ALGEBRAIC COMBINATORICS:

Using algebra to help one count

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Jan. 24, 2006

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COMBINATORICS

= study of finite or discrete objects
and their structure,

including counting them

(= **ENUMERATIVE** COMBINATORICS)

Part of

ALGEBRAIC COMBINATORICS

is using algebra to help you do

ENUMERATIVE COMBINATORICS

EXAMPLE:

Enumerating subsets of a set,
up to symmetry.

We'll observe some interesting properties,
some easy,
some harder.

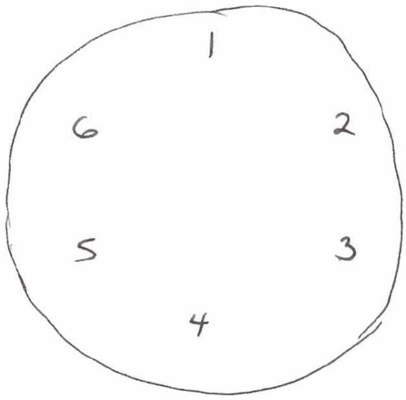
Consider the finite set

$$[n] := \{1, 2, \ldots, n\}$$

with some set G of <u>symmetries</u>

(= a subgroup of the symmetric group $\mathfrak{S}_n$ on n letters)

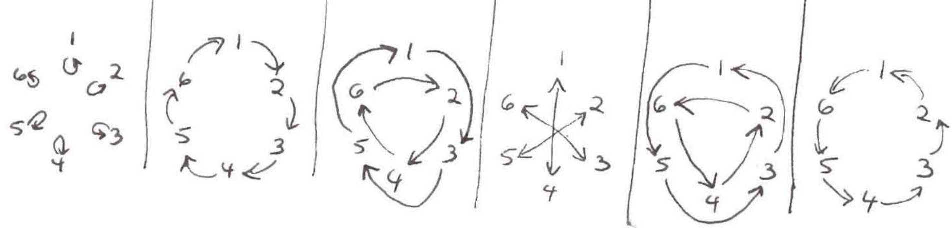
e.g. $n=6$



G = rotational symmetries = cyclic group C_6

$= \{$rotations through

0° | 60° | 120° | 180° | 240° | 300° $\}$



Let's count

$$2^{[n]} := \text{subsets of } [n]$$
$$\updownarrow$$
$$\text{black-white colorings of } [n]$$

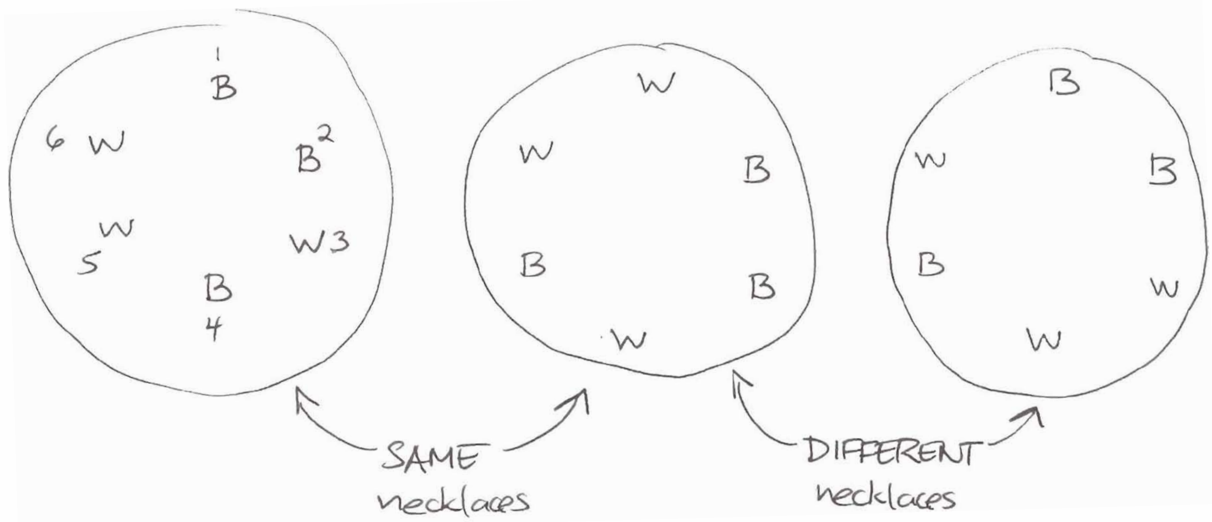
up to equivalence by elements of G,

i.e. G-orbits of subsets of $[n]$

$$=: 2^{[n]}/G$$

e.g. for $\overset{n=6}{G} = C_6$ as above,

G-orbits in $2^{[n]}/G$ are sometimes called black-white necklaces:



Let's even be more refined...

$$\binom{[n]}{k} = k\text{-element subsets of } [n]$$

The symmetries G also permute $\binom{[n]}{k}$.

Can we count

$$\binom{[n]}{k} \Big/ G := G\text{-orbits of } k\text{-subsets of } [n] \; ?$$

Let $\quad c_k := \left| \binom{[n]}{k} \Big/ G \right|$

Q: What can we say <u>in general</u> about

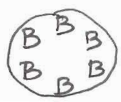
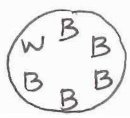
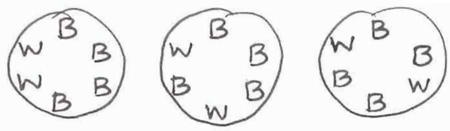
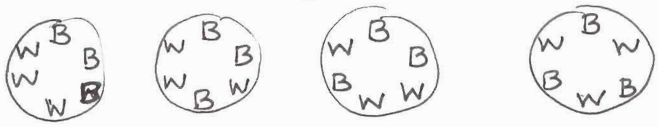
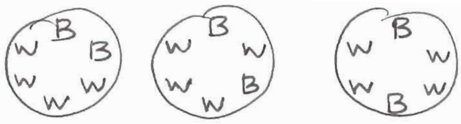
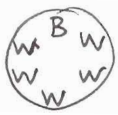
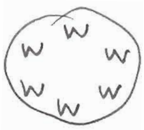
$$c_0, c_1, \ldots\ldots, c_n \; ?$$

A: They share a number of properties with <u>binomial coefficients</u> $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$ ...

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

(6)

EXAMPLE: $G = C_6$

k (= # of blacks)	C_6-orbits on $\binom{[6]}{k}$	$c_k = \left\lvert \binom{[6]}{k} / C_6 \right\rvert$
6		1
5		1
4		3
3		4
2		3
1		1
0		1

6 overlays

#1

1

1

3

4

3

1

1

$\boxed{\text{SYMMETRY}}$

#2

$1 \le 1 \le 3 \le 4 \ge 3 \ge 1 \ge 1$

$\boxed{\text{UNIMODALITY}}$

#3

+1

−1

+3

−4

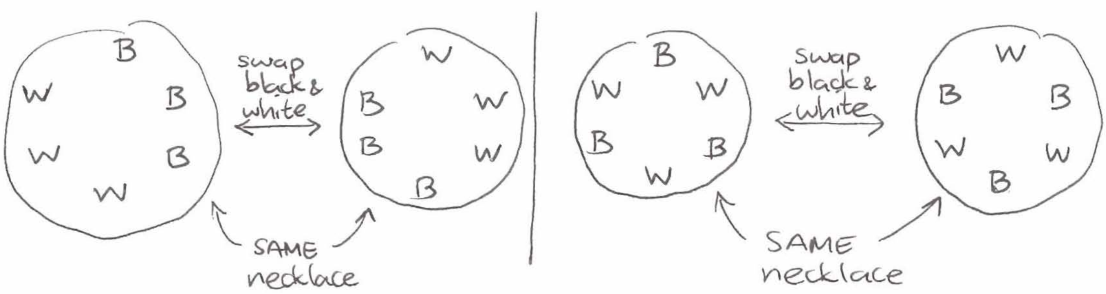
+3

−1

+1

+2 ← $\boxed{\text{ALTERNATING SUM}}$

= # of self-complementary (black ↔ white) G-orbits



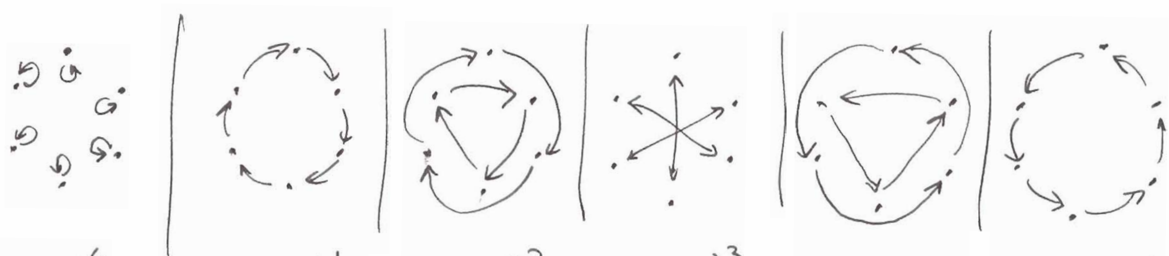
GENERATING FUNCTION

$$\sum_{k=0}^{n} c_k q^k = c_0 + c_1 q + c_2 q^2 + \dots + c_n q^n$$

is simply the <u>average</u> over the group G

of some easy products (recording cycle <u>sizes</u>)

$G = \{$



$(1+q)^6 \qquad (1+q^6)^1 \qquad (1+q^3)^2 \qquad (1+q^2)^3 \qquad (1+q^3)^2 \qquad (1+q^6)^1$

$$\sum_{k=0}^{6} c_k q^k = \text{their average}$$

$$= \frac{1}{6}\left[(1+q)^6 + 2(1+q^6) + 2(1+q^3)^2 + (1+q^2)^3 \right]$$

	q^0	q^1	q^2	q^3	q^4	q^5	q^6
$(1+q)^6 \rightarrow$	1	6	15	20	15	6	1
$2(1+q^6) \rightarrow$	2	0	0	0	0	0	2
$2(1+q^3)^2 \rightarrow$	2		0	4	0	0	2
$(1+q^2)^3 \rightarrow$	1	0	3	0	3	0	1
	6	6	18	24	18	6	6
$\times \frac{1}{6}$	$\boxed{1}$	$\boxed{1}$	$\boxed{3}$	$\boxed{4}$	$\boxed{3}$	$\boxed{1}$	$\boxed{1}$
	c_0	c_1	c_2	c_3	c_4	c_5	c_6

(8)

THEOREM:

For *any* subgroup G of $\mathfrak{S}_n$ and $c_k := \left|\binom{[n]}{k}/G\right|$,
the sequence $c_0, c_1, \ldots, c_n$

(1) has GENERATING FUNCTION

$$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{\substack{\text{cycles } C \\ \text{of } g}} (1+q^{|C|})$$

(Polya, Redfield)

(2) has ALTERNATING SUM

$$c_0 - c_1 + c_2 - \ldots + (-1)^n c_n = \#\text{ of self-complementary } G\text{-orbits}$$

(deBruijn)

(3) is SYMMETRIC: $c_i = c_{n-i}$

(nearly obvious)

(4) is UNIMODAL: $c_0 \le c_1 \le c_2 \le \ldots \le c_{\lfloor n/2 \rfloor}$

(Stanley)

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(1) – (3) are *not hard* to show directly

(4) is *hard* (in fact, not known) *without* using some sort of linear algebra, algebra or representation theory, but *easy* with them.

UNIFIED PROOF IDEA for (1) — (4)

Re-interpret subsets or black/white colorings and G-orbits in terms of (multi-)linear algebra....

Let $V = \mathbb{C}^2$ with $\mathbb{C}$-basis $\{\underset{\text{"white"}}{w}, \underset{\text{"black"}}{b}\}$

Then $$V^{\otimes n} = \overbrace{\underset{1}{V} \otimes \underset{2}{V} \otimes \cdots \otimes \underset{n}{V}}^{n \text{ times}}$$

has $\mathbb{C}$-basis elements e_S indexed by subsets $S \subseteq [n]$

e.g. $n=6$

$S = \{1,3,5\}$

$e_S = \underset{1}{b} \otimes \underset{2}{w} \otimes \underset{3}{b} \otimes \underset{4}{w} \otimes \underset{5}{b} \otimes \underset{6}{w}$, or just $bwbwbw$

G acts on $V^{\otimes n}$ by permuting the tensor positions.

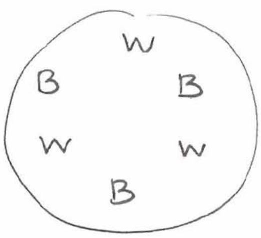
$(V^{\otimes n})^G :=$ the G-fixed subspace of $V^{\otimes n}$

has $\mathbb{C}$-basis indexed by

G-orbits of subsets or black-white colorings

e.g. $n=6$

$G=C_6$



$\longleftrightarrow$ $wbwbwb + bwbwbw \in (V^{\otimes 6})^G$

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$$V^{\otimes n} = \bigoplus_{k=0}^{n} \underbrace{\left(V^{\otimes n}\right)_k}_{\substack{\text{span of} \\ e_S \text{ with } |S|=k}}$$ is a graded $\mathbb{C}$- vector space

and G acts on each graded component $\left(V^{\otimes n}\right)_k$

so similarly

$$\left(V^{\otimes n}\right)^G = \bigoplus_{k=0}^{n} \underbrace{\left(V^{\otimes n}\right)_k^G}_{\substack{\text{with basis} \\ \text{corresponding to } \binom{[n]}{k}/G}}.$$

Thus $c_k = \dim_{\mathbb{C}} \left(V^{\otimes n}\right)_k^G$

which gives a good starting point...

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Proof of (3) (SYMMETRY): $c_k = c_{n-k}$

(A bit silly, but contains an idea useful for proof of (2)!)

Consider $\sigma := \begin{array}{c|cc} & w & b \\ \hline w & 0 & 1 \\ b & 1 & 0 \end{array} \in GL_2(\mathbb{C}) = GL(V)$

acting on $V = \mathbb{C}^2$ by swapping $\underset{\text{black}}{b} \longleftrightarrow \underset{\text{white}}{w}$

$GL(V)$ also acts on $V^{\otimes n}$ diagonally: $v_1 \otimes \ldots \otimes v_n \xmapsto{\sigma} \sigma(v_1) \otimes \ldots \otimes \sigma(v_n)$

$bbbwwb \mapsto wwwbbw$

and this commutes with the G-action,

so σ preserves $\left(V^{\otimes n}\right)^G$.

In fact, it gives a $\mathbb{C}$- vector space isomorphism

$$\left(V^{\otimes n}\right)^G_k \xrightarrow[\sim]{\sigma} \left(V^{\otimes n}\right)^G_{n-k}$$

so $$\dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_k = \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_{n-k}$$

i.e. $$c_k = c_{n-k}$$

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Sketch proof of (2) (ALTERNATING SUM):

$c_0 - c_1 + c_2 - \cdots + (-1)^n c_n$ = # self-complementary G-orbits in $2^{[n]}$

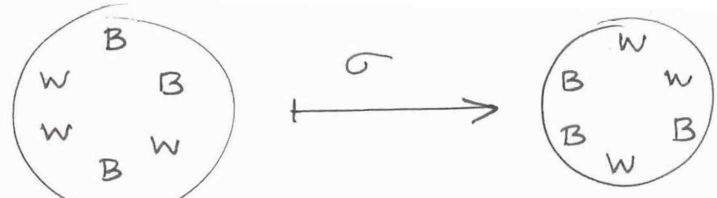
Note that $\sigma = \begin{array}{cc} & \begin{array}{cc} w & b \end{array} \\ \begin{array}{c} w \\ b \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$ is diagonalizable with eigenvalues $+1, -1$

so it is conjugate in $GL(V)$ to $\begin{array}{cc} & \begin{array}{cc} w & b \end{array} \\ \begin{array}{c} w \\ b \end{array} & \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \end{array}$

Hence $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$ should act with the same trace on $(V^{\otimes n})^G$.

Note that $\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ permutes the basis elements for $(V^{\otimes n})^G$ that are indexed by G-orbits of subsets, via complementation

e.g.



Hence

$$\mathrm{Trace}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : (V^{\otimes n})^G \to (V^{\otimes n})^G \right)$$

= # self-complementary G-orbits in $2^{[n]}$

On the other hand,

$$\begin{array}{c|cc} & w & b \\ \hline w & & \\ b & & \end{array}\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} : \quad \begin{aligned} w &\longmapsto w \\ b &\longmapsto -b \end{aligned}$$

so $\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$ acts on $\left(V^{\otimes n}\right)_k$ by the scalar $(-1)^k$

e.g. $\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} : bbbwwb \longmapsto (-b)(-b)(-b)(w)(w)(-b)$

$$= (-1)^4\, bbbwwb$$

Hence $\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}$ acts on $\left(V^{\otimes n}\right)^G_k$ by the scalar $(-1)^k$

and since $\left(V^{\otimes n}\right)^G = \bigoplus_{k=0}^{n} \left(V^{\otimes n}\right)^G_k$,

$$\mathrm{Trace}\left(\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} : \left(V^{\otimes n}\right)^G \to \left(V^{\otimes n}\right)^G \right) =$$

$$+\dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_0 - \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_1 + \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_2 - \dots =$$

$$c_0 \qquad c_1 \quad + \quad c_2 \quad - \quad \dots$$

Sketch proof of (4) (UNIMODALITY): $c_0 \le c_1 \le \dots \le c_{\lfloor n/2 \rfloor}$

We want $\quad c_k \le c_{k+1} \quad$ for $k < \frac{n}{2}$

i.e $\quad \dim_{\mathbb{C}} (V^{\otimes n})^G_k \le \dim_{\mathbb{C}} (V^{\otimes n})^G_{k+1}$

so maybe there's a $\mathbb{C}$-vector space injection

$$(V^{\otimes n})^G_k \hookrightarrow (V^{\otimes n})^G_{k+1} \qquad \text{for } k < \tfrac{n}{2}\,?$$

In fact, maybe it comes from a G-equivariant injection

$$(V^{\otimes n})_k \xhookrightarrow{U_k} (V^{\otimes n})_{k+1} \qquad \text{for } k < \tfrac{n}{2}\,??$$

Here's an obvious guess for defining U_k:

$$\underset{\text{for } |S|=k}{e_S} \longmapsto \sum_{\substack{T:\, T \supset S,\\ |T|=k+1}} e_T$$

e.g. $\quad$ bwbwww $\longmapsto$ bbbwww + bwbbww + bwbwbw + bwbwwb

It's easily seen to be G-equivariant,

but why is it injective for $k < \frac{n}{2}$?

(16)

Here is one (of several) linear algebra arguments for the injectivity of U_k if $k < \frac{n}{2}$:

Check that

$$\boxed{U_k^* U_k - U_{k-1} U_{k-1}^* = (n-2k) I_{(V^{\otimes n})_k}} \quad \forall k = 1, 2, \dots, n$$

so

$$U_k^* U_k = \underbrace{\underbrace{U_{k-1} U_{k-1}^*}_{\substack{\text{positive} \\ \text{semidefinite}}} + \underbrace{(n-2k) I_{(V^{\otimes n})_k}}_{\substack{\text{positive definite,} \\ \text{because } k < n/2}}}$$

$\Downarrow$

$U_k^* U_k$ positive definite, hence invertible

$\Downarrow$

U_k injective.

Sketch proof of (1) (GENERATING FUNCTION):

$$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{\substack{\text{cycles } C \\ \text{of } g}} (1+q^{|C|})$$

When a finite group G acts linearly on a $\mathbb{C}$-vector space $\mathcal{U}$, the <u>averageing operator</u>

$$\pi_G := \frac{1}{|G|} \sum_{g \in G} g$$

is an <u>idempotent</u> <u>projector</u> $\pi_G : \mathcal{U} \twoheadrightarrow \mathcal{U}^G$
$(\pi_G^2 = \pi_G)$

Hence its <u>trace</u> computes the dimension of the image:

$$\mathrm{Trace}(\pi_G : \mathcal{U} \to \mathcal{U}) = \dim_{\mathbb{C}}(\mathcal{U}^G).$$

Let's apply this with $\mathcal{U} = (V^{\otimes n})_k$

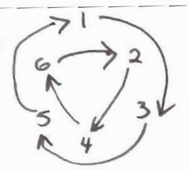
to compute $\dim_{\mathbb{C}}(\mathcal{U}^G) = \dim_{\mathbb{C}} (V^{\otimes n})_k = c_k$

$$c_k = \dim_{\mathbb{C}} \left(V^{\otimes n}\right)^G_k$$

$$= \mathrm{Trace}\left(\frac{1}{|G|} \sum_{g \in G} g : \left(V^{\otimes n}\right)_k \to \left(V^{\otimes n}\right)_k \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \mathrm{Trace}\left(g : \left(V^{\otimes n}\right)_k \to \left(V^{\otimes n}\right)_k \right)$$

Note g permutes the basis elements e_S for $\left(V^{\otimes n}\right)_k$

e.g. $g =$  sends $\underset{1\,2\,3\,4\,5\,6}{bwwwww} \overset{g}{\longmapsto} \underset{1\,2\,3\,4\,5\,6}{wwbwww}$

and so its trace counts black/white colorings of $[n]$ that it fixes, i.e. those which are monochromatic on each of g's cycles

e.g.	wwwwww	bwbwbw wbwbwb	bbbbbb	
	1 2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6	
	1	$+2q^3$	$1 \cdot q^6$	$= (1+q^3)^2$

Hence $$c_k = \frac{1}{|G|} \sum_{g \in G} \left(\begin{array}{l} \text{\# of black/white colorings of } [n] \\ \text{with } k \text{ blacks, monochromatic} \\ \text{on } g\text{'s cycles} \end{array} \right)$$

$$= \frac{1}{|G|} \sum_{g \in G} \left[\text{coefficient of } q^k \text{ in } \prod_{\substack{\text{cycles } C \\ \text{of } g}} \left(1+q^{|C|}\right) \right]$$

i.e. $$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{\substack{\text{cycles } C \\ \text{of } g}} \left(1+q^{|C|}\right).$$