Sharp representation stability for configurations of points in $\mathbb{R}^d$

Vic Reiner (U. Minnesota)
Patricia Hersh (N.C. State)

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OUTLINE:

1. Rep'n stability

2. Church's Thm. for $H^i(\mathrm{Conf}_n X)$

3. Sharpening for $X = \mathbb{R}^d$

4. The crux

5. Constraints on the characters

1. Representation Stability

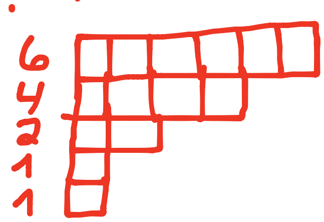
$\mathfrak{S}_n$ = symmetric group on n letters

has (complex, finite dimensional) irreducible representations $\{\chi^\lambda\}$ indexed by partitions of n

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell)$$

$$|\lambda| = \lambda_1 + \dots + \lambda_\ell = n$$

e.g. $\lambda = 64211$



EXAMPLE: $n=3$

$$\chi^{\ydiagram{3}} = \text{trivial } \mathfrak{S}_3\text{-rep'n on } \mathbb{C}^1$$

$$\chi^{\ydiagram{1,1,1}} = \text{sgn } \mathfrak{S}_3\text{-rep'n on } \mathbb{C}^1$$

$$\chi^{\ydiagram{2,1}} = \underset{\underset{\mathfrak{S}_3}{\circlearrowleft}}{\mathbb{C}^3} \Big/ \mathbb{C}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\mathfrak{S}_3$ permuting coordinates

DEF: Say $\mathfrak{S}_n$-reps $\{V_n\}_{n=1,2,\ldots}$

stabilize by n_0 if the

unique decomposition

$$V_{n_0} = \sum_{\substack{\text{partitions} \\ \lambda \text{ of } n_0}} c_\lambda \chi^{\boxed{\lambda}}$$

determines all the rest for $n \geq n_0$ via

$$V_n = \sum_{\substack{\text{partitions} \\ \lambda \text{ of } n_0}} c_\lambda \chi^{\boxed{\lambda}\underbrace{\square\cdots\square}_{n-n_0}}$$

(Say $\{V_n\}$ stabilizes sharply at n_0 if this n_0 is smallest with this property.)

EXAMPLE:

$\{V_n = \mathbb{C}^n\}$ stabilizes sharply at $n_0 = 2$

$\circlearrowleft$

$\mathfrak{S}_n$ permuting coordinates

Since $V_n = \mathbb{C}^n = \mathbb{C}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \oplus \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}^{\perp}$

$$= \chi^{\overbrace{\square\square\cdots\square}^{n}} + \chi^{\begin{smallmatrix}\square & \overbrace{\square\cdots\square}^{n-2}\\ \square & \end{smallmatrix}}$$

REMARK: Lots of examples, theory, variations have been developed by Church, Farb, Ellenberg, Sam, Snowden, Nagpal, Putman, Wilson, ...

2. Church's example

DEF: X a topological space

$\mathrm{Conf}(n,X)$ = configuration space of n distinct ~~labelled~~/ordered points in X

$$= \{(x_1,\dots,x_n) : x_i \neq x_j \text{ for } 1 \le i < j \le n\}$$

$$= X^n \setminus \underbrace{\bigcup_{1 \le i < j \le n} \{x_i = x_j\}}_{\text{thick diagonal in } X^n}$$

$\mathfrak{S}_n$ acts on $\mathrm{Conf}(n, X)$ permuting coordinates and on $H^i(\mathrm{Conf}(n, X))$ with $\mathbb{C}$-coefficients.

THM (Church 2011)

Let X be a connected, orientable d-manifold with $d \geq 2$, and $H^\bullet(X)$ finite-dimensional

Fixing $i \geq 0$, $\{V_n = H^i(\mathrm{Conf}(n, X))\}$ as $\mathfrak{S}_n$-rep'ns

- vanish unless $d-1$ divides i

- stabilize by $n_0 = \begin{cases} 2i & \text{if } d \geq 3 \\ 4i & \text{if } d = 2 \end{cases}$.

EXAMPLE: $i=1 \quad d=2$

$$H^1(\mathrm{Conf}(n,\mathbb{R}^2)) \circlearrowleft \mathfrak{S}_n =$$

$$\begin{cases} 0 & n=1 \\ \chi^{(2)} & n=2 \\ \chi^{(2,1)} + \chi^{(3)} & n=3 \\ \hdashline \chi^{(2,2)} + \chi^{(3,1)} + \chi^{(4)} & n=4 \; (=n_0) \\ \chi^{(n-2,2)} + \chi^{(n-1,1)} + \chi^{(n)} & \text{for } n \geq 5 \end{cases}$$

3. Sharpening for $X = \mathbb{R}^d$

THM (Hersh-R. 2014):

Let $d \geq 2$. Fixing $i \geq 0$,

$\{H^i(\mathrm{Conf}(n, \mathbb{R}^d))\}$ as $\mathfrak{S}_n$-rep'ns

- vanish unless $d-1$ divides i
- stabilizes sharply at $n_0 = \begin{cases} \frac{3}{d-1} \cdot i & \text{if } d \text{ odd} \\ 1 + \frac{3}{d-1} i & \text{if } d \text{ even} \end{cases}$

(cf. previous $\left\{\begin{matrix} 2i & \text{if } d \geq 3 \\ 4i & \text{if } d = 2 \end{matrix}\right\}$)

Why might we care about $X=\mathbb{R}^d$?

The $X=\mathbb{R}^2=\mathbb{C}^1$ case has

$$\mathrm{Conf}(n,\mathbb{R}^2)=K(PB_n,1)$$ (Eilenberg-Mac Lane space)

for the pure braid group PB_n

$$1\to \underset{\text{pure braid group}}{PB_n} \longrightarrow \underset{\text{braid group}}{B_n} \longrightarrow S_n \longrightarrow 1$$

So $$H^\bullet(\mathrm{Conf}(PB_n,1)) = \underset{\text{group cohomology}}{H^\bullet(PB_n)}$$

(And it also plays a crucial role in the Church-Ellenberg-Farb work on statistics on monic squarefree polynomials $f(T)$ in $\mathbb{F}_q[T]$.)

4. The crux

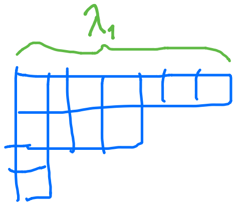
MAIN STABILITY LEMMA (Hemmer 2011):

For an $\mathfrak{S}_m$-rep'n χ, define $\mathfrak{S}_n$-rep'ns

$$M_n(\chi) := \begin{cases} 0 & \text{if } n < m \\ (\chi \otimes \mathbb{1}) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_{n-m}} & \text{if } n \geq m \end{cases}$$

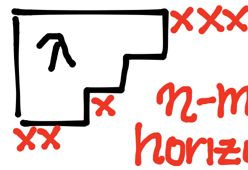
Then $\{M_n(\chi^\lambda)\}$ stabilizes sharply at

$$n_0 = \underbrace{|\lambda|}_{\text{number of cells}} + \underbrace{\lambda_1}_{\text{largest part}}$$

$\lambda =$ 

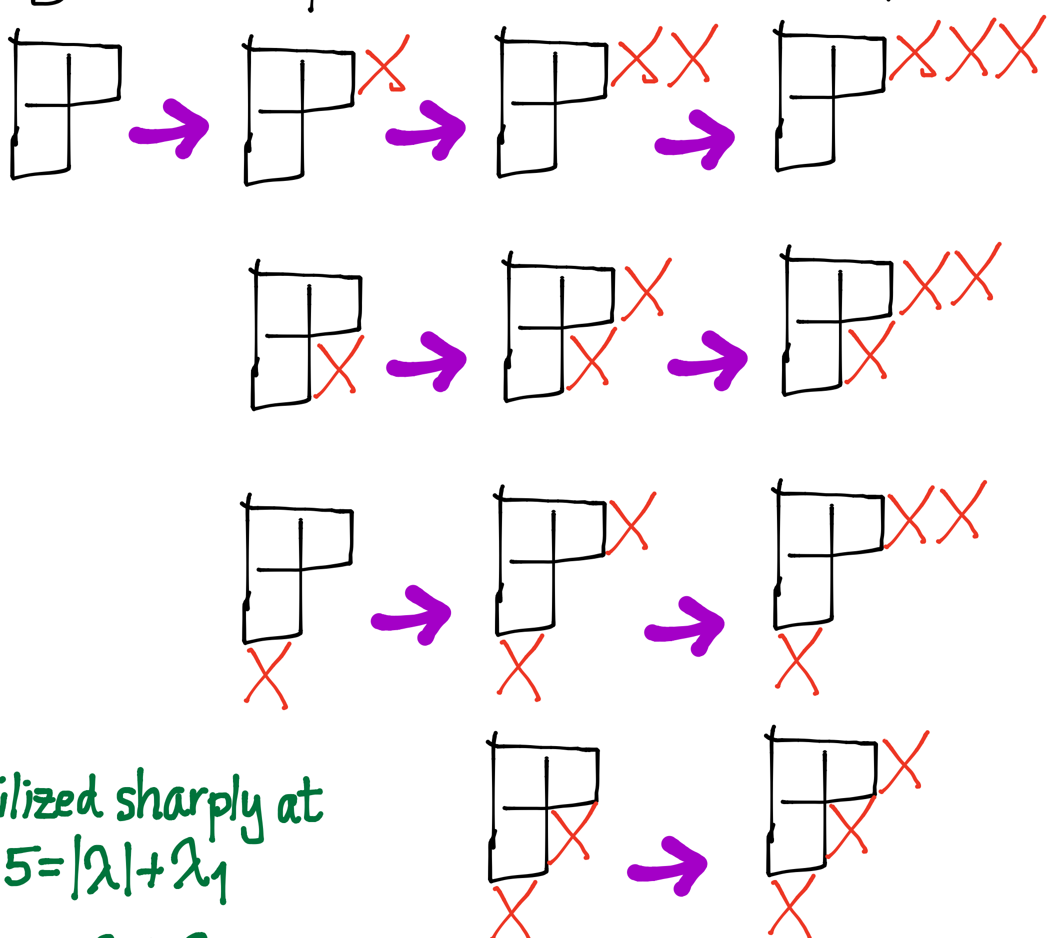
Why? Pieri formula:

$$M_n(\chi^\lambda) = \sum \chi^\mu$$

$\mu =$ 

n-m cell
horizontal strip

$n = 3 \qquad 4 \qquad 5 \qquad 6$

$\lambda =$ 

Stabilized sharply at

$n_0 = 5 = |\lambda| + \lambda_1$

$= 3 + 2$

COROLLARY: For a finite sum

$$\sum_{\mu} \underbrace{c_\mu}_{\in \mathbb{Z}_{>0}} \chi^{\mu}$$

with μ possibly of different sizes,

$$\left\{ M_n\left(\sum_{\mu} c_\mu \chi^{\mu} \right) \right\}$$

stabilizes

sharply at $n_0 = \max\{|\mu| + \mu_1\}$.

EXAMPLES

- $\mathbb{C}^n = M_n\left(\chi^{\overset{\mu}{\square}}\right)$ stabilized at
  $n_0 = 2 = 1+1 = |\mu| + \mu_1$

- We'll see that
  $$H^1(\mathrm{Conf}(n,\mathbb{R}^2)) = M_n\left(\chi^{\underset{\mu}{\square\square}}\right)$$
  explaining why it stabilized at $n_0 = 4$
  $= 2+2$
  $= |\mu| + \mu_1$

- $H^2(\mathrm{Conf}(n,\mathbb{R}^2)) = M_n\left(\chi^{(2,1)} + \chi^{(3,1)}\right)$
  will stabilize at
  $$n_0 = 7 = \max\{\underset{(2,1)}{3+2},\ \underset{(3,1)}{4+3}\}$$

Want $H^i(\mathrm{Conf}(n,\mathbb{R}^d))$ expressed as $M_n(-)$.

THM (Orlik-Solomon 1980 for $d=2$, Sundaram-Welker 1997 for all d)

$H^i(\mathrm{Conf}(n,\mathbb{R}^d))$ vanishes unless $i=(d-1)i'$ in which case it is isomorphic to

$$\begin{cases} M_n(\mathrm{Lie}^{i'}) & \text{if } d \text{ odd} \\ M_n(WH^{i'}) & \text{if } d \text{ even} \end{cases}$$

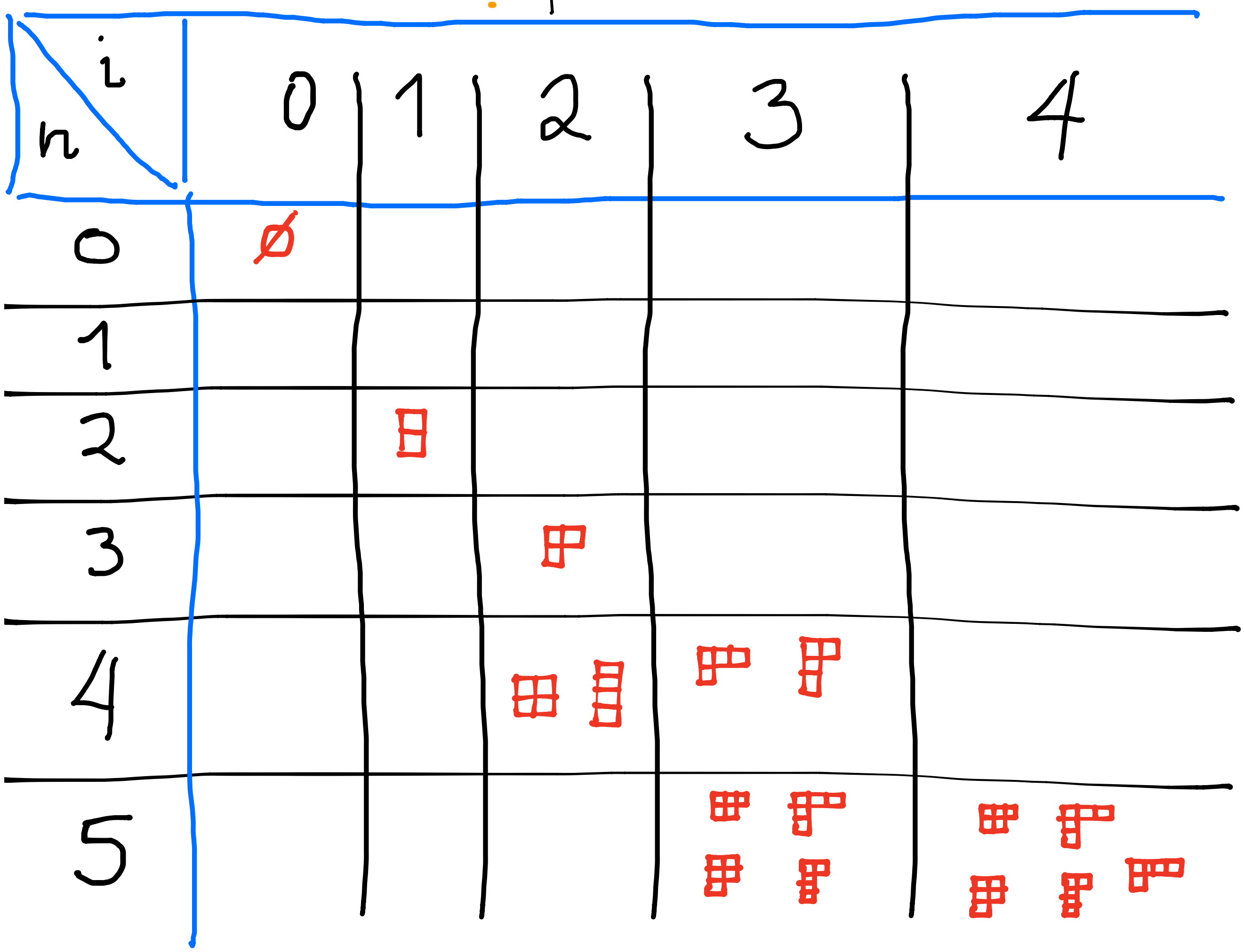
to be described more explicitly.

— CRUX: —

Lie^i, WH^i have expansions $\sum_\mu c_\mu \chi^\mu$

with $|\mu| \leq 2i$ (Church-Farb) and $\mu_1 \leq \begin{cases} i & \text{if } d \text{ odd} \\ 1+i & \text{if } d \text{ even} \end{cases}$ (New!)

Irreducible expansions of Lie^i

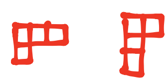
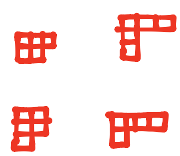
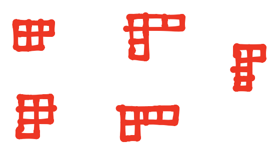


Note μ in column i have

$$|\mu| \leq 2i$$

$$\mu_1 \leq i$$

Irreducible expansions of WH^i

$n \backslash i$	0	1	2	3	4
0	$\emptyset$				
1					
2					
3					
4					
5					

Note μ in column i have

$$|\mu| \leq 2i$$

$$\mu_1 \leq 1 + i$$

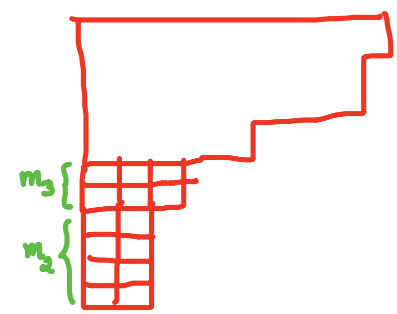
So what are Lie^i, WH^i?

They are $\sum_{\lambda} \left\{ \begin{matrix} Lie_{\lambda} & d \text{ odd} \\ WH_{\lambda} & d \text{ even} \end{matrix} \right\}$

as λ ranges over all partitions with

$\left.\begin{matrix} \cdot \ \operatorname{rank}(\lambda) = \sum_j (\lambda_j - 1) = i. \\ \cdot \ \text{no parts of size 1 in } \lambda \end{matrix}\right\} \Rightarrow |\lambda| \leq 2i$

and if $\lambda = 2^{m_2} 3^{m_3} 4^{m_4} \cdots$



then

$$ch(Lie_{\lambda}) = h_{m_2}[\ell_2] \cdot h_{m_3}[\ell_3] \cdot h_{m_4}[\ell_4] \cdot \ \cdots$$

$$ch(WH_{\lambda}) = e_{m_2}[\pi_2] \cdot h_{m_3}[\pi_3] \cdot e_{m_4}[\pi_4] \cdot \ \cdots$$

Frobenius characteristic

plethysm $f[g]$

l_n and π_n are the
(Frobenius characteristics of the)
representations of $\mathfrak{S}_n$ on

- multilinear part of the free Lie algebra on n symbols
- homology of the proper part of the poset of set partitions of $\{1, 2, \ldots, n\}$

THM: (Hanlon, Stanley 1982)

$$\pi_n = e^{2\pi i/n} \uparrow_{\mathbb{Z}/n\mathbb{Z}}^{\mathfrak{S}_n}$$

$$(\text{and } l_n = \omega(\pi_n))$$

This is enough to bound the μ_1's in their Schur function expansions $\sum_{\mu} s_\mu$:

- Can get bounds on the μ_1's for ℓ_n, π_n from previous THM

- If f_1, f_2 have μ_1 bounded by ℓ_1, ℓ_2 then $f_1 \cdot f_2$ has μ_1 bounded by $\ell_1 + \ell_2$

- If f has μ_1 bounded by ℓ then $h_m[f], e_m[f]$ have μ_1 bounded by $m\ell$

5. Constraints on the characters

We would like to know the irreducible expansions of

$$\mathrm{Lie}^i, \quad WH^i$$

but we don't.

Nevertheless, we do know a few things, e.g.

the $\mathfrak{S}_n$-rep'n component of WH^i

PROP:

$$\deg \mathrm{Lie}^i_n = \deg \underbrace{WH^i_n}_{} =$$

\# of derangements in $\mathfrak{S}_n$ with $n-i$ cycles

fixed point free permutations

$$=: d_n^{n-i}$$

THM (Wiltshire-Gordon's Conj 1):

$$WH^{i}\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \left(WH^{i-1}_{n-1}\downarrow^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}} + WH^{i-1}_{n-2} \right)\uparrow^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{n-2}}$$

(generalizes derangement recurrence

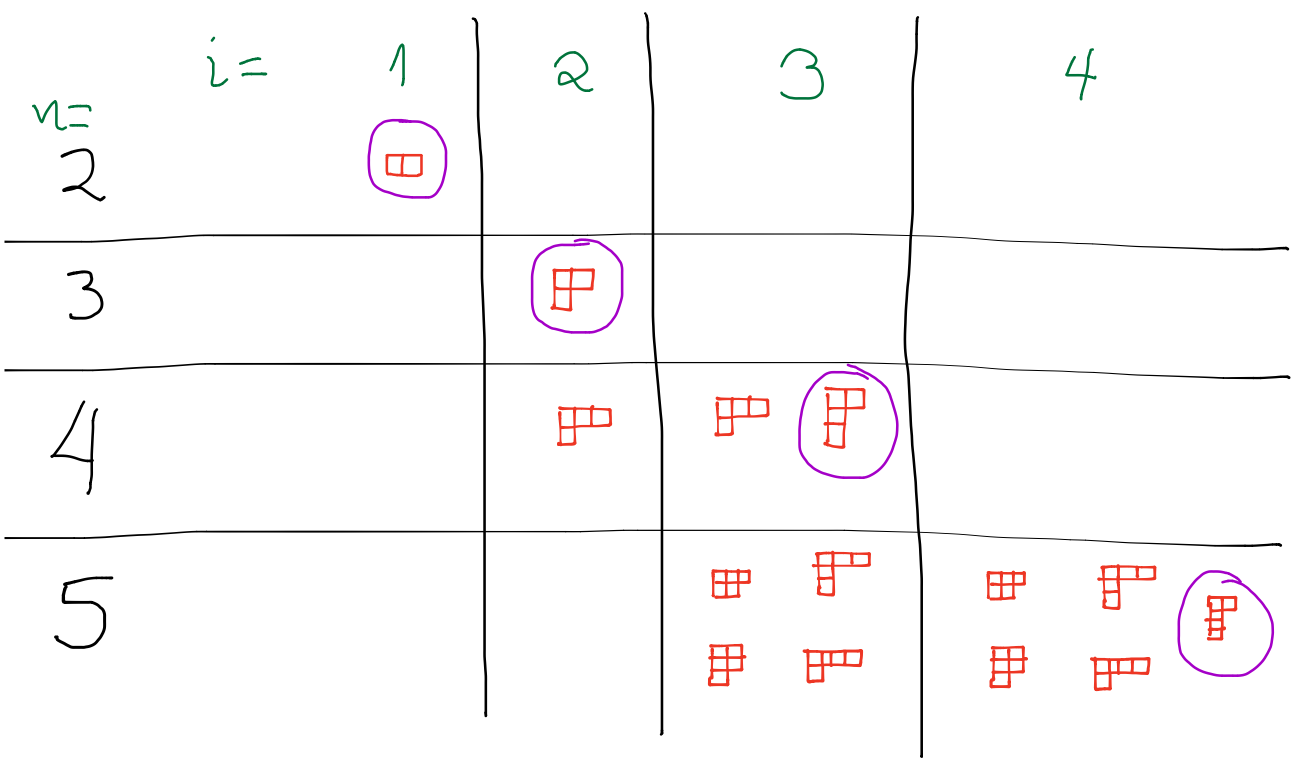
$$d_n^{k} = (n-1)\left(d^{k}_{n-1} + d^{k-1}_{n-2}\right)$$

)

THM (Wiltshire-Gordon's Conj 2):

$$\sum_i (-1)^i WH^i_n = (-1)^{n-1} \chi$$

as virtual $\mathfrak{S}_n$-rep'ns.



Method of proof?

One can collate the symmetric function

$$\sum_{\lambda} WH_{\lambda}\, x^{\mathrm{rank}(\lambda)}\, y^{|\lambda|}$$

into an infinite product, involving the power sum symmetric functions

$$p_1, p_2, p_3, \ldots\ldots$$

where $p_r = x_1^r + x_2^r + x_3^r + \cdots$

- CONJ 1 arises roughly from taking $\partial/\partial p_1$ in the generating function, corresponding to $(-)\downarrow^{\tilde{G}_n}_{\tilde{G}_{n-1}}$

- CONJ 2 arises from setting $x = -1$

Remember the derangement recurrence

$$d_n = n\, d_{n-1} + (-1)^n \;?$$

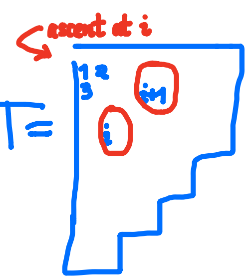
It lifts to this for $\mathrm{Lie}_n := \sum_i \mathrm{Lie}_n^i$:

PROP: $\mathrm{Lie}_n = \mathrm{Lie}_{n-1}\uparrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} + (-1)^n \chi^{(1^n)}$

From this one can fairly easily deduce this:

THM (Webb–R. 2004, related to Désarménien–Wachs 1988)

$$\mathrm{Lie}_n = \sum_{\substack{\text{standard Young}\\ \text{tableaux } T \text{ of size } n \text{ having}\\ \text{first ascent even}}} \chi^{\mathrm{shape}(T)}$$

$T=$ 

(and an analogous result for WH_n)

PROBLEM:

Refine these tableau models for the irreducible expansions of Lie_n, WH_n

to models for Lie_n^i, Lie_λ

WH_n^i, WH_λ^i.

THANK

YOU!