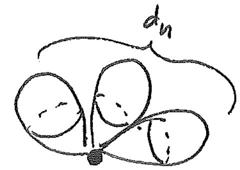
(1)

Topology & combinatorics of the complex of injective words

1. Review K_n and examples $n=1,2,3$



2. Boolean cell complexes

3. Shellability of K_n $\left(\Rightarrow K_n \approx \underbrace{S^{n-1} \vee \dots \vee S^{n-1}}_{d_n \text{ times}} \right.$
(Björner-Wachs) 1983

$\Rightarrow$ (Farmer 1978) FACT: (from Kerz's proof of Nakaoka's Thm) $\left. \tilde{H}_i(K_n) \neq 0 \text{ only for } i=n-1 \right)$

↑ TODAY?

↓ NEXT WEEK?

4. THM: (Webb-R.) 2004 $d_n = |\tilde{X}(K_n)| = \dim \tilde{H}_{n-1}(K_n)$

$= \#$ derangements in $\mathfrak{S}_n$ — no fixed points

$= \#$ desarrangements in S_n — first descent position even

5. THM: (Webb-R.) $\tilde{H}_{n-1}(K_n) \cong \bigoplus_{\substack{\text{desarrangement tableaux } Q}} \chi^{\text{shape}\,Q}$ — first descent value even

6. Generalizations

7. Other appearances of K_n

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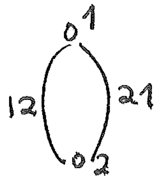
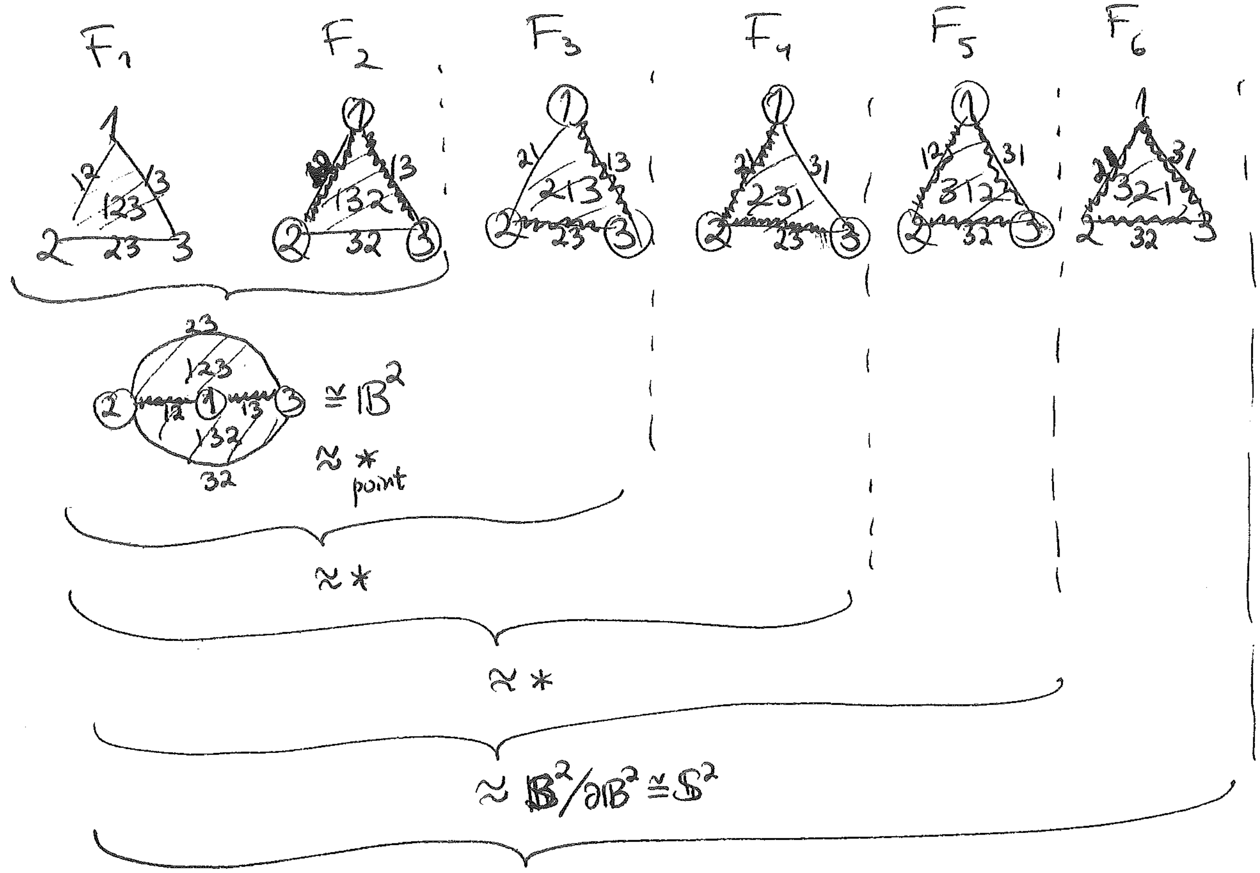
1. $K_n :=$ the (Boolean cell) complex of injective words on alphabet $[n] := \{1,2,\dots,n\}$

has (simplicial) $(k-1)$-dim'l cells indexed by words $a_1 a_2 \cdots a_k \in [n]^k$ with all a_i distinct and inclusion of cells $\longleftrightarrow$ subwords $a_{i_1} a_{i_2} \cdots a_{i_\ell}$ inside $a_1 a_2 \cdots a_k$

e.g. $K_1 = \overset{1}{\circ}$

$K_2 =$ 

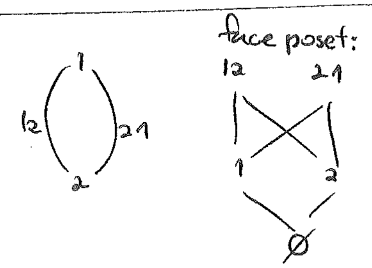
Let's assemble K_3 one facet (= maximal face) at a time in what will turn out to be a shelling order, the lexicographic order on S_n



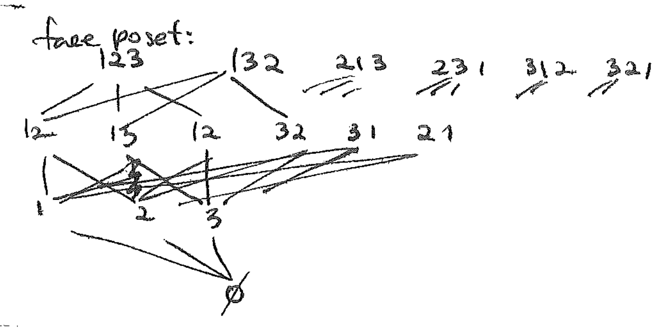
$K_3 \approx \mathbb{S}^2 \vee \mathbb{S}^2$

2. A Boolean cell complex (Björner, 1984) is a regular CW-complex whose poset of faces has every lower interval $[\emptyset, F]$ isomorphic to a Boolean algebra $2^{\dim F+1}$

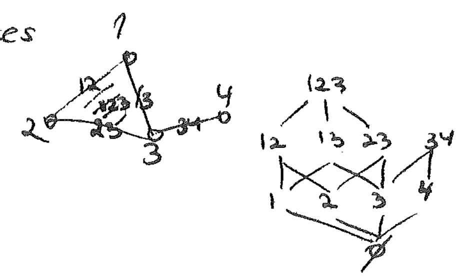
e.g. K_2



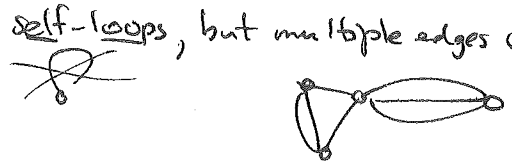
K_3



e.g. simplicial complexes are Boolean cell complexes

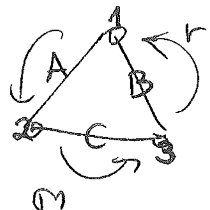


e.g. graphs with no self-loops, but multiple edges OK.



Boolean cell complexes have some nice features

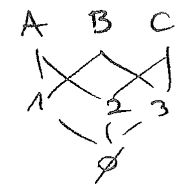
- They arise naturally from quotients of simplicial complexes K under finite group actions $G \circlearrowright K$, after taking a barycentric subdivision $K \mapsto Sd(K)$

e.g. $K =$ 

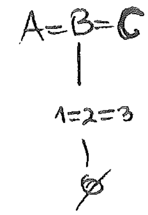
$G = \mathbb{Z}/3\mathbb{Z} = \langle r \rangle$

$\cong S^1$

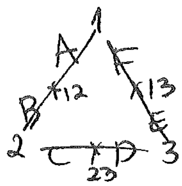
face poset P

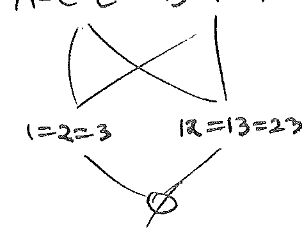


$\rightsquigarrow$ P/G



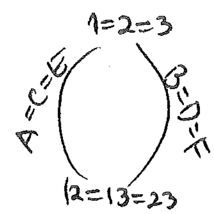
$\neq$ face poset of $K/G \cong S^1$

$Sd(K) =$ 

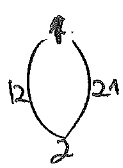
$P(Sd K)/G =$ 

= face poset of $Sd(K)/G$, a Boolean cell complex

$Sd(K)/G$



- They have a nice (generalization of the) Stanley-Reisner ring (1991)

e.g. $K_2 =$ 

has ring $k[x_1, x_2, x_{12}, x_{21}] / (x_{12}x_{21},\ x_1x_2 - (x_{12}+x_{21}))$

3. Shellability

DEF'N: A pure d-dim'l Boolean cell complex K is shellable if one can linearly order its facets $F_1, F_2, \ldots$ so that $\forall j \geq 2$

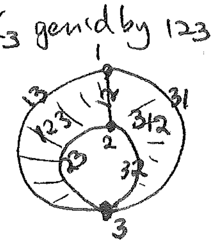
$F_j \cap (F_1 \cup F_2 \cup \ldots \cup F_{j-1})$ is a pure $(d-1)$-dim'l subcomplex of ∂F_j

(pure: all facets (= maximal faces) have dimension d)

EXAMPLE: K_3 above

NON-EXAMPLES

The subcomplex of K_3 gen'd by 123 and 312:



PROPOSITION: K a pure d-dim'l shellable Boolean cell complex

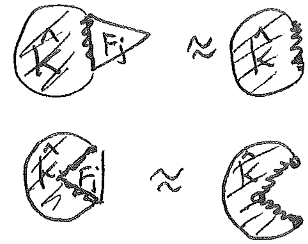
$$\Rightarrow K \underset{\text{homotopy equiv.}}{\approx} \underbrace{\mathbb{S}^d \vee \cdots \vee \mathbb{S}^d}_{\#\{\text{facets } F_j \text{ with } F_j \cap (\bigcup_{i=1}^{j-1} F_i) = \partial F_j\}}$$

(sketch) proof: Induct on # facets $F_1, F_2, \ldots, F_{j-1}$ added so far,

so $\hat{K} := F_1 \cup \cdots \cup F_{j-1}$ is shellable, $\approx \bigvee \mathbb{S}^d$ by induction

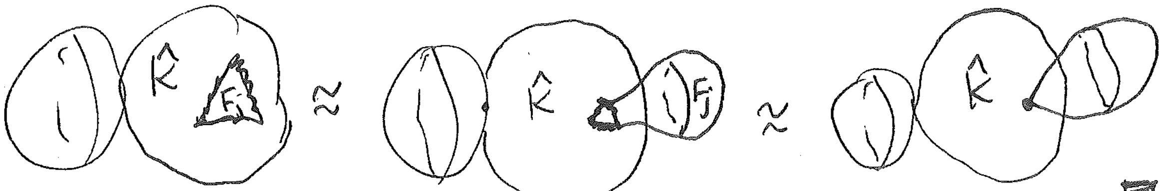
and $K = \hat{K} \cup F_j$ either has

- $\partial F_j \supsetneq \underbrace{\hat{K} \cap F_j}_{\text{pure }(d-1)\text{-dim'l}}$ so that $K \approx \hat{K}$, even by elementary collapses

e.g. 

OR

- $\partial F_j = \underbrace{\hat{K} \cap F_j}_{= \mathbb{S}^{d-1}}$ which can be contracted to a point inside $\hat{K}$, since $\pi_{d-1}(\hat{K}) = \pi_{d-1}(\bigvee \mathbb{S}^d) = 1$

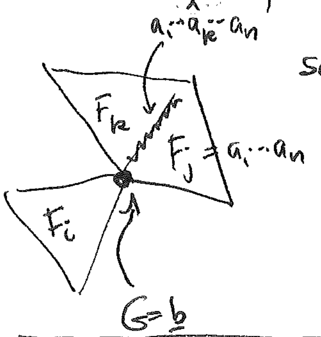


↑ 10/26/2018

↓ 10/29/2018

THEOREM (Björner-Wachs 1983): K_n is shellable, by lex order on its permutation facets.

(sketch) proof: When $F_j = a_1 \cdots a_n$ is added, for any face $G = \underline{b}$ of $F_j \cap (\bigcup_{i=1}^{j-1} F_i)$ say $G = F_i \cap F_j$ with $F_i <_{lex} F_j$, need to exhibit a letter $a_k \in a_1 \cdots a_n - \underline{b}$ that can be removed and re-inserted to create $F_k <_{lex} F_j$.



Find the leftmost position of F_j that differs from the lex-earliest permutation $\mathcal{E}(\underline{b})$ containing $\underline{b}$; this helps find a_k .... in location

e.g. $n=9$ $F_j = 1\underline{4}3\underline{2}96\underline{8}7\underline{5}$, $\underline{b} = 4285$ $\Rightarrow F_j - \underline{b} = 13679$, $\mathcal{E}(\underline{b}) = 13\underline{4}\underline{2}67\underline{8}\underline{5}9$ $\Rightarrow$ can take $a_k = 3$, $F_k = 13\underline{4}\underline{2}96\underline{8}7\underline{5}$

more examples: $F_j = 13\underline{4}\underline{2}6\underline{8}7\underline{5}9$

$F_j = 13\underline{4}6\underline{2}7\underline{8}\underline{5}9$

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4. The (reduced) Euler characteristic $\left|\tilde{\chi}(K_n)\right| =: d_n$ $\quad$ (so $K_n \simeq \underbrace{\mathbb{S}^{n-1} \vee \cdots \vee \mathbb{S}^{n-1}}_{d_n \text{ terms}}$)

$$\dim \tilde{H}_{n-1}(K_n) \stackrel{\text{Euler-Poincaré}}{=} \left| \sum_{i=-1}^{n-1} \dim \tilde{C}_i(K_n) \cdot (-1)^i \right| = ?$$

EXAMPLE: $n=3$

$$\tilde{C}_2(K_3) \xrightarrow{\partial_2} \tilde{C}_1(K_3) \xrightarrow{\partial_1} \tilde{C}_0(K_3) \xrightarrow{\partial_0} \tilde{C}_{-1}(K_3) \longrightarrow 0$$

123	12	1	$\varnothing$
132	21	2	
213	13	3	
231	31		
312	23		
321	32		

$$1\cdot 3! \;-\; 3\cdot 2! \;+\; 3\cdot 1! \;-\; 1\cdot 0! \;=\; 2$$

$$1\cdot 3! = \binom{3}{3}\cdot 3!, \quad 3\cdot 2! = \binom{3}{2}\cdot 2!, \quad 3\cdot 1! = \binom{3}{1}\cdot 1!, \quad \binom{3}{0}\cdot 0!$$

PROP: (Farmer 1978)

$$d_n = n! - n(n-1)\cdots 3\cdot 2 + n(n-1)\cdots 3 - \cdots \pm n(n-1) \mp n \pm 1$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

$= \#\{$derangements $w \in S_n\}$

= permutations with no fixed points

proof:

$$\#\{w \in S_n : \mathrm{Fix}(w) = \varnothing\} \stackrel{\boxed{\text{inclusion-exclusion}}}{=} \sum_{S \subset \{1,2,\ldots,n\}} (-1)^{|S|} \underbrace{\#\{w \in S_n : \mathrm{Fix}(w) \supseteq S\}}_{(n-|S|)!}$$

$$\stackrel{\text{group by } k=|S|}{=} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

PROP: (Désarménien 1984)

$d_n = n\,d_{n-1} + (-1)^n \Rightarrow d_n = \#\{$desarrangements $w \in S_n\}$

= permutations where first i having $i+1$ to its left (an inverse descent) is even

e.g. $n=3$

12③
13②
2①3
23①
31②
3②①

$d_3 = 2$ desarrangements

CONVENTION: $w = 12\cdots n$ has n as an inverse descent

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proof: $d_n = n\big((n-1)! - (n-1)\cdots 3\cdot 2 + (n-1)\cdots 3 - \cdots \pm (n-1) \mp 1\big) \pm 1$ $\leftarrow (-1)^n$

$= n\, d_{n-1} \pm (-1)^n$

Given a derangement $w \in S_{n-1}$, $w = w_1 w_2 \cdots w_{n-1}$, adding n to it in any position will keep it a derangement; same for removing n

EXCEPTION: when $w = 1\,2 \cdots (n-1)$, and n is added at the end / removed

$\updownarrow$

$\hat{w} = 1\,2\cdots(n-1)\,n$

This accounts for the $(-1)^n$ ∎

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e.g.

$n=2$: 1 2, ~~2 1~~, $\boxed{d_2 = 1}$

$n=3$: ~~1 2 3~~, 1 3 2, 3 1 2, $\boxed{d_3 = 2}$

$n=4$: 1 3 2 4, 1 3 4 2, 1 4 3 2, 4 1 3 2, 3 1 2 4, 3 1 4 2, 3 4 1 2, 4 3 1 2, 1 2 3 4, $\boxed{d_4 = 9}$

Now dress this up with S_n-representations.

$\tilde{C}_j(K_n)$ = transitive action of S_n on the orbit of word $a_1 a_2 \cdots a_{j+1} = 1\,2\cdots(j+1)$

with stabilizer $\underbrace{S_1 \times \cdots \times S_1}_{j+1 \text{ times}} \times S_{n-(j+1)}$

$= 1\!\!1 \uparrow_{(S_1)^{j+1} \times S_{n-(j+1)}}^{S_n}$

$= \underbrace{\chi^{\square} * \cdots * \chi^{\square}}_{j+1 \text{ times}} * \chi^{\overbrace{\square\square\square\square}^{n-(j+1)}}$ where $Rep(S_a) \times Rep(S_b) \xrightarrow{*} Rep(S_{a+b})$

$(\chi_1, \chi_2) \longmapsto \chi_1 * \chi_2 := \chi_1 \otimes \chi_2 \uparrow_{S_a \times S_b}^{S_{a+b}}$

COR: $\tilde{H}_{n-1}(K_n) = \sum_{k=0}^{n} (-1)^k \underbrace{\chi^{\square} * \cdots * \chi^{\square}}_{k \text{ times}} * \chi^{\overbrace{\square\square\square\square}^{n-k}} = \chi^{\square} * \tilde{H}_{n-2}(K_{n-1}) + (-1)^n \chi^{\overbrace{\square\square\square\square}^{n}}$

as a virtual S_n-rep'n

(7) proof by EXAMPLE: $n=3$

$$\tilde{C}_2(K_3) \longrightarrow \tilde{C}_2(K_3) \longrightarrow \tilde{C}_1(K_3) \longrightarrow \tilde{C}_0(K_3)$$

$$\tilde{H}_2(K_3) = \underbrace{\chi^{\square} * \chi^{\square} * \chi^{\square}}_{3} - \underbrace{\chi^{\square} * \chi^{\square}}_{2} * \chi^{\overset{1}{\square}} + \underbrace{\chi^{\square}}_{1} * \chi^{\overset{2}{\square\square}} - \chi^{\overset{3}{\square\square\square}}$$

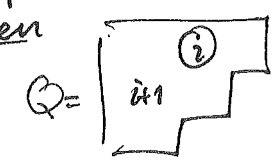
$$= \chi^{\square} * \Big(\underbrace{\chi^{\square} * \chi^{\square} - \chi^{\square} * \chi^{\overset{1}{\square}} + \chi^{\overset{2}{\square\square}}}_{\shortparallel}\Big) + (-1)^3 \chi^{\overset{3}{\square\square\square}}$$

$$= \chi^{\square} * \tilde{H}_1(K_2) + (-1)^3 \chi^{\overset{3}{\square\square\square}}$$

COR: (Webb-R. 2004)

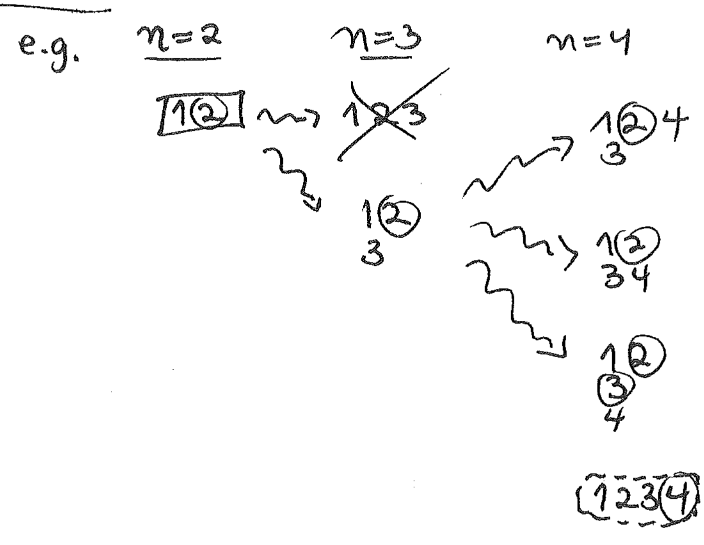
$$\tilde{H}_{n-1}(K_n) \cong \bigoplus_{\substack{\text{desarrangement} \\ \text{tableaux } Q}} \chi^{\text{shape}(Q)}$$

the first i for which $i+1$ appears in a lower row is even



CONVENTION: $Q = \boxed{1\ 2\ 3 \cdots n}$ has $n+1$ in a lower row than n

e.g.



proof of COR: Check that $\Psi_n :=$ RHS of COROLLARY also satisfies $\Psi_n = \chi^{\square} * \Psi_{n-1} + (-1)^n \chi^{\overset{n}{\square\square\cdots\square}}$ using Pieri formula: $\chi^{\square} * \chi^{\mu} = \sum'_{\lambda = \mu\square} \chi^{\lambda}$

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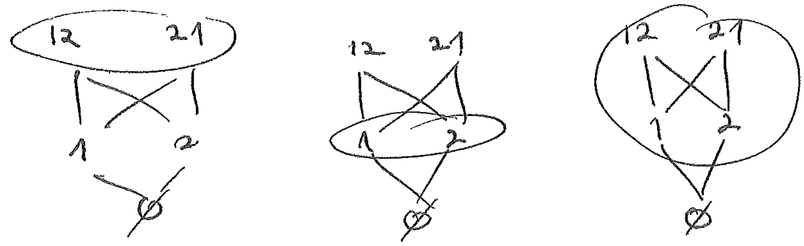
6. Generalizations

- Hanlon & Hersh (2004) decomposed the S_n-rep'n $\tilde{H}_{n-1}(K_n)$ into a Hodge decomposition according to Eulerian idempotents $\{e_k^n\}_{k=1,\dots,n}$ in $\mathbb{Q}\mathfrak{S}_n$, refining derangements according to # of cycles.

- Athanasiadis (2016) computed the S_n-irreducible decomp. for homology of all rank-selected subposets of face poset of K_n



- Ragnarsson & Tenner (2011) generalized K_n to the complex of partially commutative injective words parametrized by any finite graph on $\{1,2,\dots,n\}$, proved it is a wedge of S^{n-1}'s, and the number is counted by a subset of derangements.