Hilbert functions for Chow rings of uniform matroids and their q-analogues

Did the work!

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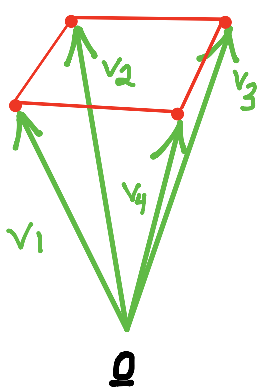
OUTLINE

1. Uniform matroids & their q-analogues
2. Chow rings of (simple) matroids
3. Hilbert series results by
   - Feichtner-Yuzvinsky (2004)
   - Adiprasito-Huh-Katz (2015)
4. MAIN RESULT
5. Proof ideas

Recall a (simple) matroid M can be specified by its lattice of flats $\mathcal{L}(M)$, a geometric lattice.
(= atomic + upper semimodular)

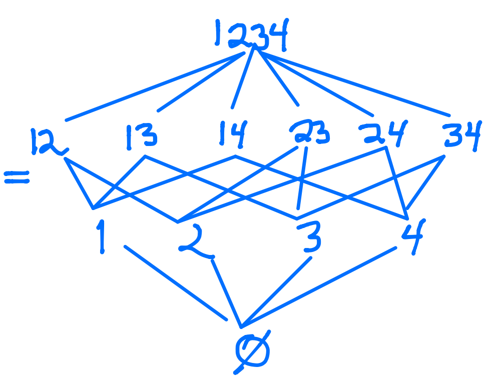
Motivating special case:

$$\mathcal{L}(M) = \left\{ \begin{array}{l} \text{linearly closed subsets} \\ \text{of a collection of} \\ \text{vectors } \{v_1, v_2, \ldots, v_n\} \subset \mathbb{F}^r \end{array} \right\}$$



$\{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^3$

$\mathcal{L}(M) =$

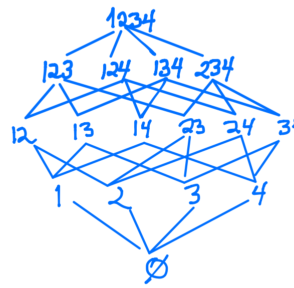


The uniform matroid $U_{n,r}$ of rank r on n elements corresponds to n generic vectors $\{v_1, v_2, \dots, v_n\} \subset \mathbb{F}^r$

$$\Rightarrow \mathcal{L}(U_{n,r}) = \left\{ \begin{array}{l} \text{subsets of } \{1,2,\dots,n\} \\ \text{of cardinality } 0,1,2,\dots,r-1 \text{ and } n \end{array} \right\}$$

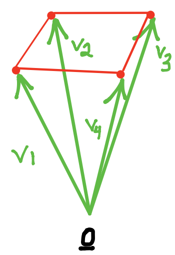
EXAMPLES with $n=4$

$r=4$: $\{v_1,v_2,v_3,v_4\}$ any basis for $\mathbb{F}^4$ ⇝ $\mathcal{L}(U_{4,4}) =$

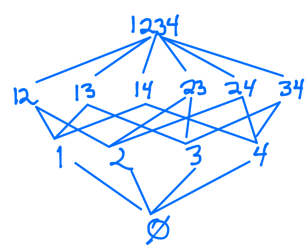


$=$ Boolean algebra $2^{\{1,2,3,4\}}$

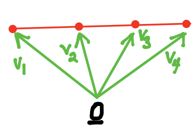
$r=3$:



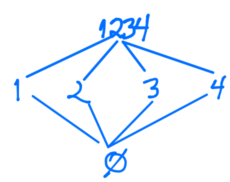
⇝ $\mathcal{L}(U_{4,3}) =$



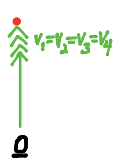
$r=2$:



⇝ $\mathcal{L}(U_{4,2}) =$



$r=1$:



⇝ $\mathcal{L}(U_{4,1}) =$

q-analogues

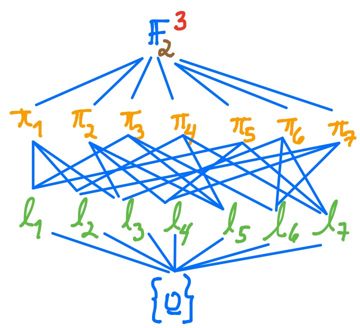
$$\mathcal{L}(U_{n,r}(q)) = \left\{ \begin{array}{l} \mathbb{F}_q\text{-linear subspaces of } \mathbb{F}_q^n \\ \text{of dimension} \\ 0,1,2,\dots,r-1 \text{ and } n \end{array} \right\}$$

"lim" $\underset{q\to 1}{\rightsquigarrow}$

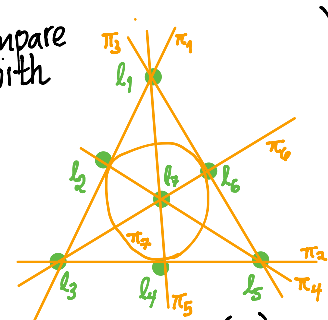
$$\mathcal{L}(U_{n,r}) = \left\{ \begin{array}{l} \text{subsets of } \{1,2,\dots,n\} \\ \text{of cardinality} \\ 0,1,2,\dots,r-1 \text{ and } n \end{array} \right\}$$

EXAMPLES

$\mathcal{L}(U_{3,3}(2)) =$ q n r

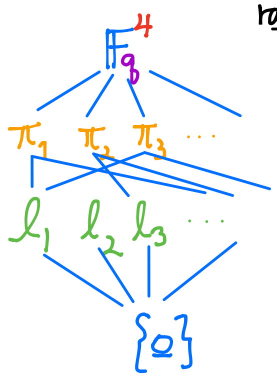


(compare with



Fano plane $= \mathbb{P}(\mathbb{F}_2^3)$)

$\mathcal{L}(U_{4,3}(q))$ "lim" $\underset{q\to 1}{\rightsquigarrow}$ $\mathcal{L}(U_{4,3})$



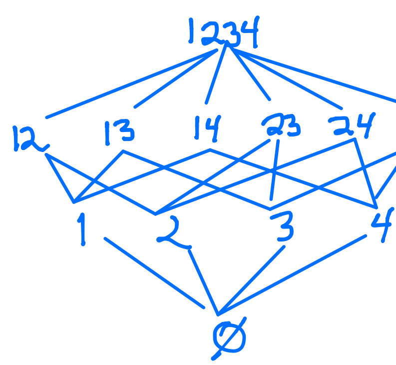
rank sizes

1

$\begin{bmatrix}4\\2\end{bmatrix}_q = 1+q+2q^2+q^3+q^4$

$\begin{bmatrix}4\\1\end{bmatrix}_q = 1+q+q^2+q^3$

$\begin{bmatrix}4\\0\end{bmatrix}_q = 1$



rank sizes

1

$\binom{4}{2} = 6$

$\binom{4}{1} = 4$

$\binom{4}{0} = 1$

---

where $\begin{bmatrix}n\\k\end{bmatrix}_q = \dfrac{[n]!_q}{[k]!_q\,[n-k]!_q}$

$[n]!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q$

$[n]_q = 1+q+q^2+\dots+q^{n-1}$

lim $\underset{q\to 1}{\rightsquigarrow}$

$\binom{n}{k}$

$n!$

n

Feichtner & Yuzvinsky's 2004

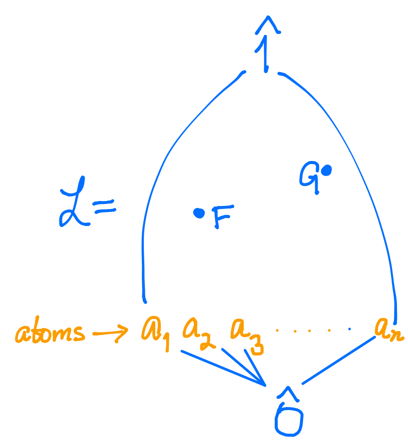
Chow ring of an atomic lattice $\mathcal{L}$

$$A(\mathcal{L}) = \mathbb{Z}[x_F : F \in \mathcal{L} - \{\hat{0}\}] \Big/ I_{\mathcal{L}} + J_{\mathcal{L}}$$

where

$$I_{\mathcal{L}} = (x_F x_G : F, G \text{ incomparable})$$

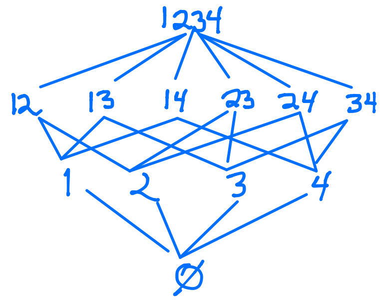
= usual Stanley-Reisner ideal for $\mathcal{L} - \{\hat{0}\}$



$$J_{\mathcal{L}} = \Big(\sum_{\substack{F \in \mathcal{L} - \{\hat{0}\} \\ a \in F}} x_F \; : \text{atoms } a \text{ of } \mathcal{L} \Big)$$

= extra linear relations, containing (more than) a linear system of parameters

EXAMPLE

$\mathcal{L} = \mathcal{L}(U_{4,3}) =$ 

$$A(\mathcal{L}) = \mathbb{Z}[x_1, x_2, x_3, x_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{1234}] \big/ I_{\mathcal{L}} + J_{\mathcal{L}}$$

$$I_{\mathcal{L}} = \Big(\underset{i \neq j}{x_i x_j} \;,\; \underset{\{i,j\} \neq \{k,l\}}{x_{ij} x_{kl}} \;,\; \underset{i \notin \{j,k\}}{x_i x_{jk}} \Big)$$

$$J_{\mathcal{L}} = \begin{pmatrix} x_1 + x_{12} + x_{13} + x_{14} \\ + x_{1234} \end{pmatrix}, \begin{pmatrix} x_2 + x_{12} + x_{23} + x_{24} \\ + x_{1234} \end{pmatrix}, \begin{pmatrix} x_3 + x_{13} + x_{23} + x_{34} \\ + x_{1234} \end{pmatrix}, \begin{pmatrix} x_4 + x_{14} + x_{24} + x_{34} \\ + x_{1234} \end{pmatrix}$$

$$A(\mathcal{L}) = A_0 \oplus A_1 \oplus A_2 \qquad \text{a graded}$$
$$= \mathbb{Z}^1 \oplus \mathbb{Z}^7 \oplus \mathbb{Z}^1 \qquad \mathbb{Z}\text{-algebra}$$

with Hilbert series

$$H(A(\mathcal{L}), t) = \sum_{d=0}^{\infty} \operatorname{rank}_{\mathbb{Z}} A_d \cdot t^d = 1 + 7t^1 + 1 \cdot t^2$$

Hilbert series results

THEOREM (Feichtner & Yuzvinsky 2004)

The ideal $I_{\mathcal{L}} + J_{\mathcal{L}} \subset \mathbb{Z}[x_F]_{F \in \mathcal{L} - \{\hat{0}\}}$

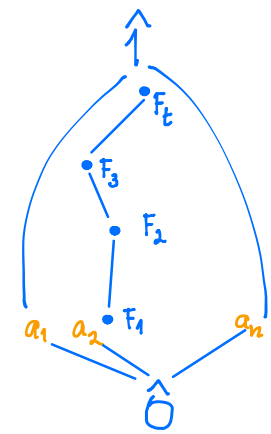
has a Gröbner basis that shows $A(\mathcal{L})$ is $\mathbb{Z}$-free,

with a $\mathbb{Z}$-basis of standard monomials

$$\left\{ x_{F_1}^{m(F_1)} \cdots x_{F_t}^{m(F_t)} : \begin{array}{l} F_1 < F_2 < \cdots < F_t \text{ in } \mathcal{L} - \{\hat{0}\}, \\ 1 \leq m(F_i) < d(F_{i-1}, F_i) \end{array} \right\}$$

$\min\{d : F_i = \bigvee_{j=1}^{d} a_j \vee F_{i-1} \text{, atoms } a_j \in \mathcal{L}\}$

$\mathcal{L} =$



EXAMPLE $A(\mathcal{L}(U_{4,3}))$ has $\mathbb{Z}$-basis

$$\{1,\ x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_{1234},\ x_{1234}^2\}$$

A_0	A_1	A_2
$\Vert$	$\Vert$	$\Vert$
$\mathbb{Z}^1$	$\mathbb{Z}^7$	$\mathbb{Z}^1$

THEOREM (Adiprasito, Huh & Katz 2015)

For a matroid M of rank r, the Chow ring $A = A(\mathcal{L}(M))$ has

- $A = A_0 \oplus A_1 \oplus \cdots \oplus A_{r-1}$ with $A_{r-1} \cong \mathbb{Z}$
- (Poincaré duality) One has a nondegenerate pairing
$$\begin{aligned} A_i \times A_{r-1-i} &\longrightarrow A_{r-1} = \mathbb{Z} \\ (x\ ,\ y) &\longmapsto x \cdot y \end{aligned}$$
- (Hard Lefschetz) Extending scalars to $\mathbb{R}$, $\exists\, \Theta$ in A_1 such that this map is an $\mathbb{R}$-linear isomorphism:
$$\begin{aligned} A_i &\longrightarrow A_{r-1-i} \\ x &\longmapsto \Theta^{r-1-2i} \cdot x \end{aligned}$$
- (Hodge-Riemann relations) The quadratic form $Q_i(x) = x \cdot \Theta^{r-1-2i} \cdot x$ on A_i when restricted to the primitive part $\ker(\Theta^{r-2i})$, has $(-1)^i Q_i(x)$ positive definite.

COROLLARY (to Poincaré duality, Hard Lefschetz)

$H(A(\mathcal{L}(M)), t)$ has symmetric, unimodal coefficient sequence.

(Hodge-Riemann was crucial for their proof of the Rota-Heron-Walsh and Mason Conjectures from the 1970's)

The main result...

THEOREM (Hameister-Rao-Simpson 2017)

$$\mathrm{Hil}\left(A(\mathcal{L}(U_{n,r})),t\right) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} - \sum_{m=1}^{n-r}\ \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \#\mathrm{Fix}(\sigma)\geq m}} t^{n-m-\mathrm{exc}(\sigma)}$$

and $\Uparrow \lim_{q\to 1}$

$$\mathrm{Hil}\left(A(\mathcal{L}(U_{n,r}(q))),t\right) = \sum_{\sigma\in\mathfrak{S}_n} \left(tq^{-1}\right)^{\mathrm{exc}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

$$- \sum_{m=1}^{n-r} q^{m-n} \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \#\mathrm{Fix}(\sigma)\geq m}} (tq)^{n-m-\mathrm{exc}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

where

$\mathrm{exc}(\sigma) = \#\{i : \sigma(i) > i\}$ = # of excedances in σ

$\mathrm{maj}(\sigma) = \sum_{i:\, \sigma(i)>\sigma(i+1)} i$ = major index of σ

$\mathrm{Fix}(\sigma) = \{i : \sigma(i) = i\}$ = fixed points of σ

SPECIAL CASES

- $H(A(\mathcal{L}(U_{n,n})),t) = \sum_{\sigma \in \mathfrak{S}_n} t^{exc(\sigma)} = n^{th}$ Eulerian polynomial $A_n(t)$

with exponential generating function

$$\sum_{n=0}^{\infty} A_n(t)\frac{x^n}{n!} = \frac{(t-1)e^t}{te^t - e^{tx}}$$

- $H(A(\mathcal{L}(U_{n,r}(q))),t) = \sum_{\sigma \in \mathfrak{S}_n} (tq^{-1})^{exc(\sigma)} q^{maj(\sigma)}$

$= n^{th}$ exc-maj q-Eulerian polynomial $A_n(t,q)$ of Shareshian-Wachs 2007

with q-exponential generating function

$$\sum_{n=0}^{\infty} A_n(t,q)\frac{x^n}{[n]!_q} = \frac{(t-1)\exp_q(t)}{t\exp_q(t) - \exp_q(tx)}$$

where $\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!_q}$

$$\bullet\ H(A(\mathcal{L}(U_{n,n-1})),t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{exc}(\sigma)} - \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \#\mathrm{Fix}(\sigma)\geq 1}} t^{n-1-\mathrm{exc}(\sigma)}$$

$$\underset{\text{Poincaré duality for } U_{n,n}}{=} \sum_{\sigma\in\mathfrak{S}_n} t^{n-1-\mathrm{exc}(\sigma)} - \sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \#\mathrm{Fix}(\sigma)\geq 1}} t^{n-1-\mathrm{exc}(\sigma)}$$

$$= \sum_{\substack{\text{derangements}\\ \sigma\in\mathfrak{S}_n}} t^{n-1-\mathrm{exc}(\sigma)} \underset{\text{Poincaré duality for } U_{n,n-1}}{=} \sum_{\substack{\text{derangements}\\ \sigma\in\mathfrak{S}_n}} t^{\mathrm{exc}(\sigma)-1}$$

EXAMPLE $n=4$

$$H(A(\mathcal{L}(U_{4,3})),t) = 1+7t+t^2 = \sum_{\substack{\text{derangements}\\ \sigma\in\mathfrak{S}_4}} t^{\mathrm{exc}(\sigma)-1}$$

derangement $\sigma\in\mathfrak{S}_4$		$\mathrm{exc}(\sigma)-1$
$(12)(34) =$	2143	1
$(13)(24) =$	3412	1
$(14)(23) =$	4321	1
$(1234) =$	2341	2
$(1243) =$	2413	1
$(1324) =$	3421	1
$(1342) =$	3142	1
$(1423) =$	4312	1
$(1432) =$	4123	0

PROOF IDEAS

$\mathcal{U}_{n,r}(q)$ case $\xRightarrow[\lim_{q\to 1}]{}$ $\mathcal{U}_{n,r}$ case

Prove $H(A(\mathcal{L}(\mathcal{U}_{n,r}(q))), t) = \sum_{\sigma \in \mathfrak{S}_n} (tq^{-1})^{\mathrm{exc}(\sigma)} q^{\mathrm{maj}(\sigma)}$

$$- \sum_{m=1}^{n-r} q^{m-n} \sum_{\substack{\sigma \in \mathfrak{S}_n: \\ \#\mathrm{Fix}(\sigma) \geq m}} (tq)^{n-m-\mathrm{exc}(\sigma)} q^{\mathrm{maj}(\sigma)}$$

by descending induction on r.

In base case $r=n$, show that both

- n^{th} exc-maj q-Eulerian polynomial $A_n(t,q)$ of Shareshian-Wachs 2007

and

- $H(A(\mathcal{L}(\mathcal{U}_{n,n}(q))), t)$

satisfy this recurrence:

$$A_n(t,q) = 1 + t \sum_{k=1}^{n} [k-1]_t \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k}(t,q)$$

The **inductive step** is equivalent to

$$\mathrm{Hilb}\left(A(\mathcal{L}(U_{n,r+1}(q))),t\right)-\mathrm{Hilb}\left(A(\mathcal{L}(U_{n,r}(q))),t\right)$$

$$=q^{-r}\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \#\mathrm{Fix}(\sigma)\geq n-r}}(tq)^{r-\mathrm{exc}(\sigma)}q^{\mathrm{maj}(\sigma)}$$

which one proves using Feichtner-Yuzvinsky's standard monomials to rewrite left side, and then at a crucial step ...

LEMMA (Wachs 1989)

For any $\tau\in\mathfrak{S}_k$ with $k\leq n$,

$$\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \text{nonfixed points of }\sigma\\ \text{"look like" }\tau}}q^{\mathrm{maj}(\sigma)}=q^{\mathrm{maj}(\tau)}\begin{bmatrix}n\\k\end{bmatrix}_q$$

The REU report has more results.

THANKS

FOR YOUR

ATTENTION

... and to the NSF

for RTG grant

support!