Seminar on Kraskiewicz–Weyman 1985

1. Shephard–Todd 1955 / Chevalley 1955 & Lusztig's fake degree formula*

2. Springer* 1974 & K–W's Thm 1.

3. K–W's 1st Corollary

4. K–W's 2nd corollary

5. Klyachko* 1974 & K–W's Thm 2

* indicates a bonus "extra for experts" proof sketch! (not for this talk)

① Shephard-Todd / Chevalley & fake degrees
1955 1955

THEOREM (S-T, C)

(a) For a finite subgroup W of $GL_n(\mathbb{C}) = GL(V)$, $V = \mathbb{C}^n$
acting via linear substitutions on $\mathbb{C}[\underline{x}] = \mathbb{C}[x_1, \ldots, x_n]$,
the invariant ring $\mathbb{C}[\underline{x}]^W = \mathbb{C}[f_1, \ldots, f_n]$ for homog. alg. indep. f_i

$\Longleftrightarrow$ W is gen'd by complex reflections

elements g whose fixed space V^g is a hyperplane

(b) In this situation,

the coinvariant algebra has a W-rep'n isomorphism

$$\mathbb{C}[\underline{x}]/(f_1, \ldots, f_n) \cong \mathbb{C}[W]$$

W acts via linear substitutions

left-regular representation

EXAMPLE: $W = S_n$ permuting coordinates in $V = \mathbb{C}^n$

$$\mathbb{C}[\underline{x}]^{S_n} = \mathbb{C}[e_1, e_2, \dots, e_n]$$

elementary symm. functions

$$\mathbb{C}[x]/(e_1, \dots, e_n) \underset{\substack{\uparrow \\ \text{as } S_n\text{-rep'ns}}}{\cong} \mathbb{C}[S_n]$$

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e.g. $n=3$

$$\mathbb{C}[x_1,x_2,x_3]/(x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3)$$

has $\mathbb{C}$-basis (not obvious, requires checking)

$x_3 = -(x_1+x_2)$ $\qquad$ $x_3^2 = -(x_1^2+x_2^2)$

$$\{ 1, \; x_1, x_2, \; x_1^2, x_2^2, \; (x_1-x_2)(x_1-x_3)(x_2-x_3) \}$$

	$\chi^{(3)}$ trivial	$\chi^{(2,1)}$	$\chi^{(2,1)}$	$\chi^{(1,1,1)}$ sign
degree:	0	1	2	3

$\mathbb{C}[S_3]$

Since each W-irreducible χ^λ will appear $\deg(\chi^\lambda)$ times in $\mathbb{C}[W]$ or $\mathbb{C}[\underline{x}]/(\underline{f})$,

one can define...

DEF'N: The *χ^λ-fake degree polynomial*

$$f^\lambda(q) := \sum_{d \geq 0} q^d \cdot \langle \chi^\lambda, \left(\mathbb{C}[\underline{x}]/(\underline{f})\right)_d \rangle_W$$

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e.g. $n=3$

$\mathbb{C}[x_1,x_2,x_3]/(x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3)$

has $\mathbb{C}$-basis (not obvious, requires checking)

		$x_3=-(x_1+x_2)$	$x_3^2=-(x_1^2+x_2^2)$	
	$\{1,$	$x_1, x_2,$	$x_1^2, x_2^2,$	$(x_1-x_2)(x_1-x_3)(x_2-x_3)\}$
	$\chi^{(3)}$	$\chi^{(2,1)}$	$\chi^{(2,1)}$	$\chi^{(1,1,1)}$
degree:	0	1	2	3

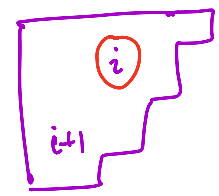
$\mathbb{C}[S_3]$

$$\Rightarrow \quad f^{(3)}(q) = q^0$$
$$f^{(2,1)}(q) = q^1 + q^2$$
$$f^{(1,1,1)}(q) = q^3$$

THEOREM (Lusztig's fake degree formula, 1978?)

For $W = S_n$ and λ a partition of n,

$$f^{\lambda}(q) = \sum_{\substack{\text{standard} \\ \text{Young tableaux } T \\ \text{of shape } \lambda}} q^{\operatorname{maj}(T)}$$

$T =$ 

where $\operatorname{maj}(T) =$ sum of i such that $i+1$ appears in a lower row of T than i

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e.g. $f^{\ydiagram{3}}(q) = \underset{q^0}{\begin{ytableau} 1 & 2 & 3 \end{ytableau}}$

$$f^{\ydiagram{2,1}}(q) = \underset{q^1}{\begin{ytableau} 1 & 3 \\ 2 \end{ytableau}} + \underset{q^2}{\begin{ytableau} 1 & 2 \\ 3 \end{ytableau}}$$

$$f^{\ydiagram{1,1,1}}(q) = \underset{q^{1+2}}{\begin{ytableau} 1 \\ 2 \\ 3 \end{ytableau}} = q^3$$

Sketch proof of Lusztig's fake-degree formula (not for this talk!)

$\mathbb{C}[\underline{x}]$ is Cohen-Macaulay, and hence a free $\mathbb{C}[e_1,\dots,e_n]$-module, so

$$f^\lambda(q) = \frac{1}{\mathrm{Hilb}(\mathbb{C}[e_1,\dots,e_n],q)} \sum_{d\geq 0} q^d \langle \chi^\lambda, \mathbb{C}[\underline{x}]_d \rangle_{S_n}.$$

$$\sum_{d\geq 0} q^d \langle \chi^\lambda, \mathbb{C}[\underline{x}]_d \rangle_{S_n} \underset{\text{Molien's Theorem}}{=} \frac{1}{n!} \sum_{w\in S_n} \frac{\chi^\lambda(w)}{\det(1-q\cdot w)}$$

and for a permutation w with cycle type $\mu(w) := (\mu_1,\mu_2,\dots)$

$$\frac{1}{\det(1-q\cdot w)} = \prod_i \frac{1}{1-q^{\mu_i}} = \prod_i (1+q^{\mu_i}+q^{2\mu_i}+\dots)$$
$$= \prod_i p_{\mu_i}(1,q,q^2,\dots) = \left[p_{\mu(w)}(\underline{x})\right]_{x_i=q^{i-1}}$$

where $p_r(\underline{x}) = x_1^r + x_2^r + x_3^r + \dots$

Hence

$$\sum_{d\geq 0} q^d \langle \chi^\lambda, \mathbb{C}[\underline{x}]_d \rangle_{S_n} = \left[\frac{1}{n!} \sum_{w\in S_n} \chi^\lambda(w) p_{\mu(w)}(\underline{x})\right]_{x_i=q^i}$$

Frobenius characteristic map (Stanley E.C. II §7.18)

$$= \left[s_\lambda(\underline{x})\right]_{x_i=q^i}$$

Stanley E.C. II Prop 7.19.11

$$= \sum_{\substack{\text{standard tab } T\\ \text{of shape } \lambda}} q^{\mathrm{maj}(T)} \Big/ (1-q)(1-q^2)\cdots(1-q^n)$$

$$= \mathrm{Hilb}(\mathbb{C}[e_1,\dots,e_n],q) \cdot \sum_T q^{\mathrm{maj}(T)} \quad \blacksquare$$

② Springer 1974 (& K-W's 1st Thm)

Springer defined a regular element $c \in W$ a $\mathbb{C}$-refl'n group to mean c has an eigenvector v with a free W-orbit:

$$c(v) = \omega \cdot v \qquad \#W \cdot v = \#W$$

eigenvalue $\omega = e^{\frac{2\pi i}{h}}$

He then proved...

THEOREM: Letting $C := \langle c \rangle = \{1, c, c^2, \ldots, c^{h-1}\} \cong \mathbb{Z}/h\mathbb{Z}$

the coinvariant algebra has a $W \times C$-rep'n isomorphism

$$\mathbb{C}[x]/(f_1, \ldots, f_n) \cong \mathbb{C}[W]$$

W acts via linear substitutions

$$(w \cdot f)(\underline{x}) = f(w^{-1}\underline{x})$$

C acts via scalar substitutions

$$c^k(x_i) = \omega^{-k} x_i$$

(in other words, c scales degree d by ω^{-d})

W multiplies on left, C multiplies on right:

$$(u, c^k) \cdot w := u w c^k$$

Sketch proof of Springer's Theorem (not for this talk!)

Consider the map $\mathbb{C}[\underline{x}] \xrightarrow{\varphi} \mathbb{C}[W]$

sending $f(\underline{x}) \longmapsto \sum_{w \in W} f(w(v)) \cdot w$

and note that it is

- $\mathbb{C}$-linear
- surjective (using easy multivariate polynomial interpolation)
- $W \times C$-equivariant, since

$$(u, c^k).f \longmapsto \sum_{w' \in W} \underbrace{f(u^{-1} w'(\omega^{-k} \cdot v))}_{f(u^{-1} w' c^{-k}(v))} \cdot w' \overset{\text{let } w := u^{-1} w' c^{-k}}{=} \sum_{w \in W} f(w(v)) \cdot u w c^k$$

and has $f(\underline{x}) - f(v)$ in its kernel for any W-invariant $f(\underline{x}) \in \mathbb{C}[\underline{x}]^W$.

Hence it induces a $W \times C$-equivariant surjection

$$R := \mathbb{C}[\underline{x}]/(f_1(\underline{x}) - f_1(v), \ldots, f_n(\underline{x}) - f_n(v)) \overset{\varphi}{\twoheadrightarrow} \mathbb{C}[W]$$

which we claim will be an isomorphism via dimension-counting

Even more strongly, the degree filtration on R

$\{\underline{0}\} \subset F_1 \subset F_2 \subset \cdots$ with $F_i = \mathrm{span}_{\mathbb{C}}\{g(\underline{x}) : \deg(g) \leq i\}$ in R

has associated graded ring $\mathrm{gr} R := F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \cdots$

($W \times C$-equivariantly) isomorphic to $\mathbb{C}[\underline{x}]/(f_1(\underline{x}), \ldots, f_n(\underline{x}))$

giving equivalences of $W \times C$-rep'ns

$$\mathbb{C}[\underline{x}]/(f_1, \ldots, f_n) = \mathrm{gr} R \cong R \xrightarrow[\sim]{\varphi} \mathbb{C}[W] \qquad \blacksquare$$

uses semisimplicity of $W \times C$-rep'ns in characteristic zero

FAVORITE EXAMPLE:

In $W = S_n$, an n-cycle such as $c = (1,2,\ldots,n)$ is a regular element,

since $v := [1, \omega, \omega^2, \ldots, \omega^{n-1}] \in V = \mathbb{C}^n$ where $\omega = e^{2\pi i/n}$ has

$$c(v) = [\omega, \omega^2, \omega^3, \ldots, \omega^{n-1}, 1] = \omega \cdot v$$

and $\#W \cdot v = n! = \#W$ (free W-orbit)

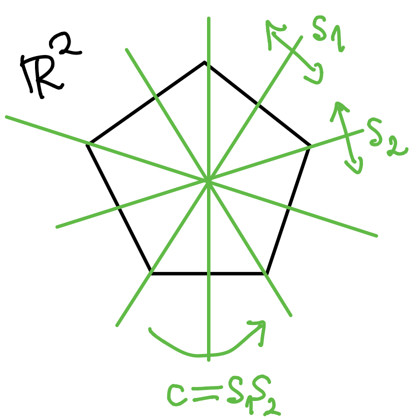
since no two coordinates of v are equal.

DIHEDRAL EXAMPLE:

W = dihedral group of symmetries of a regular n-gon

$$\cong \left\langle s_1, s_2 \;\middle|\; s_1^2 = s_2^2 = (s_1 s_2)^n = e \right\rangle$$

has $c = s_1 s_2 = \cancel{\text{rotation}}$ through $\theta = 2\pi/n$ as a regular element:



$$c = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \underset{\text{diagonalize}}{\rightsquigarrow} \begin{array}{c|cc} & v & s_2(v) \\ \hline v & \omega & 0 \\ s_2(v) & 0 & \omega^{-1} \end{array} \qquad \text{where } \omega = e^{i\theta} = e^{2\pi i/n}$$

in $\mathbb{R}^2$ in $V = \mathbb{C}^2$

MORE GENERAL FAVORITE EXAMPLE: In W a real reflection group, one always has a set of Coxeter generators $S=\{s_1,s_2,\ldots,s_n\}$ and then any Coxeter element $c=s_1s_2\cdots s_n$ (in any order) is always a regular element (of order $h =:$ the Coxeter number)

(comes from the action of c on the Coxeter plane, as in Humphreys §3.17)

Consequently, Springer generalizes...

Kraskiewicz-Weyman THEOREM 1:

For Weyl groups W of types $A, B/C, D$ and $c \in W$ a Coxeter element,

the coinvariant algebra has a $W \times C$-rep'n isomorphism

$$\mathbb{C}[x]/(f_1,\ldots,f_n) \cong \mathbb{C}[W]$$

with $W \times C$-actions as described in Springer's Thm.

③ K-W's 1st Corollary

It asserts that for any $\ell = 0, 1, 2, \dots, h-1$,

the linear character

$$\chi_\ell : \begin{array}{c} C \\ \shortparallel \\ \langle c \rangle \end{array} \longrightarrow \mathbb{C}^\times$$

sending $\quad c \longmapsto \omega^{-\ell}$

has an isomorphism of W-rep'ns

$$\mathrm{Ind}_C^W(\chi_\ell) \cong A_\ell \oplus A_{\ell+h} \oplus A_{\ell+2h} \oplus \cdots$$

$$= \bigoplus_{d \equiv \ell \bmod h} A_d$$

where $A := \mathbb{C}[\underline{x}]/(f_1, \dots, f_n)$

the coinvariant algebra

In fact,

$$\mathrm{Ind}_C^W(\chi_\ell) \cong \bigoplus_{d \equiv \ell \bmod h} A_d$$

is actually an equivalent phrasing of the K-W Thm 1 or Springer's Thm, for the following reason.

If one defines $e_\ell := \frac{1}{h}\sum_{i=0}^{h-1} \omega^{\ell i} \cdot c^i \in \mathbb{C}[C]$

then one can check it is an idempotent, projecting C-rep'ns onto their χ_ℓ-isotypic component.

(And in a $W \times C$-rep'n, this χ_ℓ-isotypic component for C is still a W-rep'n.)

So for example,

$$\mathrm{Ind}_C^W(\chi_\ell) \underset{\substack{\uparrow \\ \text{as } W\text{-rep'ns}}}{\cong} \mathbb{C}W\text{-module on } \underbrace{\mathbb{C}W \cdot e_\ell}_{\substack{\chi_\ell\text{-isotypic component} \\ \text{of } \mathbb{C}W}}$$

using your favorite definition of $\mathrm{Ind}_C^W(\chi_\ell)$.

Meanwhile,

$$\bigoplus_{d \equiv \ell \bmod h} A_d \underset{\substack{\uparrow \\ \text{as } W\text{-rep'ns}}}{\cong} \chi_\ell\text{-isotypic component of } A$$

since C acts on A_d via the linear character χ_d,

and hence A_d lies in the χ_ℓ-isotypic component for C

$\Longleftrightarrow \chi_d = \chi_\ell$

$\Longleftrightarrow d \equiv \ell \bmod h$

Thus $\mathrm{Ind}_C^W(\chi_\ell) \cong \bigoplus_{d \equiv \ell \bmod h} A_d$ as W-rep'ns $\forall \ell = 0, 1, \ldots, h-1$

$\Longleftrightarrow$ χ_ℓ-isotypic component of $\mathbb{C}W$ $\cong$ χ_ℓ-isotypic component of A as W-rep'ns $\forall \ell$

$\Longleftrightarrow$ $\mathbb{C}W \cong A$ as $W \times C$-rep'ns.

④ K-W's 2^{nd} corollary

Note for $W = S_n$, Lusztig's fake degree formula tells us

$$\mathring{A}_d \underset{\text{as } W\text{-rep'ns}}{\cong} \bigoplus_{\substack{\text{standard Young}\\ \text{tableaux } T \text{ with } n \text{ cells}\\ \text{and } maj(T)=d}} \chi^{shape(T)}$$

So one immediately obtains K-W's 2^{nd} corollary from their 1^{st} corollary:

For $C = \langle c \rangle$ where $c = (1,2,\ldots,n)$ in $W = S_n$, one has an isomorphism of W-rep'ns

$$\mathrm{Ind}_C^W(\chi_\ell) \overset{1^{st}\text{ COR}}{\cong} \bigoplus_{d \equiv \ell \bmod n} \mathring{A}_d$$

$$\overset{2^{nd}\text{ COR}}{\cong} \bigoplus_{\substack{\text{standard}\\ \text{Young tableaux } T\\ \text{with } n \text{ cells and}\\ maj(T) \equiv \ell \bmod n}} \chi^{shape(T)}$$

⑤ Klyachko 1974 & K-W's Thm 2

Klyachko proved that the S_n-rep'n

$\mathrm{Lie}_n :=$ multilinear part of the free Lie algebra $\mathrm{Lie}(V)$ on $V = \mathbb{C}^n$

has an S_n-rep'n isomorphism

$$\mathrm{Lie}_n \cong \mathrm{Ind}_C^{S_n}(\chi_1)$$

and hence by the K-W 2^{nd} corollary

$$\mathrm{Lie}_n \cong \bigoplus_{\substack{\text{standard Young}\\ \text{tableaux } T \text{ with}\\ n \text{ cells}\\ \text{and } \mathrm{maj}(T) \equiv 1 \bmod n}} \chi^{\mathrm{shape}(T)}$$

Sketch proof of Klyachko's Thm (not for this talk!)

Consider $V := \mathbb{C}^m$ with $\mathbb{C}$-basis $\{v_1, v_2, \ldots, v_m\}$ as a $GL(V)$-rep'n

and then $V^{\otimes n} := \underbrace{V \otimes V \otimes \ldots \otimes V}_{n \text{ times}}$ becomes a $GL(V) \times S_n$-rep'n:

via $\begin{cases} g(v_1 \otimes \ldots \otimes v_n) = g(v_1) \otimes \ldots \otimes g(v_n) & \text{for } g \in GL(V) \\ (v_1 \otimes \ldots \otimes v_n)\sigma = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)} & \text{for } \sigma \in S_n \end{cases}$

$GL(V)$-subrepresentations $U \subset V^{\otimes n}$ are called
polynomial rep'ns of $GL(V)$ of degree n,
and are determined up to $GL(V)$-rep'n isomorphism by
their characters $s_U(\underline{x}) = s_U(x_1, \ldots, x_m) = \mathrm{Trace}(x : U \to U)$

where $x := \begin{array}{c|c} & \begin{matrix} v_1 & v_2 & \cdots & v_m \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{matrix} & \begin{bmatrix} x_1 & & & \\ & x_2 & & 0 \\ & 0 & \ddots & \\ & & & x_m \end{bmatrix} \end{array}$

$\boxed{\text{Schur–Weyl duality – see Stanley E.C. II App. A2}}$

Two examples:

① $\mathrm{Lie}(V)_n$:= degree n component of free Lie algebra $\mathrm{Lie}(V)$
= $\mathbb{C}$-span of n-fold brackets
$[\cdots[[v_{i_1}, v_{i_2}], v_{i_3}], \ldots, v_{i_n}]$ where $[A, B] := A \otimes B - B \otimes A$

② $V^{\otimes n} \cdot \underbrace{e_1}_{}$:= $\mathbb{C}$-span of $(v_{i_1} \otimes \ldots \otimes v_{i_n}) \cdot \frac{1}{n} \sum_{i=0}^{n-1} \omega^{-i} \cdot c^i$ where $c = (1, 2, \ldots, n)$

idempotent from before inside $\mathbb{C}[C] \subset \mathbb{C}[S_n]$

Both $\mathrm{Lie}(V)_n$ and $V^{\otimes n} \cdot e_1$ have $\mathbb{C}$-bases indexed by

Lyndon words $\underline{i} := (i_1, i_2, \ldots, i_k) \in \{1,2,\ldots,m\}^n$

means $\underline{i}$ is strictly smallest lexicographically among all of its cyclic rotations $(i_k, i_{k+1}, \ldots, i_n, i_1, i_2, \ldots, i_{k-1})$

namely $\{v(\underline{i})\}_{\underline{i} \text{ Lyndon}}$ is easily seen to be a $\mathbb{C}$-basis for $V^{\otimes n} \cdot e_1$

$$v(\underline{i}) = (v_{i_1} \otimes \cdots \otimes v_{i_n}) \cdot e_1$$

see Reutenauer §5.1

while $\{b(\underline{i})\}_{\underline{i} \text{ Lyndon}}$ is a known $\mathbb{C}$-basis for $\mathrm{Lie}(V)_n$

$b(\underline{i})$ = the Lyndon bracketing of $(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$

In both cases, they are x-eigenbases, with same weight/eigenvalue:

$$x \cdot v(\underline{i}) = x_{i_1} \cdots x_{i_n} \cdot v(\underline{i})$$
$$x \cdot b(\underline{i}) = x_{i_1} \cdots x_{i_n} \cdot b(\underline{i})$$

Hence $\mathrm{Lie}(V)_n$ and $V^{\otimes n} \cdot e_1$ have same $GL(V)$-characters, and $\mathrm{Lie}(V)_n \cong V^{\otimes n} \cdot e_1$ as $GL(V)$-rep'ns.

Take $m=n$, so $V = \mathbb{C}^n$, and restrict to their multilinear parts = the $x_1 x_2 \cdots x_n$-eigenspace for x

$$\mathrm{Lie}_n \cong \mathbb{C}[S_n] e_1 \text{ as } S_n\text{-rep'ns}$$

$$\mathbb{C}[S_n] e_1 \cong \mathrm{Ind}_C^{S_n}(\chi_1)$$

discussed earlier, by def'n of induction

$\blacksquare$

Thanks for your attention!

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