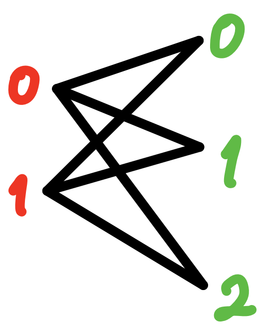
I was there as a grad student, at my first conference,

... and missed his talk!

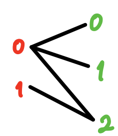
Q: (to Ron) What's a colorful multidimensional tree?

A (spanning) tree in a 1-dim'l simplicial complex $\Delta$ (graph)
is a subset $T$ of 1-simplices (edges) indexing a
basis for the column space of $C_1(\Delta,\mathbb{Q}) \xrightarrow{\partial_1} C_0(\Delta,\mathbb{Q})$

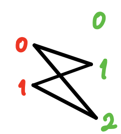
$\Delta = K_{2,3}$
complete bipartite graph




$$\partial_1 = \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{array} \begin{bmatrix} +1 & 0 & +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +1 & 0 & +1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix}$$

$T =$ 


a tree

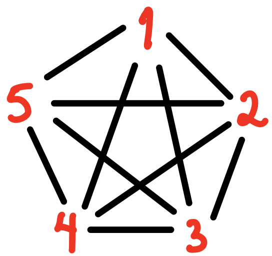



not a tree

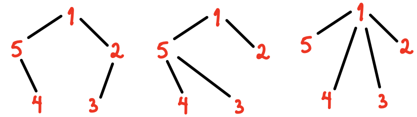
**THEOREM** (Borchardt 1860, Cayley 1889)

The complete graph $K_n$ has $n^{n-2}$ spanning trees.

e.g. $K_5$




has $5^3 = 125 = 60 + 60 + 5$ spanning trees

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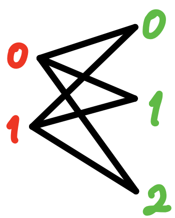
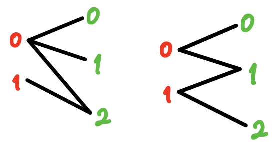
**THEOREM** (Fiedler & Sedláček 1958)

The complete bipartite graph $K_{n_1,n_2}$ has $n_1^{n_2-1} \cdot n_2^{n_1-1}$ spanning trees.

e.g. $K_{2,3}$




has $2^{3-1} \cdot 3^{2-1} = 12 = 9 + 3$ spanning trees



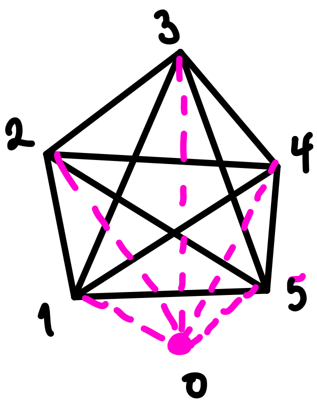
DEF'N: (Kalai 1983, Adin 1989)

A **tree** in a **$d$-dim'l** simplicial complex $\Delta$ is a subset $T$ of **$d$-simplices** indexing a **basis** for the **column space** of $C_d(\Delta, \mathbb{Q}) \xrightarrow{\partial_d} C_{d-1}(\Delta, \mathbb{Q})$

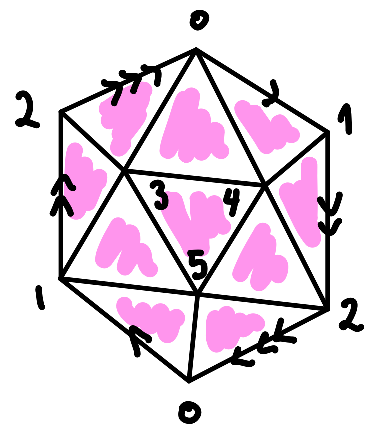
---

e.g. the 2-skeleton of a simplex on vertices $\{0,1,2,3,4,5\}$ has these two spanning trees, among others:

$T_1$ = cone over complete graph $K_5$




$T_2$ = 6-vertex triangulation of $\mathbb{RP}^2$



THEOREM (Kalai 1983) The complete $d$-complex, i.e., the $d$-skeleton $\Delta$ of the simplex on $n$ vertices, has $\leq n^{\binom{n-2}{d}}$ trees, since

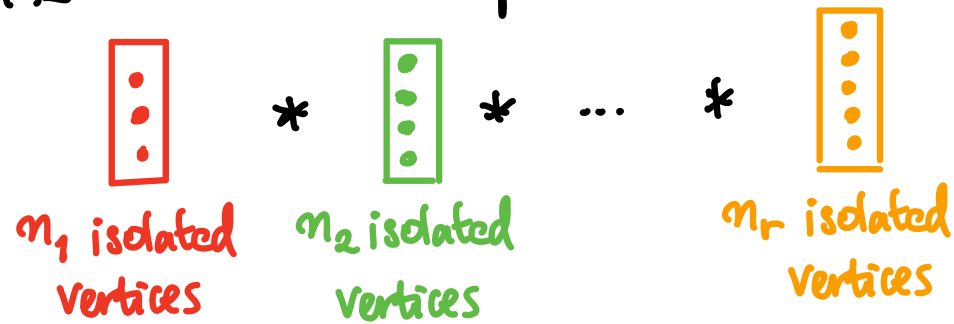
$$n^{\binom{n-2}{d}} = \sum_{\substack{\text{trees } T \\ \text{in } \Delta}} \left|\tilde{H}_{d-1}(T,\mathbb{Z})\right|^2$$

---

Gil's proof uses Binet-Cauchy Theorem to evaluate $\det\Big(\underbrace{\bar{\partial}_d\,\bar{\partial}_d^T}_{\substack{\text{(cleverly)}\\ \text{reduced}\\ \text{Laplacian}}}\Big)$ and compare it to an eigenvalue calculation.

**THEOREM** (Adin 1989)

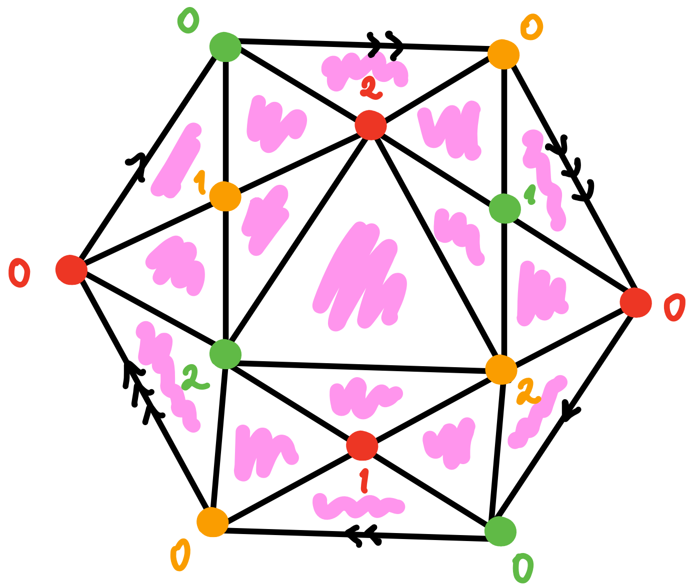
The complete $r$-colorful (or $r$-partite) complex $K_{n_1, n_2, \ldots, n_r}$ which is the $r$-fold simplicial join of 0-dim'l complexes




has $\leq n_1^{x_1} n_2^{x_2} \cdots n_r^{x_r}$ trees, where $x_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} (n_j - 1)$,

since $$n_1^{x_1} n_2^{x_2} \cdots n_r^{x_r} = \sum_{\substack{\text{trees } T \\ \text{in } \Delta}} \left| \tilde{H}_{d-1}(T, \mathbb{Z}) \right|^2$$

- Bolker 1976 working on transportation polytopes had guessed the $n_1^{x_1} n_2^{x_2} \cdots n_r^{x_r}$ count for trees was close, but wrong due to colorful trees with torsion, such as this colorful triangulation of $\mathbb{RP}^2$ inside $K_{3,3,3}$ :



- Ron's Theorem actually counts trees in any skeleton of the complete r-partite complex
- He developed a new use of the Binet-Cauchy formula to interpret pseudodeterminants of Laplacians $\partial_i \partial_i^T$ (avoiding Gil's clever choice of a reduced Laplacian)
- These methods inspired much, much later work by Bernardi, Duval, Klivans, Kook, Lee, Martin, Maxwell and others.
- It led Martin, Musiker and me to an unexpected connection with roots of unity and cyclotomic polynomials...

Consider $n = p_1 p_2 \cdots p_r$ squarefree, so $p_i$ are distinct primes.

Let $\zeta = e^{\frac{2\pi i}{n}}$ and $\mathbb{Q}(\zeta)$ the $n^{th}$ cyclotomic extension of $\mathbb{Q}$

---

**THEOREM** (Martin-R. 2005) For squarefree $n = p_1 p_2 \cdots p_r$,

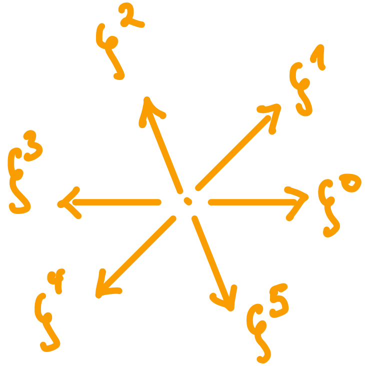
$$\left\{ \begin{array}{c} \mathbb{Q}\text{-bases} \\ B \subset \{1, \zeta, \zeta^2, \ldots, \zeta^{n-1}\} \\ \text{for } \mathbb{Q}(\zeta) \end{array} \right\} \text{ biject with } \left\{ \begin{array}{c} \left(\begin{array}{c}\text{colorful,} \\ \text{multidimensional}\end{array}\right) \\ \text{trees } T \text{ in } K_{p_1, p_2, \ldots, p_r} \end{array} \right\}$$

$$B \longmapsto \begin{array}{l} \text{tree } T \text{ whose facets} \\ \text{have simplices} \end{array}$$

$$\left\{ \left( \underset{\bmod p_1}{j}, \underset{\bmod p_2}{j}, \ldots, \underset{\bmod p_r}{j} \right) \right\}_{\substack{j \in \mathbb{Z}/n\mathbb{Z}, \\ \zeta^j \notin B}}$$

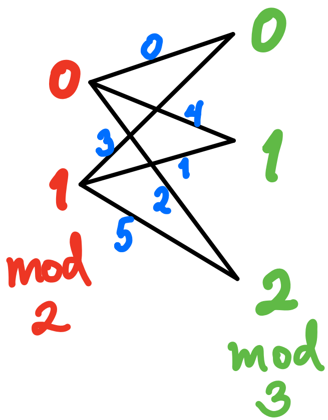
$n = 6 = 2 \cdot 3$

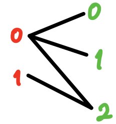
$\mathbb{Q}(\zeta)$




$$\partial_1 = \begin{array}{c} \\ 0 \\ 1 \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccccc} \zeta^0 & \zeta^1 & \zeta^2 & \zeta^3 & \zeta^4 & \zeta^5 \end{array} \\ \begin{bmatrix} +1 & 0 & +1 & 0 & +1 & 0 \\ 0 & +1 & 0 & +1 & 0 & +1 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

$K_{2,3}$
complete bipartite graph



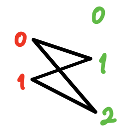

$T =$ 

a tree

$\updownarrow$

$B = \{\zeta^3, \zeta^4\}$

a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta)$



not a tree

$\updownarrow$

$\{\zeta^0, \zeta^3\}$
$= \{+1, -1\}$
not a $\mathbb{Q}$-basis for $\mathbb{Q}(\zeta)$

**THEOREM** (Musiker - R. 2014)

For $n = p_1 p_2 \cdots p_r$ squarefree, the cyclotomic polynomial

$$\Phi_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{\phi(n)} x^{\phi(n)}$$

has $c_j \neq 0$ exactly when the facets of $K_{p_1, p_2, \ldots, p_r}$ indexed by

$$\{\phi(n)+1, \phi(n)+2, \ldots, n-2, n-1\} \cup \{j\}$$

form a tree $T$, in which case $\tilde{H}_{r-2}(T, \mathbb{Z}) \cong \mathbb{Z}/|c_j|\mathbb{Z}$.

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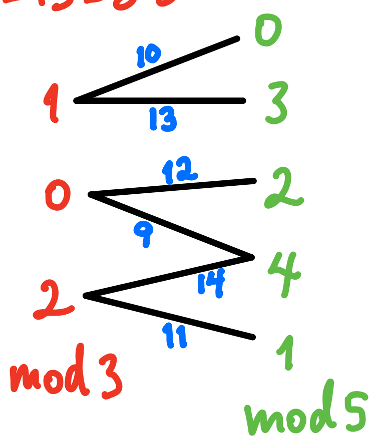
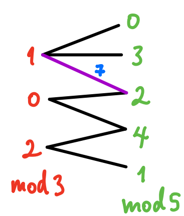
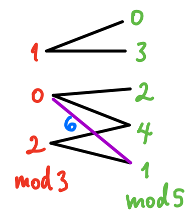
$n = 15 = 3 \cdot 5$




$$\Phi_{15}(x) = 1 - x + x^3 - x^4 + x^5 - x^7 + x^8$$




a tree since $c_7 \neq 0$




not a tree, since $c_6 = 0$

$n = 105 = 3 \cdot 5 \cdot 7$

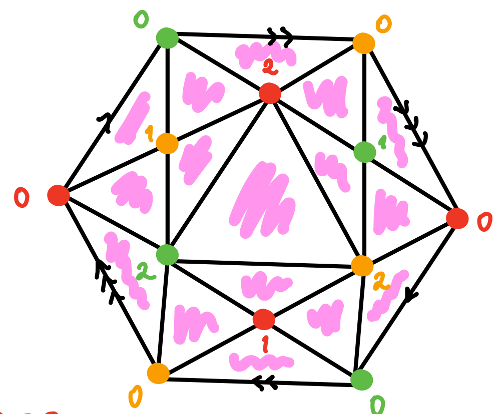
$$\begin{aligned}\Phi_{105}(x) = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34} \\ + x^{33} + x^{32} + x^{31} - x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} \\ + x^{14} + x^{13} + x^{12} - x^{9} - x^{8} - 2x^{7} - x^{6} - x^{5} + x^{2} + x + 1\end{aligned}$$

is the smallest $n$ for which coefficients $c_j \neq \pm 1$ occur,

for the same reason trees with homology torsion

can't appear inside $K_{n_1, n_2, \ldots, n_r}$

until $r \geq 3$ and $n_1, n_2, \ldots, n_r \geq 3$



$\mathbb{R}P^2 \subset K_{3,3,3}$

- Ron & Yuval help me to appreciate Escher

I met Yuval later, but have seen him more often, as he visits friends in Minneapolis. I met him first when he spoke in our seminar on this great paper

ADVANCES IN MATHEMATICS 129, 25–45 (1997)
ARTICLE NO. AI961629

# A Recursive Rule for Kazhdan–Lusztig Characters

Yuval Roichman*

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Received November 1, 1996; accepted December 30, 1996

in which descent sets of permutations and tableaux play an important role

**DEF'N:** For a permutation $w=(w_1,\ldots,w_n)$ in $S_n$ its descent set $\mathrm{Des}(w) = \{i : w_i > w_{i+1}\}$

e.g. $\mathrm{Des}\big((\overset{1}{3} > \overset{2}{1}, \overset{3}{5} > \overset{4}{2}, \overset{5}{4})\big) = \{1, 3\}$

---

For a Young tableau $Q$ with entries $\{1,2,\ldots,n\}$ its descent set $\mathrm{Des}(Q) = \{i : i+1 \text{ appears in lower row than } i\}$

e.g. $\mathrm{Des}\left(\begin{matrix} 1 & 2 & 4 & 7 & 8 \\ 3 & 5 & 9 & & \\ 6 & & & & \end{matrix}\right) = \{2, 4, 5, 8\}$

# Descent sets ...

- generalize to reflection/Coxeter groups $W$
- play a key role in Kazhdan-Lusztig theory
- have amazing connections to theory of symmetric and quasisymmetric functions through work of Stanley and Gessel

By definition, $\mathrm{Des}(w), \mathrm{Des}(Q) \subseteq \{1, 2, \ldots, n-1\}$

but an extension to cyclic descent sets for $w \in S_n$

was defined by Klyachko 1974

and Cellini 1998

$$\mathrm{cDes}(w) = \{ i : w_i > w_{i+1} \text{ with indices taken modulo } n \}$$

$$\text{e.g. } \mathrm{Des}\big((\overset{1}{3} > \overset{2}{1}, \overset{3}{5} > \overset{4}{2}, \overset{5}{4})\big) = \{1, 3, 5\}$$

Later Rhoades 2010 defined such a cyclic extension

$\mathrm{cDes}(Q) \subset \{1,2,\ldots,n\}$ of $\mathrm{Des}(Q)$

only for rectangular tableaux

$$Q: \quad \begin{tikzcd} \begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix} \arrow[rr, bend left, "\text{promotion}"] & \begin{smallmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{smallmatrix} \arrow[l] & \begin{smallmatrix} 1 & 3 & 4 \\ 2 & 5 & 6 \end{smallmatrix} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 & 6 \end{smallmatrix} \arrow[r, shift left] & \begin{smallmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{smallmatrix} \arrow[l, shift left] \end{tikzcd}$$

$$\mathrm{cDes}(Q): \quad \begin{tikzcd} \{3,6\} \arrow[rr, bend right, "+1 \bmod 6"'] & \{2,5\} \arrow[l] & \{1,4\} \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \{2,4,6\} \arrow[r, shift left] & \{1,3,5\} \arrow[l, shift left, "+1 \bmod 6"] \end{tikzcd}$$

using equivariance with respect to adding 1 mod $n$

and Schützenberger's promotion.
1972

**DEF'N:** Given a descent map $\mathcal{A} \xrightarrow{\text{Des}} 2^{\{1,2,\ldots,n-1\}}$

Say a map $\mathcal{A} \xrightarrow{\text{cDes}} 2^{\{1,2,\ldots,n-1,n\}}$

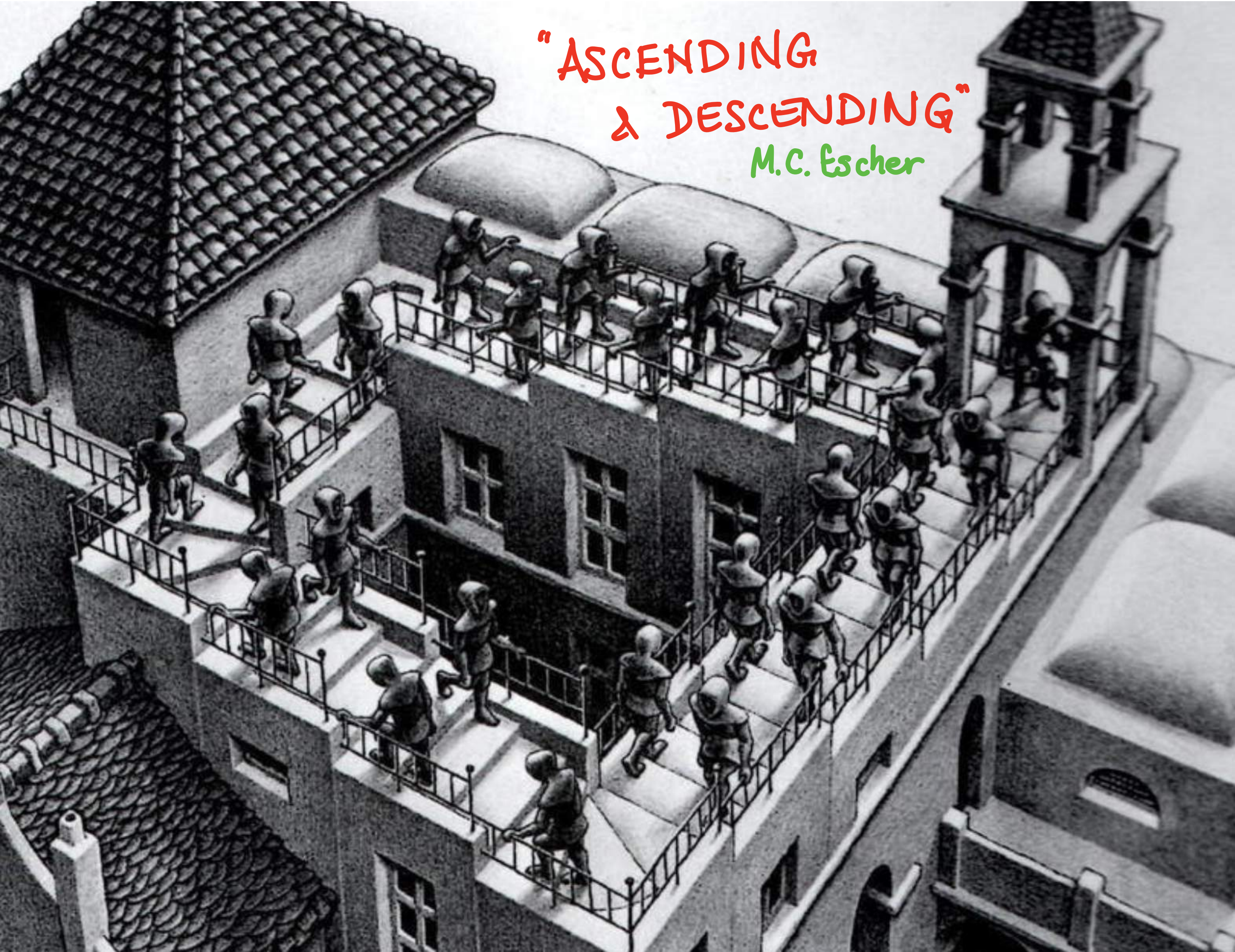
and a bijection $\mathcal{A} \xrightarrow{p} \mathcal{A}$

give a cyclic descent extension of Des if

- $\text{cDes}(a) \cap \{1,2,\ldots,n-1\} = \text{Des}(a)$ (extension)
- $\text{cDes}(p(a)) = \text{cDes}(a) + 1 \mod n$ (equivariance)
- $\emptyset \neq \text{cDes}(a) \neq \{1,2,\ldots,n\}$ (non-Escher)

---

The non-Escher axiom makes the distribution of cDes unique.

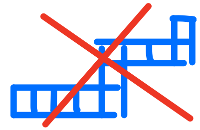

"ASCENDING & DESCENDING"
M.C. Escher

From "Inception"
C. Nolan

Q: (Ron & Yuval)

When does $\mathcal{A} \xrightarrow{\text{Des}} 2^{\{1,2,\dots,n-1\}}$ have a cyclic extension?

In particular, for which skew shapes $\lambda/\mu$ does it exist for $\mathcal{A} = \{\text{standard tableaux of shape } \lambda/\mu\}$?

(A: All but the ribbons  )

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They developed better and better techniques for this:

Elizalde-Roichman 2015 2016

Adin-R.-Roichman 2017

Adin-Elizalde-Roichman 2018

Adin-Gessel-R.-Roichman 2018

Bloom-Elizalde-Roichman 2019

Adin-Hegedüs-Roichman 2019

## TECHNIQUE 1:

Gessel's fundamental quasisymmetric function

$$F_{\mathrm{Des}(a)} = \sum_{\substack{k_1 \le k_2 \le \cdots \le k_n \\ k_i < k_{i+1} \text{ if } i \in \mathrm{Des}(a)}} x_{k_1} x_{k_2} \cdots x_{k_n}$$

let's one test if $\mathcal{A} \xrightarrow{\mathrm{Des}} 2^{\{1,2,\ldots,n-1\}}$ has a cyclic extension assuming $Q_{\mathcal{A}} := \sum_{a \in \mathcal{A}} F_{\mathrm{Des}(a)}$ is a symmetric function:

---

cDes exists $\iff \left\langle Q_{\mathcal{A}}, \tilde{s}_{\alpha^{cyc}(J,n)} \right\rangle \ge 0$ for $\emptyset \subsetneq J \subsetneq \{1,2,\ldots,n\}$

cyclic ribbon function of Ron & Yuval,

closely related to a toric Schur function of Postnikov 2005.

## TECHNIQUE 2: (Adin-Hegedüs-Roichman 2019)

Assuming $Q_A$ is not only symmetric, but also Schur-positive, then

cyclic extension cDes for $A \xrightarrow{\text{Des}} 2^{\{1,2,\ldots,n-1\}}$ exists

$$\Longleftrightarrow$$

the generating function

hook Schur function

$$M_A(x) = \sum_{k=0}^{n} \left\langle Q_A, s_{(n-k,1^k)} \right\rangle x^k$$

lies in $(1+x)\cdot \mathbb{N}[x]$

---

They applied this to prove ...

THEOREM (Adin-Hegedüs-Roichman 2019)

The usual descent map on

$$\mathcal{C}_\lambda = \{\text{permutations of cycle type } \lambda\} \xrightarrow{\mathrm{Des}} 2^{\{1,2,\ldots,n-1\}}$$

has a cyclic extension cDes

$\iff$ $\lambda \neq (r,r,\ldots,r)$ for some square free $r$

---

Here the symmetric function $Q(\mathcal{C}_\lambda)$ is known,
via a result of Gessel - Reutenauer 1993,
to be the higher Lie character $\mathrm{Lie}_\lambda$ of Thrall 1942

---

And this is how ...

- Ron & Yuval beat me (and Gerhard, Götz and Matt) to the punch!

---

Douglass (Matt), Pfeiffer (Götz) and Röhrle (Gerhard) have extensively studied the $S_n$-rep'ns on

$H^i(\mathrm{Conf}(n,\mathbb{R}^d))$, which vanishes unless $i = r(d-1)$

$\mathrm{Conf}(n,\mathbb{R}^d)$: configuration space of $n$ distinct points $(x_1,\ldots,x_n)$ in $\mathbb{R}^d$

and then gives $S_n$-rep'ns
$$\begin{cases} \bigoplus_{\lambda:\ \mathrm{rank}(\lambda)=r} \mathrm{Lie}_\lambda, & \text{if } d \text{ odd} \\ \bigoplus_{\lambda:\ \mathrm{rank}(\lambda)=r} \underbrace{OS_\lambda}_{\substack{\text{Orlik-Solomon rep.} \\ \text{or Whitney homology} \\ \text{of type } \lambda}}, & \text{if } d \text{ even} \end{cases}$$

where $\mathrm{rank}(\lambda) = \sum_i (\lambda_i - 1)$.

---

Also studied extensively by Sundaram and others.

Douglass, Pfeiffer and Röhrle more generally study the $W$-rep'ns on

$$H^i(\underbrace{M_W}_{\text{real hyperplane arrangement complement for reflection group } W} \otimes_{\mathbb{R}} \mathbb{R}^d),$$

which vanishes unless $i = r(d-1)$

and then gives $W$-rep'ns

$$\begin{cases} \displaystyle\bigoplus_{\substack{W\text{-orbits} \\ [X] \text{ of} \\ \text{codimension } r \\ \text{flats}}} L_{[X]}^W, & \text{if } d \text{ odd} \\ \displaystyle\bigoplus_{\substack{W\text{-orbits} \\ [X] \text{ of} \\ \text{codimension } r \\ \text{flats}}} OS_{[X]}^W, & \text{if } d \text{ even} \end{cases}$$

Orlik-Solomon rep. or Whitney homology corresponding to $[X]$

– see talk by Sarah Brauner tomorrow

For example, in 2019 D-P-R proved a conjecture of Felder-Veselov 2005 on where to find the copies of the trivial W-rep'n in each rep'n $OS^W_{[x]}$.

---

Q: (asked Röhrle in May 2019)

For each of the exterior powers $\Lambda^k V$ of the reflection rep'n $V$ of $W$,

how many times does $\Lambda^k V$ occur in $L^W_{[x]}$, $OS^W_{[x]}$?

special cases: $\Lambda^0 V$ = trivial W-rep'n

$\Lambda^n V$ = det or sign W-rep'n

All four of us looked at the data, noticed patterns, for example ...

---

THEOREM (D-P-R-R, unpublished)

The total number of occurrences of all $\Lambda^k V$ is always the same for $L^W_{[x]}$ as for $OS^W_{[x]}$,

that is $\langle \Lambda V, L^W_{[x]} \rangle = \langle \Lambda V, OS^W_{[x]} \rangle$.

I later heard Yuval speak in Spring 2020 at (Mittag-Leffler ACOW) on Adin-Hegedüs-Roichman 2019 and realized they had already answered some of our questions for $W = \mathfrak{S}_n$, since there

$$\langle L^W_{[X]}, \wedge^k V \rangle = \langle Q_{C_\lambda}, S^{(n-k+1,\,1^k)} \rangle$$

exactly what A-H-R needed to analyze!

They had also noticed other patterns, e.g.

---

CONJECTURE (Adin-Hegedüs-Roichman 2019)

For $k=0,1,\ldots,n$, $\left\langle Q_{C_\lambda}, S^{\overbrace{\phantom{xxx}}^{n-k}}_{k\{} \right\rangle$

is unimodal as a sequence.

---

Looks true so far, and particularly interesting since the analogue for $W=D_n$ seems to fail at $n=7$, i.e. $\langle L^{D_n}_{[X]}, \Lambda^k V \rangle$ is not unimodal in $k$ for a certain $W$-orbit $[X]$.

Thank you both
Ron and Yuval
for teaching me
so much, and
happy 60th birthdays!