

On the coincidental reflection groups

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1. A few of my favorite
product formulas
2. Reflection group
generalizations with
invariant theory
explanations
3. The coincidental groups
and two theorems

1. Product formulas

Start with three that
generalize the fact that the
symmetric group

$$\mathfrak{S}_n = \left\{ \begin{array}{l} \text{permutations} \\ \omega = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \omega_1 & \omega_2 & \omega_3 & \dots & \omega_n \end{pmatrix} \end{array} \right\}$$

has cardinality

$$|\mathfrak{S}_n| = n! = 1 \cdot 2 \cdot \dots \cdot n$$

$$\sum_{w \in \mathfrak{S}_n} q^{\text{inv}(w)} = [n]_q! \quad \text{Mahonian distribution}$$

$$:= [1]_q [2]_q \cdots [n]_q$$

where $[m]_q := 1 + q + q^2 + \dots + q^{m-1}$

and $\text{inv}(w) := \#\{1 \leq i < j \leq n : w_i > w_j\}$
 = inversion number of w

$n=3$

<u>w</u>	<u>inv(w)</u>	
123	0	} $1 + 2q + 2q^2 + q^3$ $= 1 \cdot (1+q)(1+q+q^2)$ $= [1]_q [2]_q [3]_q$
132	1	
213	1	
231	2	
312	2	
321	3	

$$\sum_{w \in \tilde{G}_n} q^{\#cycles(w)} = q(q+1)(q+2) \cdots (q+(n-1))$$

$$= \sum_k c(n, k) q^k$$

↑ signless Stirling numbers of the 1st kind

$n=3$

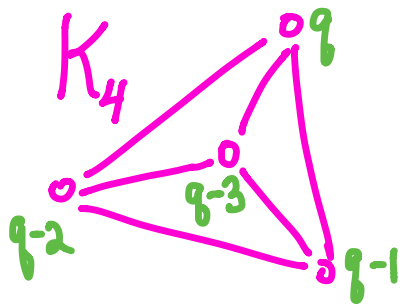
<u>w</u>	<u>#cycles(w)</u>
123 = (1)(2)(3)	3
132 = (1)(23)	2
213 = (12)(3)	2
231 = (123)	1
312 = (132)	1
321 = (13)(2)	2

$$q^3 + 3q^2 + 2q^1 = q(q+1)(q+2)$$

$$\sum_{w \in \mathcal{G}_n} \text{sgn}(w) q^{\#\text{aydes}(w)} = q(q-1)(q-2) \cdots (q-(n-1))$$

$$= \sum_k s(n, k) q^k$$

↑
(signed) Stirling
numbers of the
1st kind



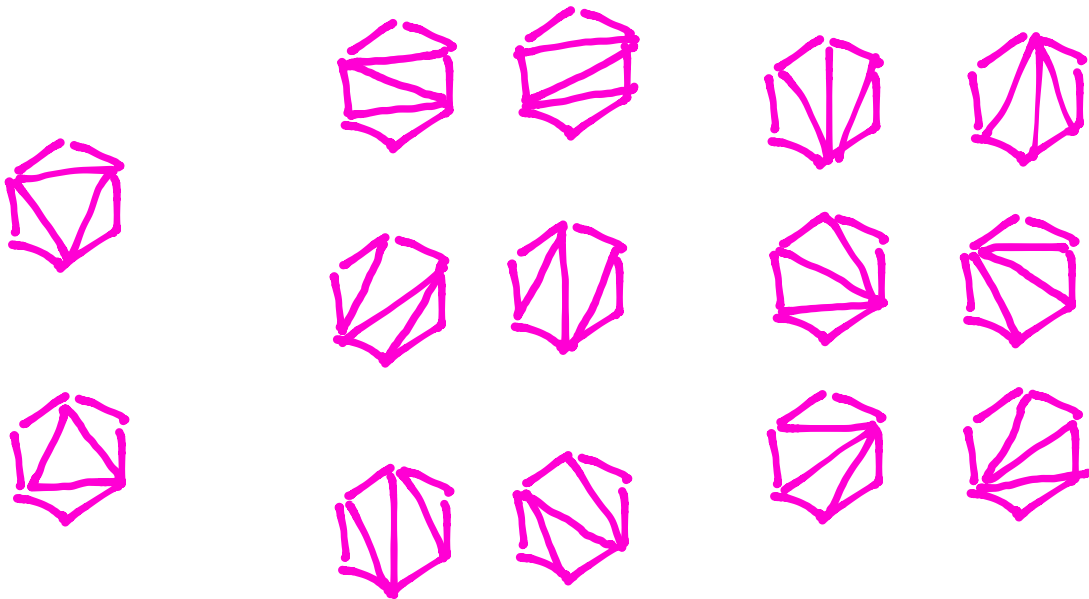
= chromatic polynomial of
complete graph K_n ,
counting proper q -colorings

$$\Downarrow q = -1$$

$$(-1)^n |\mathcal{G}_n| = (-1)(-2) \cdots (-n)$$

$$\begin{aligned}
 \text{\# triangulations} &= \frac{1}{n+1} \binom{2n}{n} \\
 \text{of an } (n+2)\text{-gon} & \\
 =: \text{Catalan number} &= \frac{(n+2)(n+3)\cdots(2n)}{2 \cdot 3 \cdots n}
 \end{aligned}$$

$$n=4 \quad \frac{1}{4+1} \binom{2 \cdot 4}{4} = \frac{6 \cdot 7 \cdot 8}{2 \cdot 3 \cdot 4} = 14$$

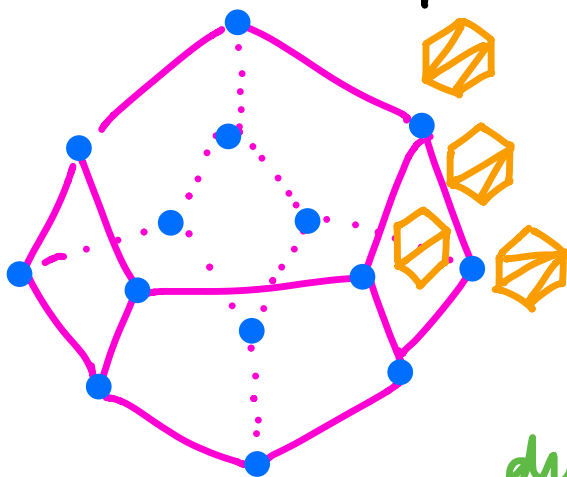


More generally,
 #dissections of
 $(n+2)$ -gon using
 $n-1-k$ diagonals

Kirkman-Cayley
 or 'little' Schröder numbers

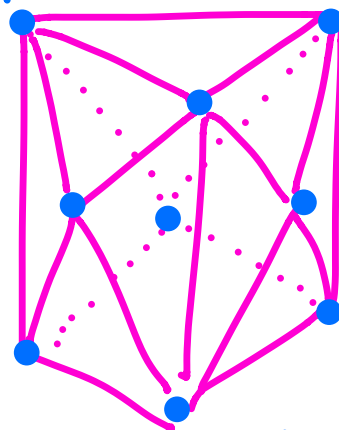
$$= \frac{1}{n} \binom{n}{k} \binom{2n-k}{n-k-1}$$

gives the **f-vector** (= face numbers)
 for the (simple) **associahedron**
 (Stasheff polytope)



simple
 associahedron

dual



simplicial
 associahedron

$$(f_0, f_1, f_2, f_3) = (14, 21, 9, 1)$$

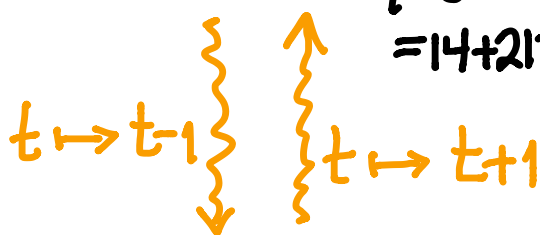
Vertices
 edges
 polygons
 3-face

The corresponding **h-vector** entries are celebrated products too!

f-vector

$$(f_0, f_1, f_2, f_3) = (14, 21, 9, 1) \quad f(t) = \sum_{i=0}^{n-1} f_i t^i$$

$$= 14 + 21t + 9t^2 + t^3$$



h-vector

$$(h_0, h_1, h_2, h_3) = (1, 6, 6, 1) \quad h(t) = \sum_{i=0}^{n-1} h_i t^i$$

$$= 1 + 6t + 6t^2 + t^3$$

$$h_k = \frac{1}{n} \binom{n}{k} \binom{n}{k+1}$$

Narayana numbers

Catalan, Kirkman-Cayley, Narayana numbers also have q -analogues with (essentially) same product formulas, that hide counts for objects with q -cyclic symmetry in their $q = e^{2\pi i/d}$ evaluations. [Cyclic sieving phenomenon; R. Stanton-White 2004]

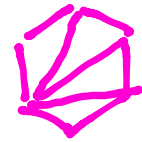
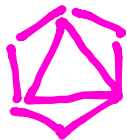
E.g. q -Catalan
$$\frac{[n+2]_q [n+3]_q \cdots [2n]_q}{[2]_q [3]_q \cdots [n]_q}$$

evaluated at $q = \left(e^{\frac{2\pi i}{n+2}}\right)^d$
 counts triangulations of $(n+2)$ -gon
 fixed by $\frac{2\pi}{n} \cdot d$ rotation

$$n=4$$

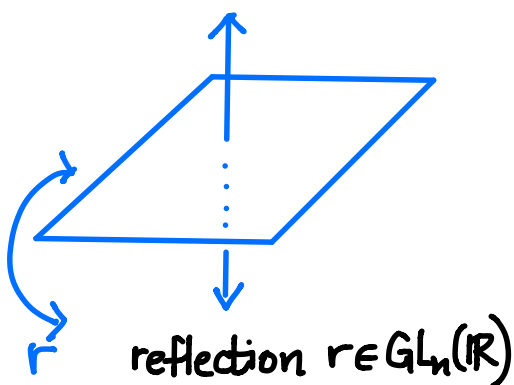
$$\frac{[6]_q, [7]_q, [8]_q}{[2]_q, [3]_q, [4]_q} = 1 + q + q^2 + 2q^3 + q^4 + 2q^5 + q^6 + 2q^7 + q^8 + q^9 + q^{10} + q^{12}$$

$$0 \quad q = e^{\frac{2\pi i}{6}} \quad 2 \quad q = e^{\frac{2\pi i}{3}} \quad 6 \quad q = -1 \quad q = 1 \quad 14$$



2. Reflection group generalizations

A **reflection group** is a finite subgroup $W \subset GL_n(\mathbb{C}) = GL(V)$ generated by **reflections** $r \in W$, which are elements whose fixed space $V^r := \{v \in V : r(v) = v\}$ is a **hyperplane**.



r diagonalizes to

$$\begin{bmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & -1 \end{bmatrix}$$

in $GL_n(\mathbb{C})$

Shephard & Todd (1955) **classified** the irreducible reflection groups into

- one infinite family $G(d, e, n)$
- 34 exceptional groups

on their way to proving this:

THEOREM: When a finite subgroup $W \subset GL_n(\mathbb{C})$ acts on the polynomials $S = \mathbb{C}[x_1, \dots, x_n]$, the W -invariant subalgebra S^W is again a polynomial algebra $S^W = \mathbb{C}[f_1, \dots, f_n]$

\Leftrightarrow W is a **reflection group**

In this case, the **degrees**

$$\begin{array}{ccccccc} d_1 & \leq & d_2 & \leq & \dots & \leq & d_n \\ \parallel & & \parallel & & & & \parallel \\ \deg(f_1) & & \deg(f_2) & & & & \deg(f_n) \end{array}$$

in $S^W = \mathbb{C}[f_1, \dots, f_n]$ are **unique**, and give

$$\begin{aligned} \text{Hilb}(S^W, \mathfrak{f}) &:= \sum_{d=0}^{\infty} q^d \cdot \dim_{\mathbb{C}}(S^W)_d \\ &= \prod_{i=1}^n \frac{1}{1 - q^{d_i}} \end{aligned}$$

EXAMPLE $\mathbb{C}[x_1, \dots, x_n]^{\mathbb{G}_n} = \mathbb{C}[e_1, e_2, \dots, e_n]$

$$\begin{array}{ccccccc} \text{degrees } d_1 & \leq & d_2 & \leq & \dots & \leq & d_n \\ \parallel & & \parallel & & & & \parallel \\ 1 & & 2 & & \dots & & n \end{array}$$

$$\text{and } \text{Hilb}(S^{\mathbb{G}_n}, \mathfrak{f}) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$$

Shephard and Todd deduced this:

COROLLARY: For a finite ref'n group W ,

$$d_1 d_2 \cdots d_n = |W|$$

and more generally,

$$[d_1]_g [d_2]_g \cdots [d_n]_g = \text{Hilb}(S/(f_1, \dots, f_n), g)$$

↳ the **coinvariant algebra**,
a graded version of the
regular rep'n $\mathbb{C}[W]$

when W is a
real ref'n group

$$= \sum_{w \in W} g^{l(w)}$$

$l(w)$ = Coxeter
group length
of w



Shephard & Todd also noted this...

THEOREM: If a reflection group has

$$\sum_{w \in W} q^{\dim(V^w)} = (q+e_1)(q+e_2) \cdots (q+e_n)$$

where the **exponents** (e_1, e_2, \dots, e_n)
can be defined by $e_i = d_i - 1$

EXAMPLE $W = \mathfrak{S}_n$ acting on $V = \mathbb{C}^n$
has exponents $(0, 1, 2, \dots, n-1)$, and so

$$\sum_{w \in \mathfrak{S}_n} q^{\#\text{cycles}(w)} = q(q+1)(q+2) \cdots (q+n-1)$$

since $\dim(V^w) = \#\text{cycles}(w)$

This formula wasn't well-understood until...

THEOREM (Solomon 1963)

W a reflection group has

$(S \otimes \wedge V^*)^W$ an S^W -exterior algebra

on exterior generators df_1, \dots, df_n

where $df := \sum_{j=1}^n \frac{\partial f}{\partial x_j} \otimes x_j$



df_i contributes $q^{e_i} t$

$\text{Hilb}((S \otimes \wedge V^*)^W, q, t) = \prod_{i=1}^n \frac{1 + q^{e_i} t}{1 - q^{d_i}}$

f_i contributes q^{d_i}

The missing pieces are **Molien formulas**:

for finite subgroups $W \subset GL_n(\mathbb{C})$,

$$\frac{1}{|W|} \sum_{\omega \in W} \text{Trace}_V(\omega) = \dim(V^W)$$

$$\frac{1}{|W|} \sum_{\omega \in W} \frac{1}{\det(1 - q\omega)} = \text{Hilb}(S^W, q)$$

$$\frac{1}{|W|} \sum_{\omega \in W} \frac{\det(1 + t\omega)}{\det(1 - q\omega)} = \text{Hilb}((S \otimes V^*)^W, q, t)$$

From here, the **product formulas**
come from **coefficient extraction**.

Later came this...

THEOREM (Orlik & Solomon 1980)

W a reflection group has

$$\sum_{w \in W} \det(w) q^{\dim(V^w)} = \sum_{X} \mu(V, X) q^{\dim X}$$

reflecting hyperplane intersections X

$$= (q - e_1^*) (q - e_2^*) \cdots (q - e_n^*)$$

characteristic polynomial for the reflection arrangement

where $(e_1^*, e_2^*, \dots, e_n^*)$ are the ω -exponents

giving the degrees of an

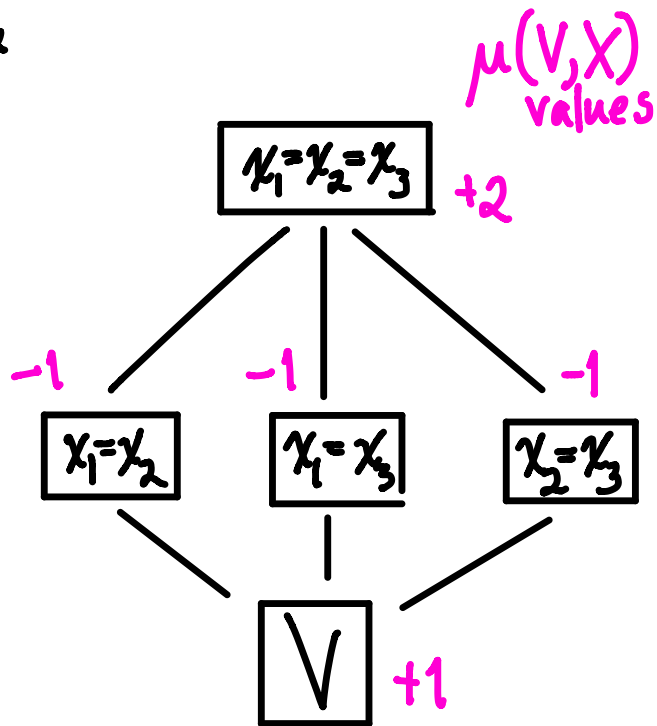
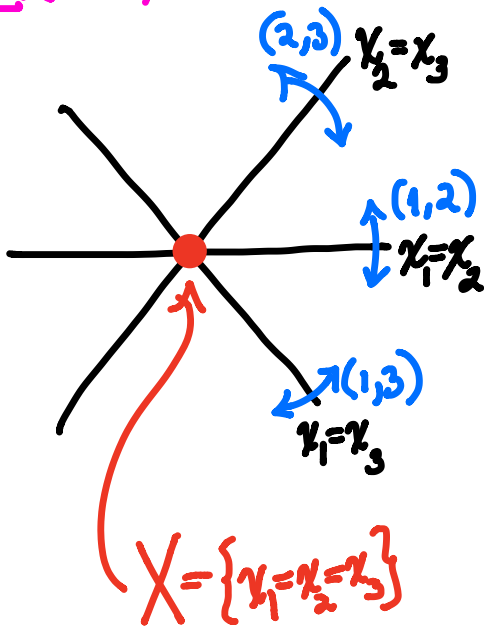
S^W -basis $\Theta_1, \Theta_2, \dots, \Theta_n$ for $(S \otimes V)^W$

i.e. $\Theta_i = \sum_{j=1}^n \Theta_i^{(j)} \otimes y_j$

has polynomial degree e_i^*

For real retn groups W , one has $V \cong V^*$
 and co-exponents $e_i^* = -\text{exponents } e_i$

EXAMPLE $W = \mathfrak{S}_3$ acting on $V = \mathbb{R}^3$



$$\sum_X \mu(V, X) q^{\dim X} = q^3 - 3q^2 + 2q = q(q-1)(q-2)$$

Orlik & Solomon deduced it from this result...

THEOREM W a reflection group has

$(S \otimes \wedge V)^W$ an S^W -exterior algebra
on exterior generators $\theta_1, \theta_2, \dots, \theta_n$



θ_i contributes $q^{e_i^*} u$

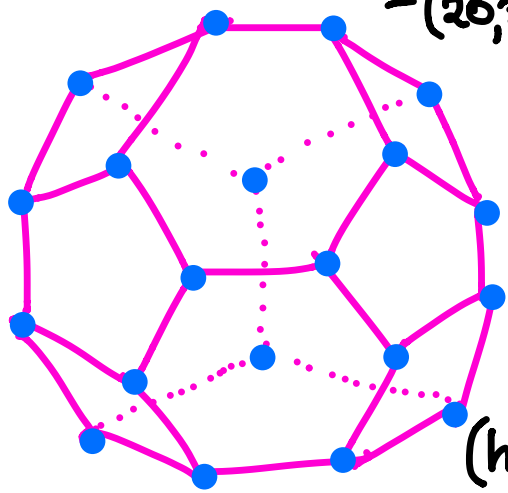
$$\text{Hilb}((S \otimes \wedge V)^W, q, u) = \prod_{i=1}^n \frac{1 + q^{e_i^*} u}{1 - q^{d_i}}$$

f_i contributes q^{d_i}

Catalan, Kirkman-Cayley, Narayana numbers generalize to **real reflection groups** W as **f-vectors** and **h-vectors** for finite type cluster complexes and Cambrian fans (Fomin & Zelevinsky 2001) (Reading 2004)

$W = B_3$

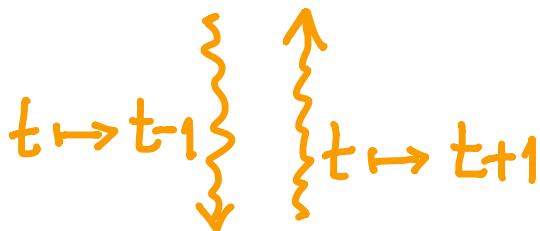
$(f_0, f_1, f_2, f_3) = (20, 30, 12, 1)$



$(h_0, h_1, h_2, h_3) = (1, 9, 9, 1)$

cyclodhedron or Bott-Taubes polytope

$f(t) = 20 + 30t + 12t^2 + t^3$



$h(t) = 1 + 9t + 9t^2 + t^3$

These f_k have a nice q -analogue
with an invariant-theory connection:

THEOREM (Armstrong-R. Rhoades 2015)

A real reflection group W with $h := d_n$

has $f_k = \lim_{q \rightarrow 1} f_k(q)$ where $f_k(q)$ is

$$\frac{\text{Hilb}((S \otimes V^* \otimes V)^{\otimes k})^W}{\text{Hilb}(S^W, q)} \Bigg|_{t = -\frac{h+1}{q}}$$

Coxeter number

(not so well understood, though!)

The $k=0$ case is the q -Catalan number for W
and has a **product formula**

$$f_0(q) = \prod_{i=1}^n \frac{[h+d_i]_q}{[d_i]_q}$$

derived from Solomon's Theorem

EXAMPLE for $W = \tilde{E}_n$,

$$f_0(q) = \frac{[n+2]_q [n+3]_q \cdots [2n]_q}{[2]_q [3]_q \cdots [n]_q} = q\text{-Catalan}$$

The $k=0$ case also was checked ^[Eu & Fu 2006]
to have a **cyclic sieving phenomenon**
for a natural cyclic action on clusters.

However the other $f_k(q)$
outside of the boundary cases

$$k = 0, 1, n-1, n$$

seem to have

- no product formula
- no cyclic sieving phenomenon

... except in the

coincidental

types W ...

3. The coincidental types

Shephard & Todd's
classification dichotomy...

- One infinite family

$G(d, e, n) = n \times n$ monomial
matrices with nonzero
entries in $\sqrt[d]{1}$
and product in $\sqrt[e]{1}$

$$\begin{bmatrix} 0 & 0 & \xi^2 & 0 \\ \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi \\ 0 & \xi^4 & 0 & 0 \end{bmatrix}$$

- 34 exceptions
-

... is deceptive.

all \mathbb{C} ref'n groups

all \mathbb{C} ref'n groups

$$G(d, e, n) \\ n, d, e \geq 2$$

$$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$$

duality groups $\{e_i^*\} = \{h - e_i\}$
(=well-generated)

all \mathbb{C} ref'n groups

$G(d, e, n)$
 $n, d, e \geq 2$

$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$

duality groups $\{e_i^*\} = \{h - e_i\}$
(=well-generated)

$G(e, n)$
 $e \geq 2, n \geq 3$
 $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$

Coxeter
= real

Shephard
= symmetry of
regular
 \mathbb{C} -polytopes

all \mathbb{C} ref'n groups

$G(d, e, n)$
 $n, d, e \geq 2$

$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$

duality groups $\{e_i^*\} = \{h - e_i\}$
(= well-generated)

$G(e, n)$ $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$
 $e \geq 2, n \geq 3$

Shephard
= symmetry of
regular
 \mathbb{C} -polytopes

Coxeter
= real

$D_n = G(2, 2, n)$



all \mathbb{C} ref'n groups

$G(d, e, n)$
 $n, d, e \geq 2$

$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$

duality groups $\{e_i^*\} = \{h - e_i\}$
 (=well-generated)

$G(e, n)$ $G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$
 e_{22}, n_{33}

Coxeter
 = real

$D_n = G(2, 2, n)$



...4... F_4

...5... H_4

Shephard
 = symmetry of
 regular
 \mathbb{C} -polytopes

Coincidental
 = e_i, e_i^* in arithmetic
 progressions

all \mathbb{C} ref'n groups

$$G(d, e, n)$$

$n, d, e \geq 2$

$$G_7, G_{11}, G_{12}, G_{13}, G_{15}, G_{19}, G_{22}, G_{31}$$

duality groups $\{e_i^*\} = \{h - e_i\}$
 (=well-generated)

$$G(e, e, n) \quad G_{24}, G_{27}, G_{29}, G_{33}, G_{34}$$

e_{22}, n_{33}

Coxeter
= real

$$D_n = G(2, 2, n)$$



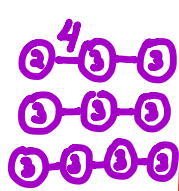
$$\dots 4 \dots F_4$$

$$\dots 5 \dots H_4$$

Shephard
= symmetry of
regular
 \mathbb{C} -polytopes

$$G(d, 1, n)$$

$d \geq 3$



$$\frac{1}{p_0} + \frac{1}{p_1} + \frac{2}{m} > 1$$

Coincidental
= e_i, e_i^* in arithmetic
progressions

$$A_{n-1} = G(1, 1, n)$$

$$B_n = G(2, 1, n)$$

$$I_2(m)$$

$\frac{m}{m}$

$$H_3 \dots 5 \dots$$

DEFINITION:

A reflection group $W \subset GL_n(\mathbb{C})$ is

coincidental

if its exponents and coexponents
form arithmetic progressions

$$(e_1, \dots, e_n) = (e_1, e_1 + a, e_1 + 2a, \dots, e_1 + (n-1)a)$$

$$(e_1^*, \dots, e_n^*) = (1, 1+a, 1+2a, \dots, 1+(n-1)a)$$

with the same

$$\text{gap } a := e_i - e_{i-1} \\ = e_i^* - e_{i-1}^*$$

EXAMPLES

$$\begin{aligned} A_{n-1} = \mathfrak{S}_n \text{ has } (e_1, e_2, \dots, e_{n-1}) \\ = (e_1^*, e_2^*, \dots, e_{n-1}^*) \\ = (1, 2, \dots, n-1) \text{ so } a=1 \end{aligned}$$

$$\begin{aligned} B_n = G(a, 1, n) \text{ has } (e_1, e_2, \dots, e_n) \\ = (e_1^*, e_2^*, \dots, e_n^*) \\ = (1, 3, 5, \dots, 2n-1) \text{ so } a=2 \end{aligned}$$

$$\begin{aligned} G(d, 1, n) \text{ has } (e_1, \dots, e_n) = (d-1, 2d-1, \dots, nd-1) \\ (e_1^*, \dots, e_n^*) = (1, d+1, 2d+1, \dots, (n-1)d+1) \\ \text{so } a=d \end{aligned}$$

THEOREM (R.-Sheper-Sommers 2019)
 For coincidental W ,

$$\text{Hilb}((S \otimes \wedge^r V^* \otimes \wedge^r V)^W, q, t) = \frac{\sigma_r(q^{e_1^*}, \dots, q^{e_n^*}) \prod_{i=1}^r (1 + q^{e_i^*} t) \prod_{i=1}^{n-r} (1 + q^{e_i} t)}{\prod_{i=1}^n (1 - q^{d_i})}$$

σ_r elementary symmetric function

\swarrow $r=0$
 Solomon's
 Thm.

\searrow set $t=0$
 Orlik-Solomon
 Thm.

It is really a **product formula** that depends only on e_1 and a :

$$\text{Hilb}((S \otimes V^* \otimes V)^W, q, t) =$$

$$q^{r+a} \binom{r}{2} \binom{n}{r}_q \frac{(-tq^{-1}; q^{-a})_r (-tq^{e_1}; q^a)_{n-r}}{(q^{e_1+1}; q^a)_n}$$

where $(z; q)_n := (1-z)(1-zq) \cdots (1-zq^{n-1})$

and $\binom{n}{k}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$

Thus if we call

$$f_r(W, q, t) := q^{r+a(2)} \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{(-tq^{-1}; q^{-a})_r (-tq^{e_1}; q^a)_{n-r}}{(q^{e_{i+1}}; q^a)_n}$$

then it gives a **product formula** for the **f-vector** entries of the cluster complex or Cambrian fan in **coincidental** types (by setting $t = -q^{h+1}$ and then $q = 1$).

Is there a **product formula** for the **h-vector** entries in **coincidental** types?

THEOREM (R.-Sheper-Sommers 2019)
 For coincidental W with gap a , setting

$$h_r(W, q, t) :=$$

$$(-tq^{-ar-1})^{n-r} \begin{bmatrix} n \\ r \end{bmatrix}_q \frac{(-tq; q^{-a})_r}{(q^{a+1}; q^a)_r}$$

one has

$$\sum_{r=0}^n f_r(W, q, t) \cdot s^r = \sum_{r=0}^n h_r(W, q, t) \cdot \underbrace{(-sq; q^a)_r}_{(1+sq)(1+sq^{a+1}) \dots (1+sq^{(r-1)a+1})}$$

(specializing to

$$\sum_{r=0}^n f_r \cdot s^r = \sum_{r=0}^n h_r (s+1)^r$$

)

Thus this $h_r(W, q, t)$ does give
a **product-formula** for the h-vector
in the **coincidental** types
(again by setting $t = -q^{h+1}$ and then $q = 1$).

The q -analogue $h_r(W, q, t)|_{t = -q^{h+1}}$
is a **q -Narayana** number
happily agreeing with precursors by...

- Furlinger & Hofbauer
 - R. Sommers
 - Krattenthaler & Wachs
- ... and exhibiting C.S.P.'s

The theorem on
 $\text{Hilb}((S \otimes \Lambda^k V^* \otimes \Lambda^k V)^w, q, t)$
is proven **case-by-case**
in coincidental types,
with $G(d, 1, n)$ modeled on a result of
Kirillov & Pak 1990

The theorem generalizing
h-vector to f-vector uses a
q-transformation of F.H. Jackson
(thanks Dennis Stanton!)

MYSTERY: why do coincidental types
behave better not only here, but also

in formulas of Reading 2007
counting Coxeter element
reflection factorizations $c = t_1 t_2 \dots t_k w$

in the Coxeter-biCatalan combinatorics
of Barnard & Reading 2016

in doppelgängers of root posets, studied
by Hamaker-Patrias-Pechenik-Williams 2016
and by S. Hopkins 2019

in work of Abramenko 1994
and Alex Miller 2018
on walls in Coxeter complexes
and Milnor fiber complexes

in work of Alex Miller 2015
on Foulkes characters

??

Thanks for
your
attention,
and thanks

MAAGC!