Hop on the left-regular band-wagon!

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partly based on arXiv:2206.11406
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1. What is a left-regular band (LRB) ?

EXAMPLES

- free LRB $\mathcal{F}_n$ (and q-analogue $\mathcal{F}_n^{(q)}$)
- Tits face semigroup $\mathcal{F}(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$
- The space of phylogenetic trees

2. What is invariant theory ?

3. Invariant theory for the free LRB

(featuring: the derangement representations of $\mathfrak{S}_n$)

1. What is a left-regular band (LRB)?

In case, we don't get it across ...

Ken Brown — Semigroups, rings, and Markov chains
J. Theoret. Probab. 13 (2000)

Stuart Margolis
Franco Saliola
Ben Steinberg — Cell complexes, Poset Topology and the Representation Theory of Algebras Arising in Combinatorics and Discrete Geometry
Mem. Amer. Math. Soc. 1345 (2021)

What is a left-regular band (LRB)?

A semigroup M in which

$$\begin{cases} xyx = xy & \forall x,y \in M \\ x^2 = x & \forall x \in M \end{cases}$$

$\leftarrow$ defines a band

=idempotent semigroup

Studied by Bidigare, Bidigare-Hanlon-Rockmore, Brown, Brown & Diaconis, Saliola, Margolis-Saliola-Steinberg, Aguiar-Mahajan, ...

What does $xyx = xy$ mean?

Roughly in examples,

xy = "x perturbed to make more decisions in the direction that y has made them"

Like x is a swing voter who may skip some races and ballot questions,

y is trying to influence them and get them to vote their way...

EXAMPLE The free LRB $\mathcal{F}_n$ on letters $a_1, a_2, \ldots, a_n$

$\mathcal{F}_n = \{$ injective words on the letters $\}$

no repeated letters

with multiplication

$$a_1 a_2 \cdots a_\ell \cdot b_1 b_2 \cdots b_m = \underbrace{(a_1 a_2 \cdots a_\ell b_1 b_2 \cdots b_m)^{\wedge}}_{\text{concatenation}}$$

$^{\wedge}$ means remove 2^{nd}, 3^{rd}, ... occurrences of letters e.g. $n=3$ On letters $\{a,b,c\}$, $F_3 = \{1,\ a,\ b,\ c,\ ab,\ ac,\ ba,\ bc,\ ca,\ cb,\ abc,\ acb,\ bac,\ bca,\ cab,\ cba\}$

with $1 = \begin{pmatrix}\text{empty}\\ \text{word}\end{pmatrix}$

$a \cdot a = a$

$a \cdot ca = ac$

$a \cdot cab = acb$

$ac \cdot bc = acb$

$bac \cdot ab = bac$

$ab \cdot bca = abc$

EXAMPLE

$\mathcal{F}_n^{(q)}$ = q-analogue of the free LRB $\mathcal{F}_n$

= {initial partial flags $(V_1, V_2, \ldots, V_\ell)$ of subspaces

$\underset{\text{line}}{V_1} \subset \underset{\text{plane}}{V_2} \subset \underset{\cdots}{\ldots} \subset \underset{\ell\text{-subspace}}{V_\ell}$ in $\mathbb{F}_q^n$ with $\dim V_i = i$ }

with $(V_1, V_2, \ldots, V_\ell) \cdot (W_1, W_2, \ldots, W_m) :=$

$$\left(V_1, V_2, \ldots, V_\ell, V_\ell + W_1, V_\ell + W_2, \ldots, V_\ell + W_m\right)^{\wedge}$$

$^\wedge$ means remove any subspace that appears earlier in the list

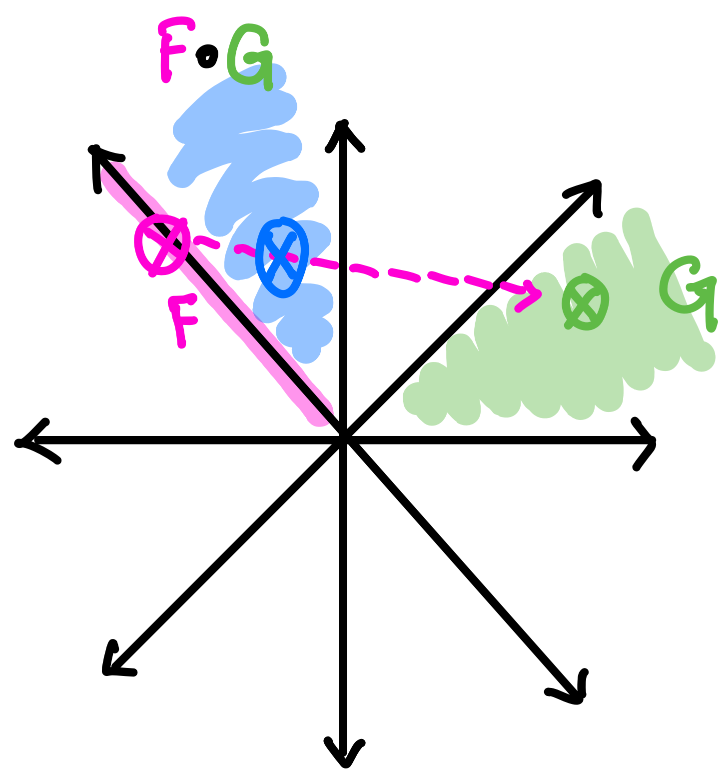
MOTIVATING EXAMPLE

J. Tits's face semigroup (1974) of a hyperpane arrangement $\mathcal{A} \subset \mathbb{R}^n$

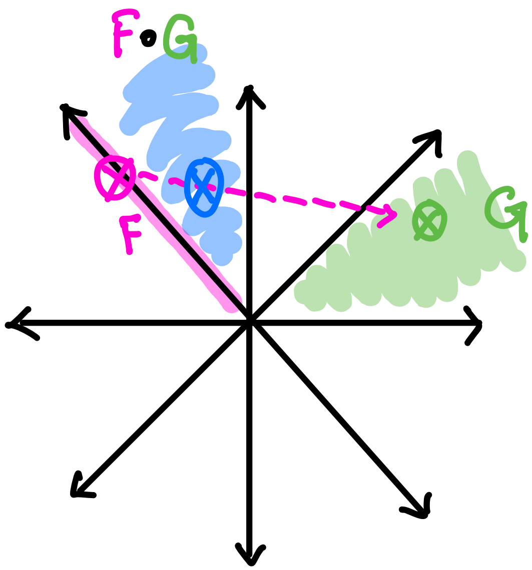
$$\mathcal{F}(\mathcal{A}) = \{\text{faces } F \text{ of } \mathcal{A}\}$$

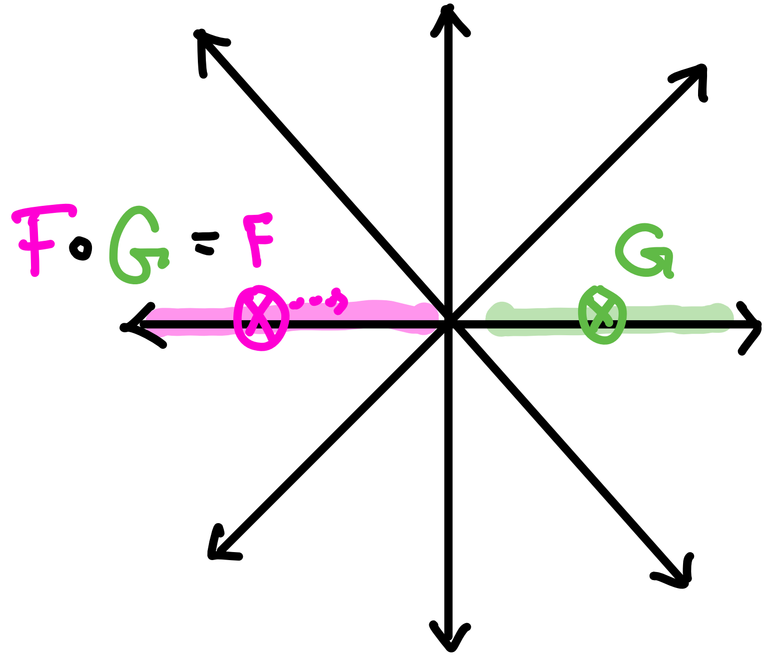
chambers and all their subface cones

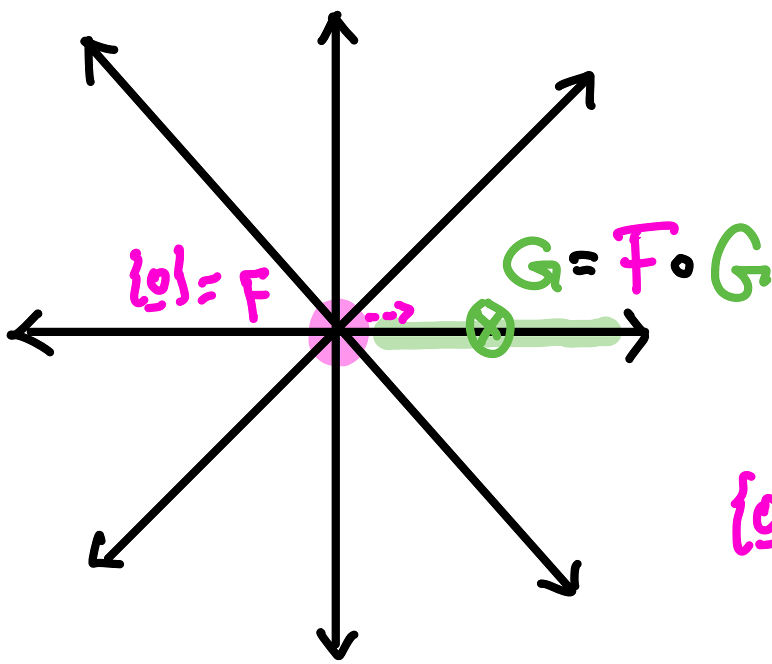
with $F \circ G$ = "face F perturbed toward face G"



$F \circ G$ = "face F perturbed toward face G"







$\{0\} \circ G = G \quad \forall$ faces G

MODERN LRB MOTIVATION:

Inside the monoid algebra $\Bbbk\mathcal{M} := \left\{ \sum_{m \in \mathcal{M}} c_m m : c_m \in \Bbbk \right\}$

one can model card-shuffling Markov chains

and use representation theory of $\Bbbk\mathcal{M}$ to analyze

eigenvalues and mixing times.

EXAMPLE: Random-to-top shuffling on $\mathfrak{S}_n = \left\{ \begin{array}{l} \text{permutations} \\ \text{of } a_1, \ldots, a_n \end{array} \right\}$

$$R2T(abc) = \frac{1}{3}(abc + bac + cab)$$

move each letter to the front,
with equal probability

R2T on $\mathfrak{S}_n$ can be modeled inside $\mathbb{Q}\mathcal{F}_n$ $\leftarrow$ free LRB $\mathcal{F}_n$

as left-multiplication by

$\frac{1}{n} \cdot x$ where $x := a_1 + a_2 + \dots + a_n$.

---

e.g. $n=3$: In $\mathbb{Q}\mathcal{F}_3$,

$$\frac{1}{3}(\overbrace{a+b+c}^{x}) \cdot (abc) = \frac{1}{3}(abc + bac + cab)$$

R2T on $\mathfrak{S}_n$ can also be modeled inside $\mathbb{Q}\mathcal{F}(\mathcal{A}_n)$

braid arrangement $\bigcup_{1\le i<j\le n}\{x_i = x_j\}$

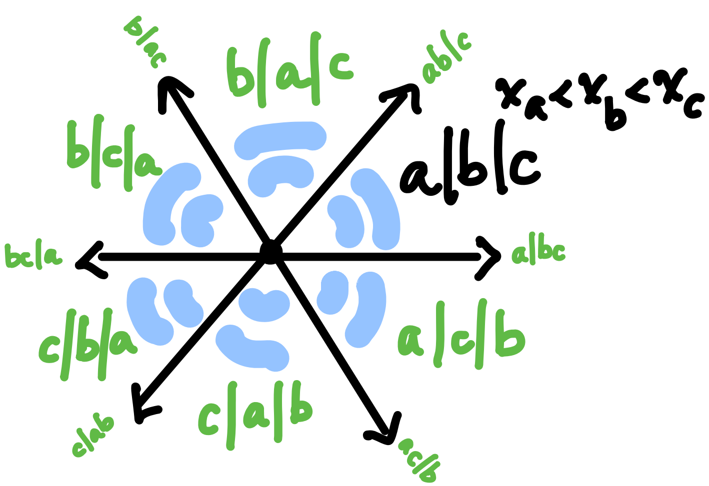
as left-multiplication by $\frac{1}{n}\cdot x$

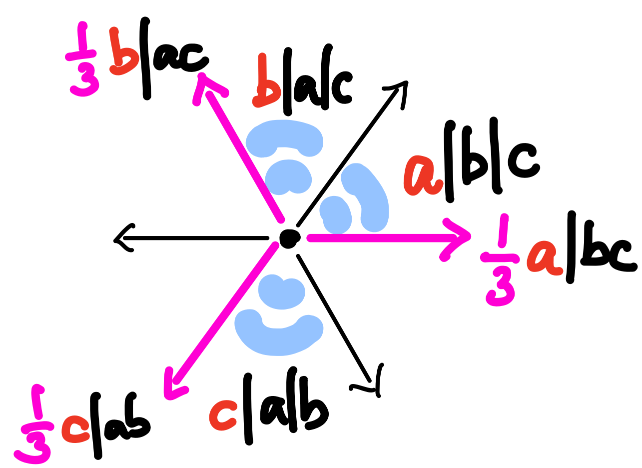
where $x = 1|23\cdots n + 2|134\cdots n + \cdots + n|12\cdots n{-}1$

e.g. $n=3$:

$$\frac{1}{3}\big(\overbrace{a|bc + b|ac + c|ab}^{x}\big) \circ (a|b|c) = \frac{1}{3}\big(a|b|c + b|a|c + c|a|b\big)$$

$\mathcal{A}_3$





Note those elements $\frac{1}{n}x$ are invariant under $\mathfrak{S}_n$.
Similarly for models of other shuffling algorithms, like inverse riffle shuffles, motivating this result:

THEOREM (Bidigare 1997)

When $\mathfrak{S}_n$ acts on $kF(\mathcal{A}_n)$, the $\mathfrak{S}_n$-invariant subalgebra

$$kF(\mathcal{A}_n)^{\mathfrak{S}_n} \cong \underbrace{\mathrm{Sol}(\mathfrak{S}_n)}^{opp}$$

Solomon's descent algebra for $\mathfrak{S}_n$
(a non-semisimple algebra)

(and same for all finite reflection groups W with reflection hyperplane arrangement $\mathcal{A}_W$)

Bidigare and later Bidigare-Hanlon-Rockmore 1999 used

used the representation theory of $kF(A_n)$ to analyze

rep theory of its invariant subalgebra $kF(A_n)^{\mathfrak{S}_n}$,

applying it to analyze random-to-top R2T inverse riffle shuffles

and other symmetric shuffling algorithms.

Further work on $kF(A_n)$ as $\mathfrak{S}_n$-rep and $kF(A_n)^{\mathfrak{S}_n}$-module by

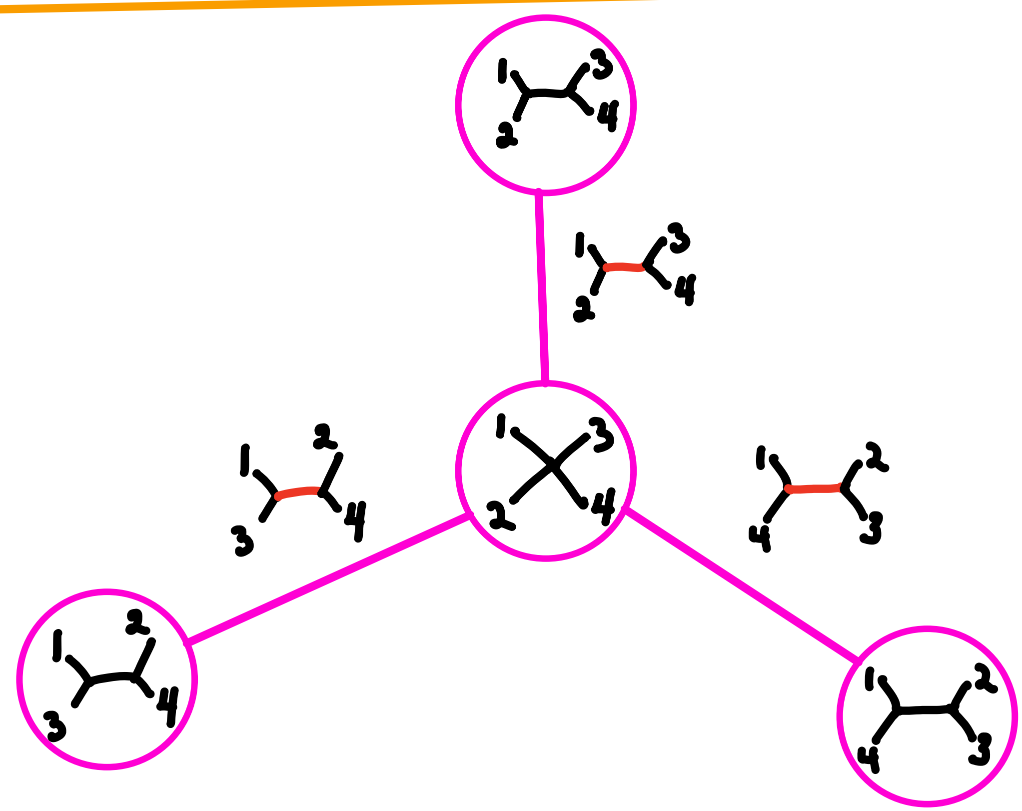
Garsia & Reutenauer 1989
Uyemura-Reyes 2002
Commins 2022+ (ongoing thesis work)

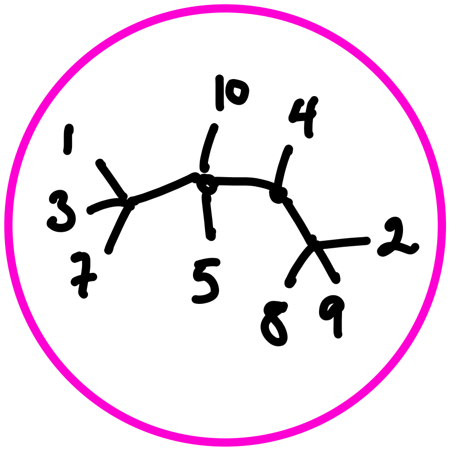
(FUN!)
EXAMPLE Billera-Holmes-Vogtmann (2001) introduced the space $\mathcal{T}_n$ of phylogenetic trees with n leaves, a CAT(0) cubical complex.

vertices (0-faces) = trees with
- leaves $1, 2, \ldots, n$
- internal vertices at least trivalent
- internal edges all of length 1

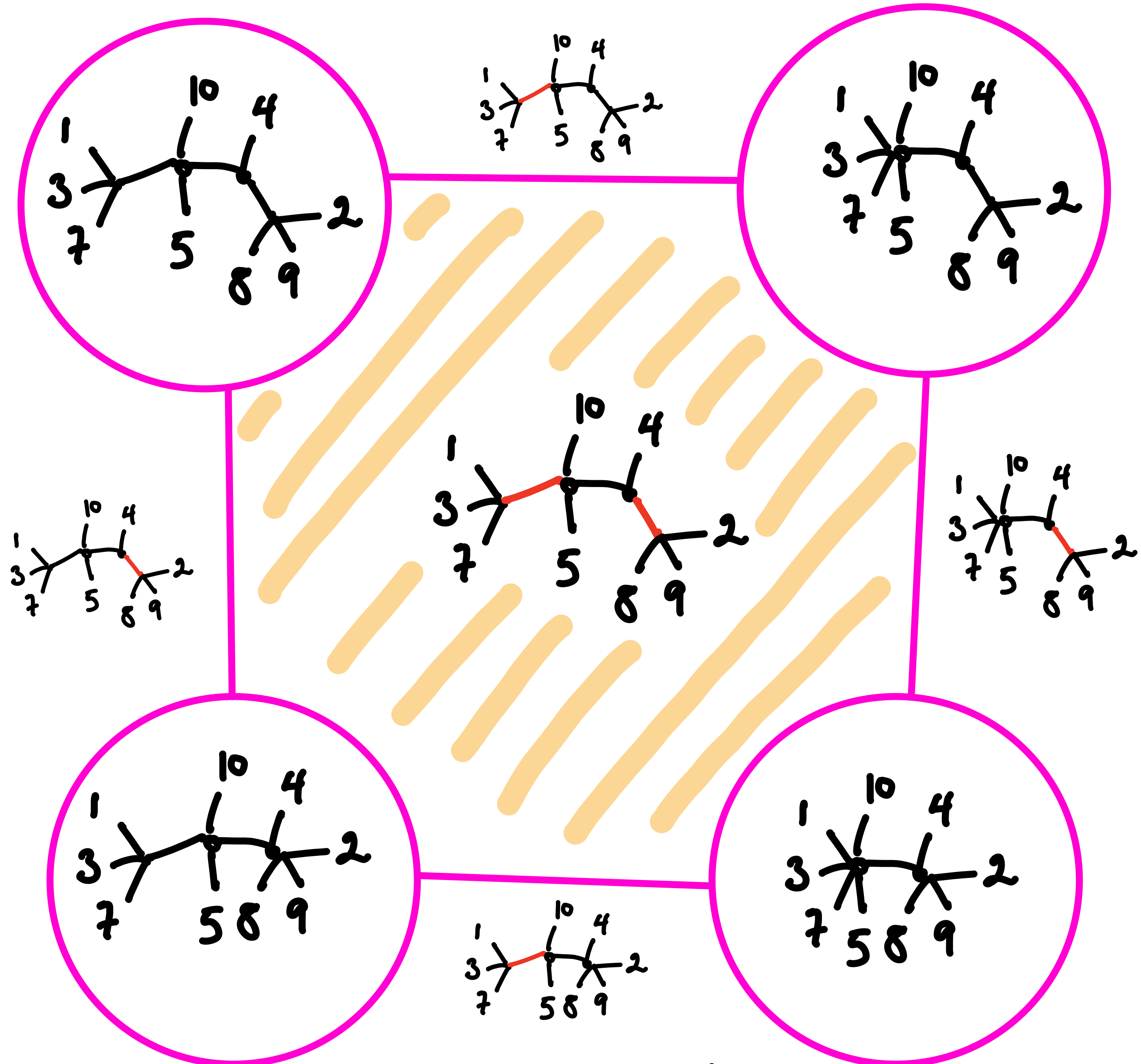
d-cube faces = same except d internal edge lengths float in $[0,1]$

$\mathcal{T}_4$ =





a vertex in $\widetilde{\mathcal{T}}_{10}$



a 2-cubical face in $\mathcal{T}_{10}$

THEOREM: All Cat(0) cubical complexes have a semigroup structure on their faces
$$(F,G) \mapsto F \circ G$$
making them an LRB.

Margolis-Saliola-Steinberg 2021

Bandelt-Chepoi-Knauer 2018

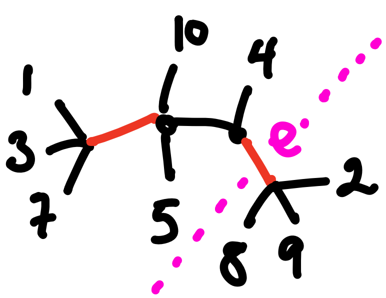
Commins: (this month)

In the tree space T_n:

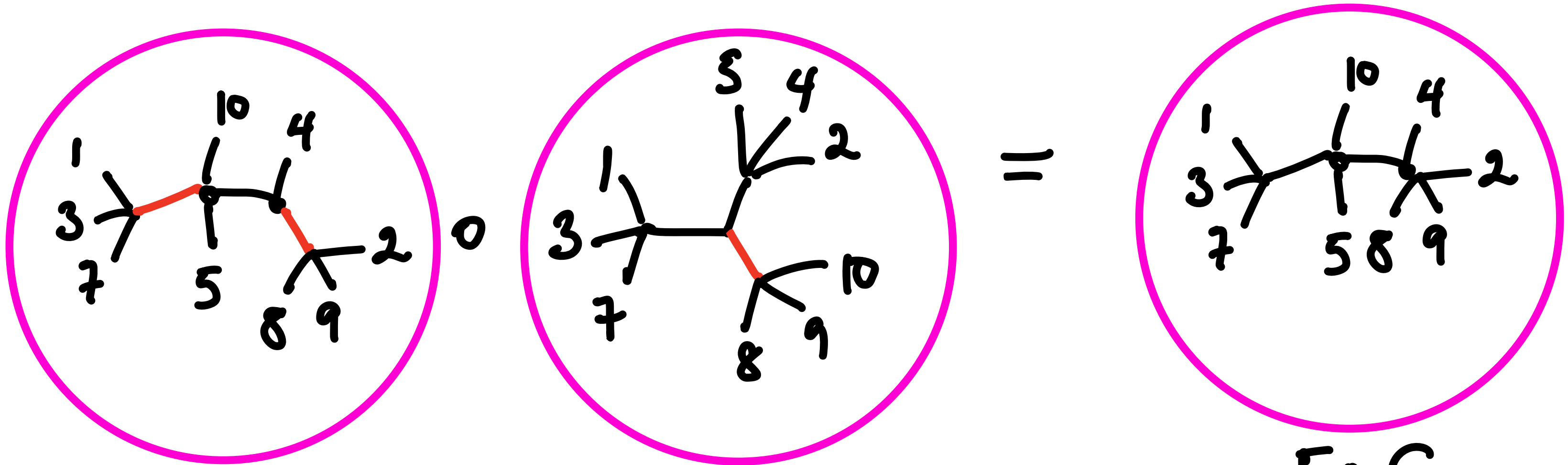
$F \circ G$ starts with the tree F, and for each edge e whose length is floating in $[0,1]$, it looks for an edge e' in G separating the same sets of leaves: $A \overset{e}{—} B$ $\quad$ $A \overset{e'}{—} B$

- If G has no such e', $F \circ G$ contracts e to length 0
- If G has such e' floating, $F \circ G$ leaves e floating
- If G has e' of length 1, $F \circ G$ gives e length 1

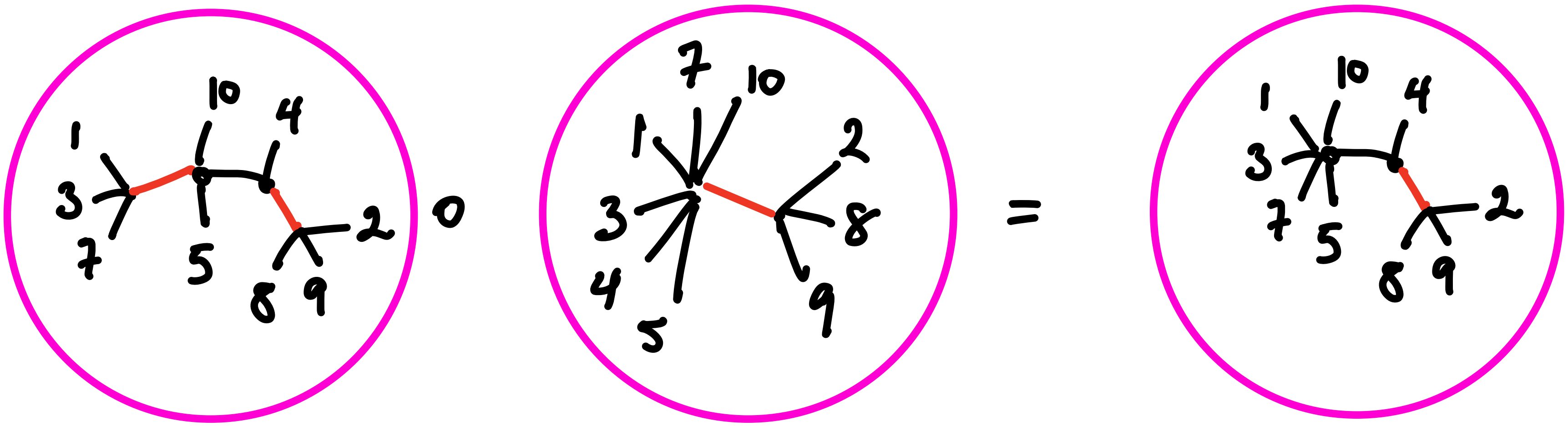
$A = \{1,3,4,5,7,10\}$



$B = \{2,8,9\}$



F $\quad$ G $\quad$ $F \circ G$



F $\quad$ G $\quad$ $F \circ G$

2. What is invariant theory?

Classically it asks, for a subgroup $G \subset GL_n(k)$ acting on $S = k[x_1, \dots, x_n]$ by linear substitutions

$$g(x_j) = \sum_j g_{ij} x_i \quad \dots$$

- Structure of the G-invariants
  $S^G := \{ f(x) \in S : f(g\underline{x}) = f(\underline{x}) \ \forall g \in G \}$ as a ring?
  Generators, relations?

- Structure of the whole ring S as an S^G-module and simultaneously as a G-representation?

Simplest answers for finite reflection groups $G \subset GL_n(\mathbb{C})$:

- $S^G = \mathbb{C}[f_1, f_2, \ldots, f_n]$ is also a polynomial algebra
  (n generators, 0 relations)

  e.g. $G = \mathfrak{S}_n$ permuting variables in $\mathbb{C}[x_1, \ldots, x_n]$
  has $\mathbb{C}[x_1, x_2, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

  elementary symmetric polynomials

- $S = \mathbb{C}[x_1, \ldots, x_n]$ is a free S^G-module

  $$S = \bigoplus_{\substack{G\text{-irreducible} \\ \text{characters } \chi}} S^{G,\chi}$$

  each χ-isotypic component is a free S^G-module, with $\chi(1)^2$ basis elements in known degrees.

3. Invariant theory for the free LRB

PROPOSITION (easy) The free LRB $\mathcal{F}_n$ has $\mathfrak{S}_n$-invariant subalgebra $(k\mathcal{F}_n)^{\mathfrak{S}_n}$ with k-basis of orbit sums (for any coefficients k):

NOTE: $x_1 = x$ from before (having $R2T = \frac{1}{n}\cdot x$)

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= a_1 + a_2 + \cdots + a_n \\
x_2 &= a_1a_2 + a_2a_1 + a_1a_3 + \cdots + a_na_{n-1} \\
&\ \vdots \\
x_n &= a_1a_2\cdots a_n + \cdots + a_n\cdots a_2a_1
\end{aligned}
$$

EXAMPLE $(k\mathcal{F}_3)^{\mathfrak{S}_3}$ has k-basis

$$
\begin{aligned}
x_0 &= 1 \\
x_1 &= a+b+c \\
x_2 &= ab+ba+ac+ca+bc+cb \\
x_3 &= abc+acb+bac+bca+cab+cba
\end{aligned}
$$

(easy)

PROPOSITION: $x := x_1 = a_1 + a_2 + \dots + a_n$ left-multiplies in this basis triangularly

$$x \cdot x_\ell = \ell \cdot x_\ell + x_{\ell+1}$$

EXAMPLE $n=4$ so F_4 has letters $\{a,b,c,d\}$

$$x \cdot x_2 = (a+b+c+d)(ab+ba+ac+ca+\dots+cd+dc)$$

$$= \underbrace{2(ab}_{\text{comes from } a\cdot ab,\ a \cdot ba}+ba+ac+ca+\dots+cd+dc) + (\underbrace{abc}_{\text{comes from } a\cdot bc}+acb+\dots+bcd)$$

$$= 2x_2 + x_3$$

(easy)

COROLLARY: The powers $\{1, x, x^2, \ldots, x^n\}$ expand unitriangularly in the orbit sum k-basis $\{1, x_1, x_2, \ldots, x_n\}$ for $(kF_n)^{\mathfrak{S}_n}$ with Stirling numbers $S(n,k)$ of 2nd kind as coefficients: $x^m = \sum_k S(m,k)\, x_k$

EXAMPLE:

$$x^0 = 1 \qquad = 1\cdot x_0$$
$$x^1 = a+b+c \quad = 1\cdot x_1$$
$$x^2 = (a+b+c)^2 = 1\cdot x_1 + 1\cdot x_2$$
$$x^3 = (a+b+c)^3 = 1\cdot x_1 + 3\cdot x_2 + 1\cdot x_3$$

Stirling numbers $S(n,k)$

n \ k	0	1	2	3	4
0	1				
1		1			
2		1	1		
3		1	3	1	
4		1	6	7	1

$S(n,k) = \#$ of set partitions of $\{1,2,\ldots,n\} = B_1 \sqcup B_2 \sqcup \ldots \sqcup B_k$ with k blocks

$S(n,k) = S(n-1,k-1) + k\,S(n-1,k)$

COROLLARY: $x = a_1 + a_2 + \cdots + a_n$ generates $(k\mathfrak{S}_n)^{\mathfrak{S}_n}$, with minimal polynomial $f(X) = X(X-1)(X-2)\cdots(X-n)$.

(Brauner - Commins - R. 2022)

Hence one has a ring isomorphism

$$\begin{array}{ccc} k[X]/(f(X)) & \xrightarrow{\sim} & (k\mathfrak{S}_n)^{\mathfrak{S}_n} \\ X & \longmapsto & x \end{array}$$

In particular, when $n! \in k^\times$, the invariant ring $(k\mathfrak{S}_n)^{\mathfrak{S}_n}$ is

- commutative
- semisimple
- and x acts with eigenvalues $0, 1, \ldots, n$

in finite dimensional $(k\mathfrak{S}_n)^{\mathfrak{S}_n}$-modules.

CONCLUSION: To complete the second invariant theory goal of describing kF_n simultaneously as

$$\begin{cases} (kF_n)^{\mathfrak{S}_n}\text{-module} \\ \text{and} \\ \mathfrak{S}_n\text{-representation} \end{cases}$$

one only needs to describe the

$\mathfrak{S}_n$-rep on each eigenspace $\ker(x - m)$ on kF_n

for each $m = 0, 1, 2, \ldots, n$.

... and same story for the q-analogue $\mathcal{F}_n^{(q)}$ with the action of $GL_n(\mathbb{F}_q)$:

- $x \rightsquigarrow x^{(q)} = \sum_{\substack{\text{lines } L \\ \text{in } \mathbb{F}_q^n}} (L) = (L_1) + (L_2) + \cdots + (L_{[n]_q})$

  where $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$

- Stirling numbers $S(n,k) \rightsquigarrow$ q-Stirling numbers (of Milne 1982)

- $\left(k\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)} \cong k[X] / \left(X(X-[1]_q)(X-[2]_q)\cdots(X-[n]_q)\right)$

- $\left(k\mathcal{F}_n^{(q)}\right)^{GL_n(\mathbb{F}_q)}$ is commutative, semisimple, and $x^{(q)}$ acts with eigenvalues $[0]_q, [1]_q, \ldots, [n]_q$ on modules.

  (when $\# GL_n(\mathbb{F}_q) \in k^\times$)

BUILDING BLOCK: The derangement representation of $\mathfrak{S}_n$

The derangement numbers

$$d_n := n!\left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + \frac{(-1)^n}{n!}\right)$$

count permutations $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ in $\mathfrak{S}_n$

with no fixed points $\sigma_i = i$

n	0	1	2	3	4
d_n	1	0	1	2	9
derangements in $\mathfrak{S}_n$	()		(12)	(123) (132)	(1234) (12)(34) (1243) (13)(24) (1324) (1342) (14)(23) (1423) (1432)

But d_n are also dimensions of an $\mathfrak{S}_n$-rep D_n whose associated symmetric function d_n was introduced by Désarménien & Wachs 1993.

(EQUIVALENT) DEFINITIONS:

- $$d_n = \sum_{\substack{\text{standard Young tableaux } Q \\ \text{whose 1st ascent } i \text{ is even}}} s_{\lambda(Q)}$$ explicit $\mathfrak{S}_n$ irreducible decomposition of D_n

- $$d_n = h_{1^n} - e_1 \cdot h_{1^{n-1}} + e_2 h_{1^{n-2}} + e_3 h_{1^{n-3}} - \dots + (-1)^n e_n$$

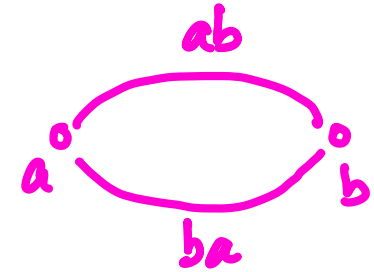
  e_i := elem. symm. fn.
  h_i := complete homog. symm. fn.

- $$h_{1^n} = d_n + h_1 d_{n-1} + h_2 d_{n-2} + \dots + h_{n-1} d_1 + h_n$$

- $$D_n \cong \ker(R2T : \mathbb{Q}\mathfrak{S}_n \to \mathbb{Q}\mathfrak{S}_n)$$

- $D_n \cong \mathrm{sgn}_{\mathfrak{S}_n} \otimes$ (top homology of the (cell) complex K_n of injective words on $a_1, a_2, \dots, a_n$)

K_2



n	standard Young tableaux Q with 1st ascent i even	symmetric function d_n	derangement number d_n
0	$\emptyset$	1	1
1	–	0	0
2	1 2	$s_{(1,1)}$	1
3	1 3 2	$s_{(2,1)}$	2
4	1 2 3 4 1 3 2 4 1 3 2 4 1 3 4 2	$s_{(1,1,1,1)} + s_{(2,1,1)} + s_{(2,2)} + s_{(3,1)}$	9

Analyzing the whole ring $\Bbbk\mathcal{F}_n$ for the free LRB $\mathcal{F}_n$:

Filter $\mathcal{F}_n$ by word length:

$\mathcal{F}_{\geq \ell}$ = $\Bbbk$-span of injective words of length $\geq \ell$

$$\mathcal{F}_n = \mathcal{F}_{\geq 0} \supset \mathcal{F}_{\geq 1} \supset \mathcal{F}_{\geq 2} \supset \ldots \supset \mathcal{F}_{\geq n-1} \supset \mathcal{F}_n$$

Semisimplicity of $(\Bbbk\mathcal{F}_n)^{\mathfrak{S}_n}$ and of $\Bbbk\mathfrak{S}_n$

$\Rightarrow$ sufficient to describe the $\mathfrak{S}_n$-rep on

- each x-eigenspace $\ker(x-m)$ for $m=0,1,\ldots,n$
- acting on each filtration factor $\mathcal{F}_{\geq \ell}/\mathcal{F}_{\geq \ell+1}$

THEOREM (Brauner-Commins-R. 2022)

In kF_n, the x-eigenspace $\ker(x-m)$ for $m=0,1,\ldots,n$ when x acts on $F_{\geq \ell}/F_{\geq \ell+1}$ for $\ell=0,1,\ldots,n$ carries $\mathfrak{S}_n$-rep with symmetric function

$$h_{n-\ell}\cdot h_m \cdot d_{\ell-m}$$

(that is, the induction

derangement rep

$$\left(\mathbb{1}_{\mathfrak{S}_{n-\ell}} \otimes \mathbb{1}_{\mathfrak{S}_m} \otimes D_{\ell-m}\right)\Big\uparrow_{\mathfrak{S}_{n-\ell}\times\mathfrak{S}_m\times\mathfrak{S}_{\ell-m}}^{\mathfrak{S}_n}$$
)

... and the exact same holds for the q-analogue $k\mathcal{F}_n^{(q)}$

but replacing ...

- $\mathfrak{S}_n$-irreducibles $\rightsquigarrow$ $GL_n(\mathbb{F}_q)$ unipotent irreducibles

- induction $\mathfrak{S}_a \times \mathfrak{S}_b$ to $\mathfrak{S}_{a+b}$

  $\rightsquigarrow$ parabolic induction $GL_a \times GL_b$ to GL_{a+b}

In other words, the q-analogy runs perfectly here!

Proof ideas:

- In bottom of filtration, $kF_{\geq n} \cong k\mathfrak{S}_n$ = regular rep, and can construct m-eigenvectors for x on $k\mathfrak{S}_n$ by inducing $(\mathbb{1}_{\mathfrak{S}_m} \otimes -)\Big\uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}}^{\mathfrak{S}_n}$ nullvectors for x on $k\mathfrak{S}_{n-m}$

- Then use $h_{1^n} = d_n + h_1 d_{n-1} + h_2 d_{n-2} + \cdots + h_{n-1} d_1 + h_n$

  to show nullspace must carry d_n,

  m-eigenspace must carry $h_m d_{n-m}$.

- j-eigenspace for x on $F_{\geq \ell}/F_{\geq \ell+1}$ is $\mathfrak{S}_n$-isomorphic to

$$\left(\begin{matrix}j\text{-eigenspace for } x\\ \text{on } k\mathfrak{S}_\ell\end{matrix}\right) \otimes \mathbb{1}_{\mathfrak{S}_{n-\ell}} \Bigg\uparrow_{\mathfrak{S}_\ell \times \mathfrak{S}_{n-\ell}}^{\mathfrak{S}_n} \rightsquigarrow \left(h_m \cdot d_{\ell-m}\right) \cdot h_{n-\ell}$$

What's next?

- Invariant theory for $\mathfrak{S}_n$ on $k\mathcal{T}_n$? tree space
  (Commins, ongoing)

- Brauner, Commins and Summer 2022 REU students brought (type A) Hecke algebra $\mathcal{H}_n(q)$ acting on flags into the q-analogue story, hoping to gain leverage on symmetrized shuffling operators, like

  random-to-random shuffling $= (R2T)^t \circ R2T \ : \ \mathbb{Q}\mathfrak{S}_n \longrightarrow \mathbb{Q}\mathfrak{S}_n$

- What other LRBs with symmetry are out there?
  Symmetric CAT(0) cube complexes?

Thanks for
your attention!