Counting trees and nilpotent endomorphisms

(based on Tom Leinster's
arXiv:1912.12562)

U. Minnesota
Combinatorics Seminar
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1. Cayley's formula counting trees
& reformulations

2. Transients/recurrents
& Joyal's proof

3. Fitting's Lemma

4. Fine-Herstein Theorem
counting nilpotents

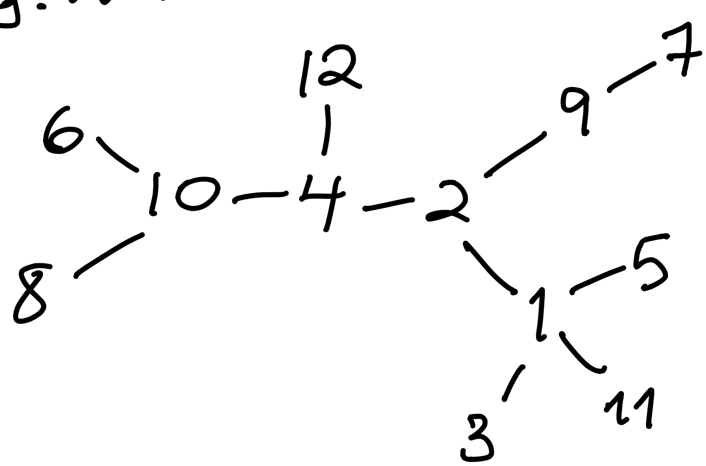
5. Leinster's proof

1. Cayley's tree formula & reformulations

THEOREM (Borchardt 1860, Cayley 1889)

$$\#\left\{\text{trees on vertex set } [n]:=\{1,2,\dots,n\}\right\} = n^{n-2}$$

e.g. $n=12$

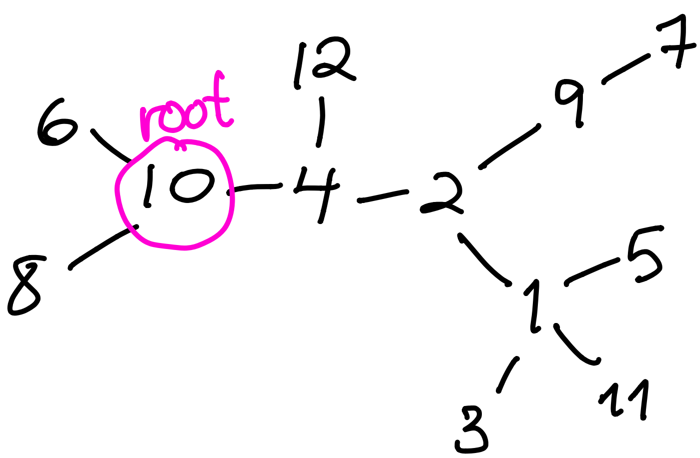


There are $\underset{n^{n-2}}{\underset{\parallel}{12^{10}}}$ of these.

A reformulation:

THEOREM

$$\#\left\{\begin{array}{c}\text{(vertex-)rooted} \\ \text{trees on } [n]\end{array}\right\} = n \cdot n^{n-2} = n^{n-1}$$



So there are 12^{11} of these.

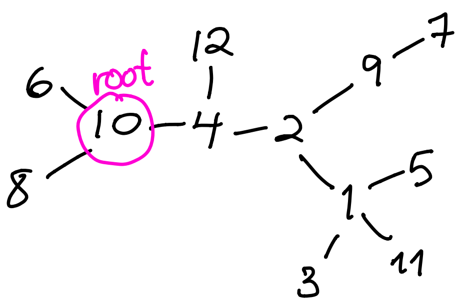
Another reformulation:

THEOREM

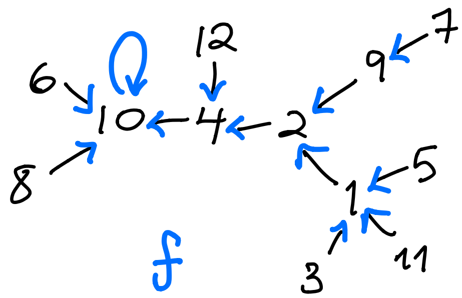
$$\#\left\{\begin{array}{c}\text{eventually constant}\\ \text{endofunctions}\\ f:[n]\to[n]\end{array}\right\} = n^{n-1}$$

because there is an easy bijection

$$\left\{\begin{array}{c}\text{rooted trees}\\ \text{on } [n]\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{eventually constant}\\ f:[n]\to[n]\end{array}\right\}$$



rooted tree



eventually constant f

Or equivalently,

since there are n^n

endofunctions $f : [n] \to [n]$ total...

THEOREM

For any finite set X,

$$\mathrm{Prob}\left(\begin{array}{c} f : X \to X \\ \text{is eventually} \\ \text{constant} \end{array} \right) = \frac{1}{\# X}$$

$$\left(\begin{array}{l} \text{since if } n = \# X \\ \text{then LHS} = \dfrac{n^{n-1}}{n^n} = \dfrac{1}{n} \end{array} \right)$$

2. Transients / recurrents & Joyal's proof (1981)

One more reformulation...

THEOREM

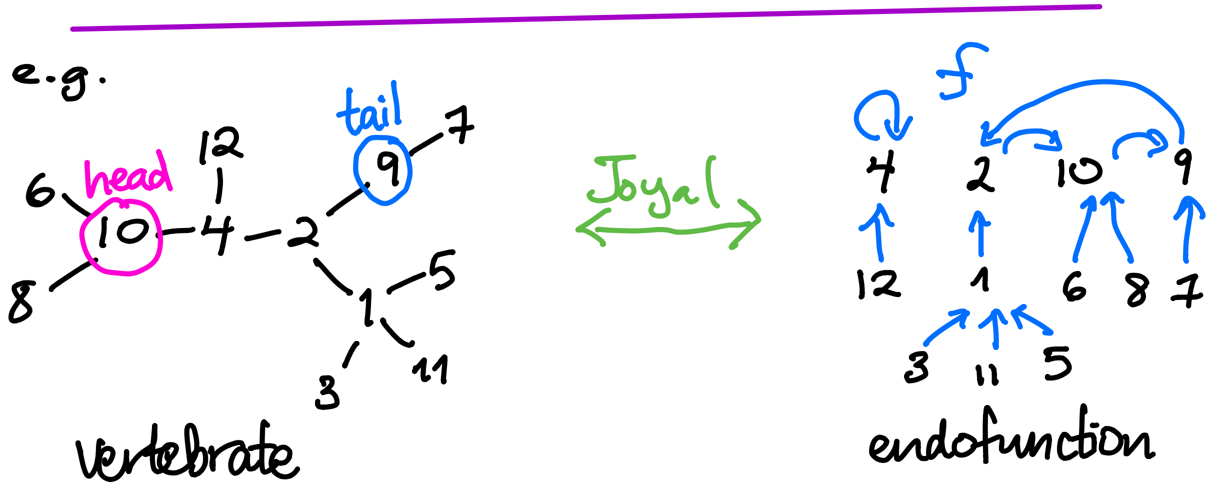
$$\#\left\{ \begin{array}{l} \text{vertebrates on } [n] \\ := \text{trees with a choice of head vertex and tail vertex} \end{array} \right\} = n \cdot n \cdot n^{n-2} = n^n$$

Joyal bijects these

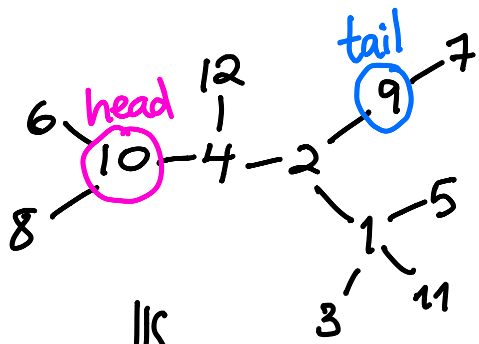
$$\| $$

$$\#\left\{ \begin{array}{c} \text{all endofunctions} \\ f : [n] \to [n] \end{array} \right\}$$

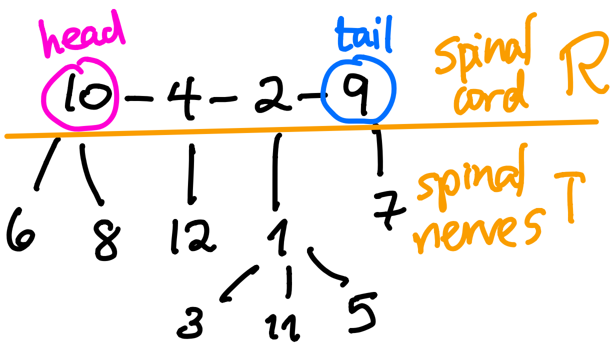
e.g.



$$\{\text{vertebrates on } [n]\} \overset{\text{Joyal}}{\longleftrightarrow} \{\text{all endofunctions } f : [n] \to [n]\}$$



$\cong$

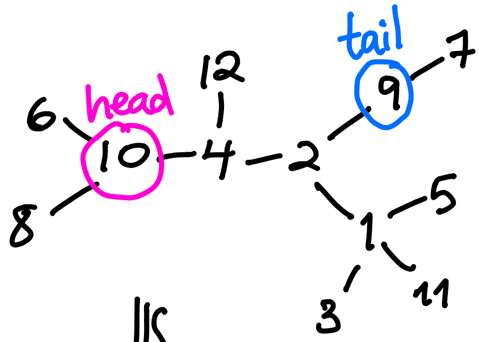


$$\{\text{vertebrates on } [n]\}$$

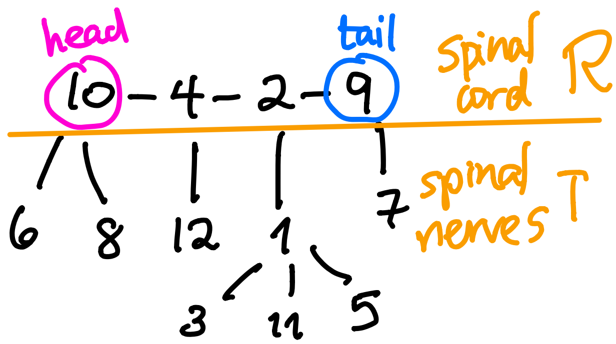
$$\Updownarrow$$

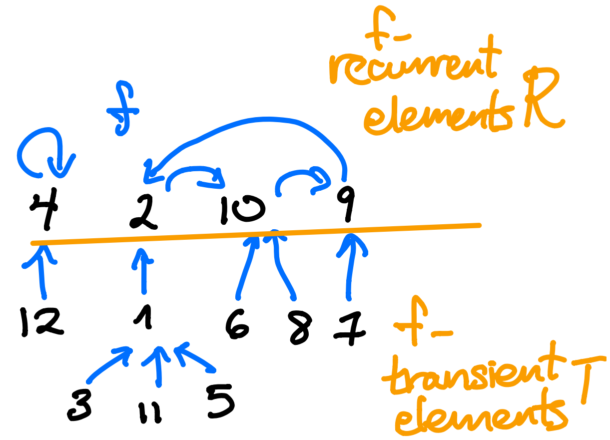
$\{$ decompositions $[n] := R \sqcup T$
with a linear order on R,
and T a collection of trees rooted at elements of R $\}$

$\{\text{vertebrates on } [n]\} \overset{\text{Joyal}}{\longleftrightarrow} \{\text{all endofunctions } f : [n] \to [n]\}$



$\cong$





$\{\text{vertebrates on } [n]\}$

$\updownarrow$

$\{$decompositions $[n] := R \sqcup T$ with a linear order on R, and T a collection of trees rooted at elements of $R\}$

$\{\text{endofunctions on } [n]\}$

$\updownarrow$

$\{$decompositions $[n] := R \sqcup T$ with a permutation of R, and T a collection of trees rooted at elements of $R\}$

Need for every subset $R \subseteq [n]$, a bijection

$$\left\{ \begin{matrix} \text{linear} \\ \text{orders} \end{matrix} \text{ on } R \right\} \leftrightarrow \left\{ \begin{matrix} \text{permutations} \\ \text{of } R \end{matrix} \right\}$$

So just pick a reference linear order $(r_1, r_2, \ldots, r_{\#R})$ for R and biject

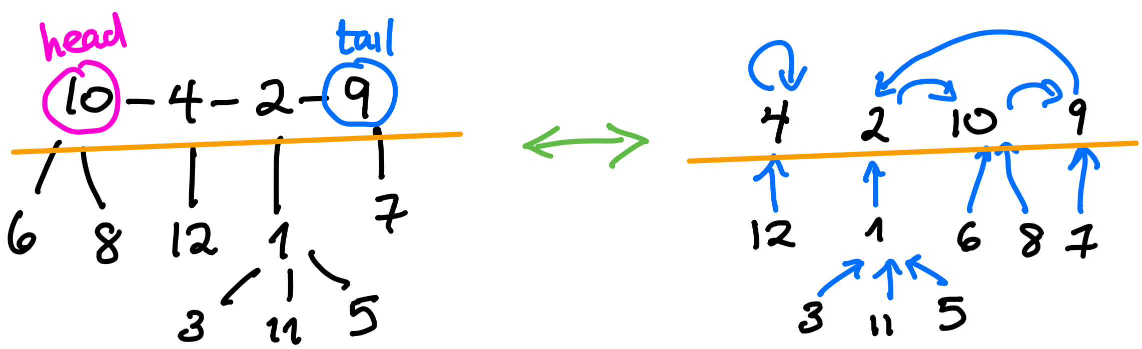
$$\left(r_{\sigma(1)}, r_{\sigma(2)}, \ldots, r_{\sigma(\#R)}\right) \longleftrightarrow \begin{pmatrix} r_1 & r_2 & \cdots & r_{\#R} \\ r_{\sigma(1)} & r_{\sigma(2)} & \cdots & r_{\sigma(\#R)} \end{pmatrix}$$

---

head $\quad$ tail

$$10 - 4 - 2 - 9 \quad \longleftrightarrow \quad \begin{pmatrix} 2 & 4 & 9 & 10 \\ 10 & 4 & 2 & 9 \end{pmatrix} \quad \leftarrow \text{reference order on } R = \{2,4,9,10\}$$

$$= (4)(2\ 10\ 9)$$



3. Fitting's Lemma (1930's)

X a finite dimensional vector space
and $f : X \to X$ in $\mathrm{End}(X)$

gives rise to a unique
f-stable decomposition

$$X = V \oplus W$$

with $f|_V$ invertible, $f|_W$ nilpotent

$\in \mathrm{Aut}(V)$ $\in \mathrm{Nilp}(V)$

$= GL(V)$

Fitting's lemma

X a finite dimensional vector space and $f : X \to X$ in $\mathrm{End}(X)$ gives rise to a unique f-stable decomposition

$$X = V \oplus W$$

with $f|_V$ invertible, $f|_W$ nilpotent

---

proof: These chains stabilize in $\leq \dim(X)$ steps:

$$X \supseteq \mathrm{im}(f) \supseteq \mathrm{im}(f^2) \supseteq \dots \mathrm{im}(f^\infty)$$

$$\{0\} \subseteq \ker(f) \subseteq \ker(f^2) \subseteq \dots \ker(f^\infty)$$

because any equality persists thereafter.

Fitting's lemma

X a finite dimensional vector space and $f : X \to X$ in $\mathrm{End}(X)$

gives rise to a unique f-stable decomposition

$$X = V \oplus W$$

with $f|_V$ invertible, $f|_W$ nilpotent

---

proof: Once they stabilize...

$$\begin{array}{ccccccccccc}
X & \supseteq & \mathrm{im}(f) & \supseteq & \mathrm{im}(f^2) & \supseteq & \cdots & \mathrm{im}(f^\infty) & =: & V \\
 & & & & & & & & & \oplus \\
\{0\} & \subseteq & \ker(f) & \subseteq & \ker(f^2) & \subseteq & \cdots & \ker(f^\infty) & =: & W \\
 & & & & & & & & & \parallel \\
 & & & & & & & & & X
\end{array}$$

dims sum to $\dim X$ via rank-nullity formula

$\mathrm{im}(f^\infty) \cap \ker(f^\infty) = \{0\}$

$\blacksquare$

4. The Fine-Herstein Theorem

Recall Cayley's Theorem was equivalent to saying for all finite sets X

$$\mathrm{Prob}\left(f : X \to X \text{ is eventually constant}\right) = \frac{1}{\# X}$$

THEOREM (Fine & Herstein 1958)

For all finite vector spaces X (so $X \cong \mathbb{F}_q^n$),

$$\mathrm{Prob}\left(\begin{array}{c}\text{linear map} \\ f : X \to X \\ \text{is eventually} \\ \text{constant,} \\ \text{i.e. nilpotent}\end{array}\right) = \frac{1}{\# X}$$

In other words,

$$\#\left\{\begin{array}{l}\text{nilpotent linear} \\ \text{maps } f : \mathbb{F}_q^n \to \mathbb{F}_q^n\end{array}\right\} = \frac{q^{n \times n}}{q^n} = q^{n(n-1)}$$

5. Leinster's proof (of Fine-Herstein Thm)

To prove

$$\mathrm{Prob}\left(\underset{\text{is nilpotent}}{f \in \mathrm{End}(X)} \right) = \frac{1}{\#X}$$

Leinster gives a bijection

$$\underset{\substack{\text{nilpotent} \\ \text{linear maps } N: X \to X}}{\mathrm{Nilp}(X)} \times X \longrightarrow \underset{\substack{\text{all linear} \\ \text{maps } f: X \to X}}{\mathrm{End}(X)}$$

But analogous to Joyal's choice of a reference linear order on each subset $R \subset [n]$, he needs a choice for every subspace $V \subset X$ of

- a reference ordered basis $(v_1, v_2, \ldots, v_k)$ of V,
- a reference complement space $V^\perp$ with $X = V \oplus V^\perp$

no counterpart for $R \subseteq [n]$
where $[n] = R \sqcup R^\perp$
forces $R^\perp = [n] \setminus R$

GOAL: A bijection

$$
\begin{tikzcd}[column sep=large, row sep=huge]
\mathrm{Nilp}(X) \times X \arrow[r, "\sim"] \arrow[dr, dashed, "\substack{\text{a bijection}\\ \text{here}\\ \text{would}\\ \text{suffice}}"'] & \mathrm{End}(X) \arrow[d, equal, "\substack{\text{Fitting's}\\ \text{Lemma}}"] \\
 & \left\{ \begin{array}{c} \text{decompositions} \\ X = V \oplus W \\ \text{with } g \in \mathrm{Aut}(V) \quad n \in \mathrm{Nilp}(W) \end{array} \right\}
\end{tikzcd}
$$

REVISED GOAL: A bijection

$$\mathrm{Nilp}(X)\times X \xrightarrow{\sim} \left\{ \begin{array}{c} \text{decompositions} \\ X = V \oplus W \\ \text{with } g\in \mathrm{Aut}(V) \circlearrowleft V,\ n\in \mathrm{Nilp}(W) \circlearrowleft W \end{array} \right\}$$

Given $(N, v) \in \mathrm{Nilp}(X)\times X$,

- let $k :=$ smallest power with $N^k(v)=0$

  and let $V := \mathrm{span}\{v, N(v), N^2(v), \ldots, N^{k-1}(v)\}$

  an N-stable subspace, annihilated by N^k.

- Since N acts nilpotently on V, its minimal polynomial is some power of x, dividing x^k, so equal to x^k, by definition of k.

- Hence $(v, N(v), N^2(v), \ldots, N^{k-1}(v))$ is an ordered basis for V, and we can define $g\in \mathrm{Aut}(V)$ by

$$\begin{array}{ccccccccc} & (v_1, & v_2, & v_3, & \ldots, & v_k) \\ g & \downarrow & \downarrow & \downarrow & & \downarrow \\ & (v, & N(v), & N^2(v), & \ldots, & N^{k-1}(v)) \end{array}$$

the chosen reference ordered basis for V

REVISED GOAL: A bijection

$$\mathrm{Nilp}(X)\times X \xrightarrow{\sim} \left\{ \begin{array}{c} \text{decompositions} \\ X = \underset{\substack{\circlearrowleft \\ g\,\in\,\mathrm{Aut}(V)}}{V} \oplus \underset{\substack{\circlearrowleft \\ n\,\in\,\mathrm{Nilp}(W)}}{W} \\ \text{with} \end{array} \right\}$$

Given (N,v) we found $V \circlearrowleft g\in \mathrm{Aut}(V)$.

For the rest, consider N acting on

$$X = V \oplus V^{\perp}$$

the chosen reference complement of V

$$\begin{array}{ccc} & & \begin{array}{cc} v, N(v), \dots, N^{k-1}(v) & \qquad\qquad \end{array} \\ \begin{array}{c} V \\ \\ \oplus \\ \\ V^{\perp} \end{array} & \begin{array}{c} v \\ N(v) \\ \vdots \\ N^{k-1}(v) \\ \\ \end{array} & \left[\begin{array}{c|c} \begin{matrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots & 1 \\ 0 & & & 0 \end{matrix} & N_{V,V^{\perp}} \\ \hline 0 & N_{V^{\perp},V^{\perp}} \end{array}\right] \end{array}$$

Considering N acting on

$$X = V \oplus V^{\perp}$$

$$\begin{array}{ccc}
 & & \begin{array}{cc} v, N(v), \dots, N^{k-1}(v) & \phantom{N_{V,V^\perp}} \end{array} \\
\begin{array}{c} V \\ \\ \oplus \\ \\ V^{\perp} \end{array} & \begin{array}{c} v \\ N(v) \\ \vdots \\ N^{k-1}(v) \\ \\ \\ \end{array} & \left[\begin{array}{c|c}
\begin{matrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & 0 \end{matrix} & N_{V,V^{\perp}} \\ \hline
0 & N_{V^{\perp},V^{\perp}}
\end{array}\right]
\end{array}$$

an arbitrary nilpotent in $\mathrm{Nilp}(V^{\perp})$

an arbitrary linear map $V^{\perp} \xrightarrow{\varphi} V$

So far we achieved

$$(N, v) \in \mathrm{Nilp}(X) \times X$$

$$(N, v) \longmapsto \left(\begin{array}{c} X = V \oplus V^{\perp} \\ g \in \mathrm{Aut}(V) \qquad N_{V^{\perp}, V^{\perp}} \in \mathrm{Nilp}(V^{\perp}) \\ \underline{\text{plus}} \;\; N_{V V^{\perp}} : V^{\perp} \to V \end{array} \right)$$

which looks almost, but not quite right,

since we really want

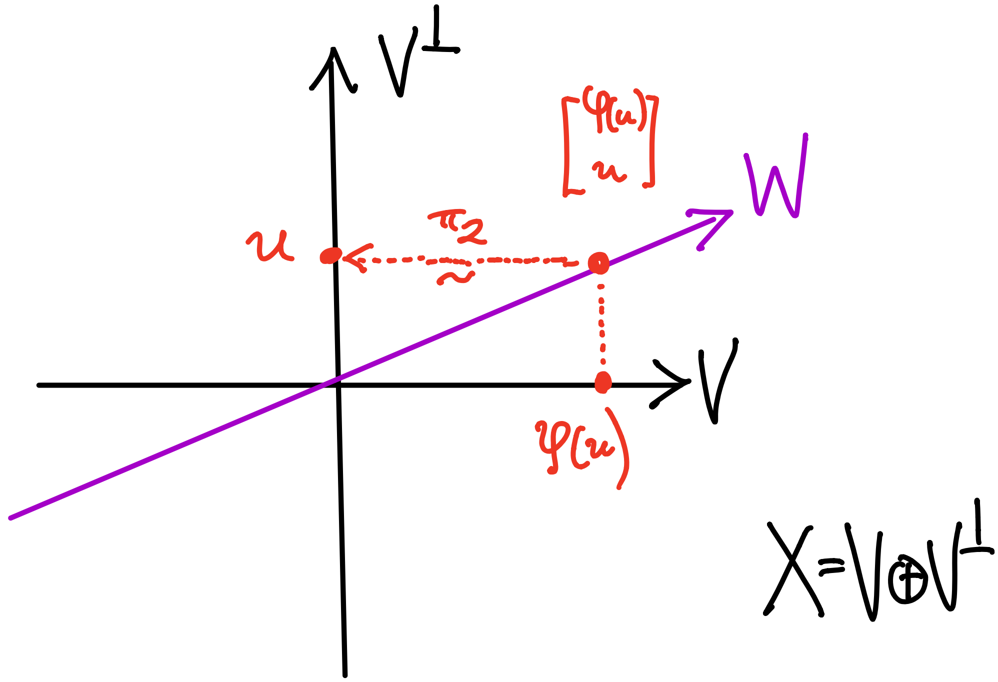
$$(N, v) \longmapsto \left(\begin{array}{c} X = V \oplus W \\ g \in \mathrm{Aut}(V) \qquad n \in \mathrm{Nilp}(W) \end{array} \right)$$

LINEAR ALGEBRA FACT:

$$\left\{ \begin{array}{c} \text{linear maps} \\ V^\perp \xrightarrow{\varphi} V \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{complements } W \\ \text{for } V \text{ in} \\ V \oplus V^\perp =: X \end{array} \right\}$$

$$\varphi \longmapsto W := \mathrm{graph}(\varphi) = \left\{ \begin{bmatrix} \varphi(u) \\ u \end{bmatrix} : u \in V^\perp \right\}$$

(along with an isomorphism $W \xrightarrow[\sim]{\pi_2} V^\perp$)



So now we can fix ...

$$(N, v) \longmapsto \left(\begin{array}{l} X = \underset{\substack{\circlearrowleft \\ g \in \mathrm{Aut}(V)}}{V} \oplus \underset{\substack{\circlearrowright \\ N_{V^\perp, V^\perp} \in \mathrm{Nilp}(V^\perp)}}{V^\perp} \\[2ex] \text{plus } N_{V V^\perp} : V^\perp \to V \end{array} \right)$$

by replacing ...

- $N_{V, V^\perp}$ by $W = \mathrm{graph}(N_{V, V^\perp}) \cong V^\perp$
- $N_{V^\perp, V^\perp} \in \mathrm{Nilp}(V^\perp)$ by the corresponding $n \in \mathrm{Nilp}(W)$

$$(N, v) \longmapsto \left(X = \underset{\substack{\circlearrowleft \\ g \in \mathrm{Aut}(V)}}{V} \oplus \underset{\substack{\circlearrowright \\ n \in \mathrm{Nilp}(W)}}{W} \right)$$

... and check it's all reversible! $\blacksquare$

REMARKS:

- Leinster's preprint is
  - only 5 pages (!)
  - beautifully written
  - has more history and useful comments

- His proof should lend itself to more geometry, maybe of nilpotent cone

$$\mathrm{Nilp}(X) \subset \mathrm{End}(X) \; ?$$

Thanks for your attention!

(... and contact me or Chris Fraser if you would like to speak.)