Sandpiles and Representation Theory

Vic Reiner
Univ. of Minnesota

(joint with Benkart & Klivans,
Gaetz,
Grinberg & Huang)

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OUTLINE

Laplacian & sandpile group for a...

- ... graph

- ... group representation

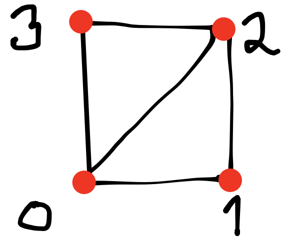
[– suppressed –

- ... module over a Hopf algebra]

Graphs

$\Gamma = (V, E)$ an undirected (multi-) graph

$$V = \{0, 1, 2, \dots, \ell\}$$

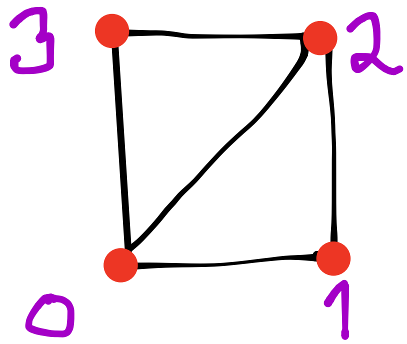


$$\underset{\text{graph Laplacian}}{L_\Gamma} := \underset{\text{diagonal matrix of vertex degrees}}{D_\Gamma} - \underset{\text{adjacency matrix}}{A_\Gamma}$$

$$(L_\Gamma)_{i,j} = \begin{cases} \deg_\Gamma(i) & \text{if } i = j \\ -\#\{\text{edges } i \text{ to } j\} & \text{if } i \neq j \end{cases}$$

EXAMPLE

$\Gamma =$

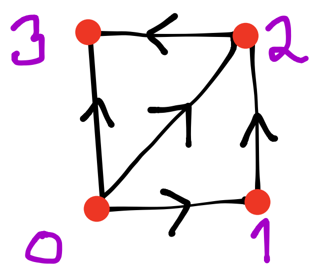


$$L_\Gamma = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

The graph Laplacian L_Γ is positive semidefinite, since

$L_\Gamma = \partial\partial^T$ where

$$\begin{tikzcd} \mathbb{R}^E \arrow[r, "\partial"] \arrow[d, equal] & \mathbb{R}^V \arrow[d, equal] \\ C_1(\Gamma,\mathbb{R}) & C_0(\Gamma,\mathbb{R}) \end{tikzcd}$$

$\Gamma =$ 

$$L_\Gamma = \partial\partial^T = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 01 & 02 & 03 & 12 & 23 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & 0 & 0 \\ +1 & 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & +1 & -1 \\ 0 & 0 & +1 & 0 & +1 \end{bmatrix} \end{array} \begin{bmatrix} -1 & +1 & 0 & 0 \\ -1 & 0 & +1 & 0 \\ -1 & 0 & 0 & +1 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & -1 & +1 \end{bmatrix}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

The graph Laplacian L_Γ has

$$\mathbb{R}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix} \subseteq \ker(L_\Gamma)$$

with equality $\boxed{\text{here}}$ $\Leftrightarrow$ Γ connected

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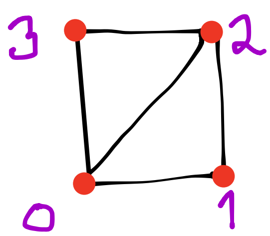
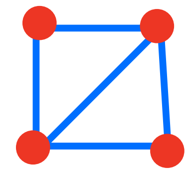
$\Gamma =$



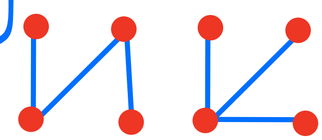
$$\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

- From spectrum (=eigenvalues) of L_Γ

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\ell$$



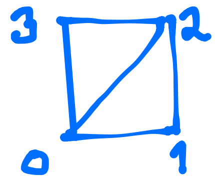
One can count the spanning trees in Γ:

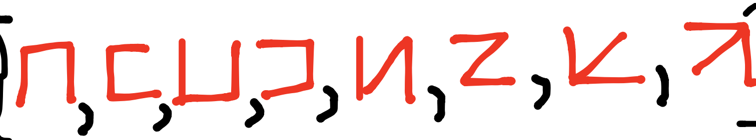


$$\underbrace{\tau(\Gamma)}_{\substack{\#\text{spanning}\\ \text{trees in } \Gamma}} = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell+1}$$

- Alternatively,

$$\tau(\Gamma) \underset{\substack{\uparrow\\ \text{Kirchhoff's}\\ \text{Matrix-Tree}\\ \text{Theorem}\\ (1845)}}{=} \det\Big(\underbrace{L_\Gamma - \begin{Bmatrix} 0^{th}\text{ row,} \\ 0^{th}\text{ column} \end{Bmatrix}}_{\substack{\text{reduced Laplacian}\\ \overline{L}_\Gamma}}\Big)$$

EXAMPLE $\Gamma =$  has

$\tau(\Gamma) = \#\{$  $\} = 8$

$$L_\Gamma = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$ has eigenvalues

$$\underset{\lambda_0}{0} \le \underset{\lambda_1}{2} \le \underset{\lambda_2}{4} \le \underset{\lambda_3}{4}$$

so $\tau(\Gamma) = \dfrac{\lambda_1 \lambda_2 \lambda_3}{4} = \dfrac{2 \cdot 4 \cdot 4}{4} = 8$ ✓

Or, $\tau(\Gamma) = \det \bar{L}_\Gamma = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$= 2(3 \cdot 2 - 1) + (-1 \cdot 2 - 0)$

$= 10 - 2 = 8$ ✓

REMARK:

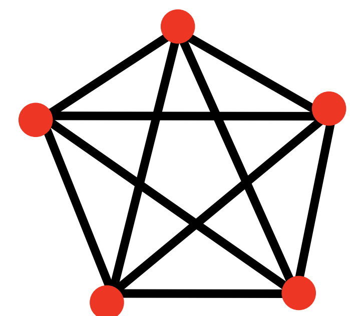
Eigenvalues of L_Γ are known for several families of graphs, letting one compute $\tau(\Gamma)$:

usually graphs with large symmetry or with inductive structure

- complete graphs, complete multipartite graphs
- cubes, Cartesian products
- distance-regular graphs
- threshold graphs, co-graphs

EXAMPLE

Complete graphs K_n



have L_{K_n} eigenvalues

$$\lambda_0 < \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$$

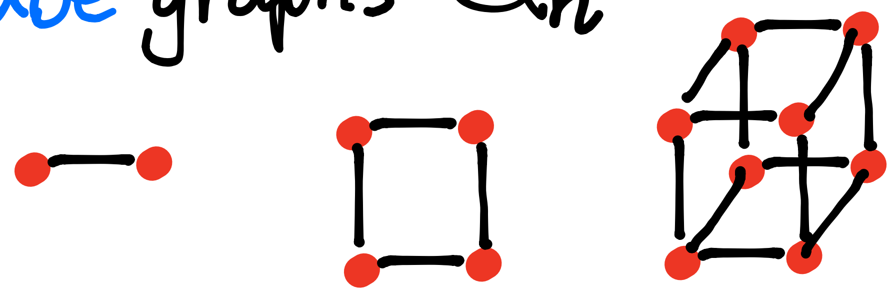
$$(0, n, n, \ldots, n)$$

COROLLARY

$$\tau(K_n) = \frac{n^{n-1}}{n} = n^{n-2}$$

Cayley 1889
Borchardt 1860

EXAMPLE n-dimensional **cube** graphs Q_n



Q_1 Q_2 Q_3

have L_{Q_n} eigenvalues

λ	0	2	4	$\cdots$	$2n-2$	$2n$
mult.	1	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

COROLLARY

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

REMARK

Eigenvectors of L_Π are also important in applications to

- optimal graph-drawing
- clustering of data

(see articles and surveys by Dan Spielman)

What about the Laplacian L_Γ considered as a map $R^V \xrightarrow{L_\Gamma} R^V$ for other rings R, e.g. what is $\mathrm{rank}(L_\Gamma)$ when reduced mod p?

To answer this, one can work with $R = \mathbb{Z}$ and compute a diagonal form over $\mathbb{Z}$

$$P L_\Gamma Q = \begin{bmatrix} d_1 & & & & \\ & d_2 & & 0 & \\ & & \ddots & & \\ & 0 & & d_\ell & \\ & & & & 0 \end{bmatrix}$$

$P, Q \in GL_{\#V}(\mathbb{Z})$

row operations   column operations

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e.g. Smith form has d_i dividing d_{i+1}

This is equivalent to computing the integer cokernel of L_Γ

$$\operatorname{coker}\left(\mathbb{Z}^V \xrightarrow{L_\Gamma} \mathbb{Z}^V\right) := \mathbb{Z}^V / \operatorname{im}(L_\Gamma)$$

$$\cong \mathbb{Z} \oplus \underset{\substack{\text{critical group} \\ \text{or sandpile group}}}{K(\Gamma)}$$

$$\cong \mathbb{Z} \oplus \overbrace{\mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_\ell\mathbb{Z}}$$

$$\text{if } P L_\Gamma Q = \begin{bmatrix} d_1 & & & 0 \\ & \ddots & & \\ & & d_\ell & \\ 0 & & & 0 \end{bmatrix}$$

$K(\Gamma)$ finite $\iff$ Γ connected

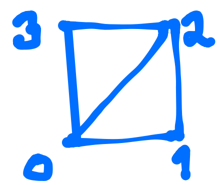
Alternatively, one can show that

$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^{\ell} \xrightarrow{L_\Gamma} \mathbb{Z}^{\ell}\right)$$

and hence

$$\# K(\Gamma) = \det(L_\Gamma)$$
$$= \#\underset{\text{in } \Gamma}{\text{spanning trees}} =: \tau(\Gamma)$$

Kirchhoff's Thm.

EXAMPLE $\Gamma =$ 

has
$$L_\Gamma = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$
with

$$\operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{L_\Gamma} \mathbb{Z}^4\right) \cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/8\mathbb{Z}}_{K(\Gamma)}$$

because one can compute L_Γ has

Smith normal form

$$P L_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $P, Q \in GL_4(\mathbb{Z})$

Alternatively, using the reduced Laplacian $\overline{L}_\Gamma$

$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/8\mathbb{Z}$$

via an equivalent Smith form calculation $P\overline{L}_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

So, for example,

$$\operatorname{rank}_{\mathbb{F}_2}(L_\Gamma) = 2 \quad (\text{not } 0 \text{ or } 1)$$

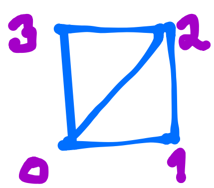
Why sandpile group?

The reduced Laplacian $\tilde{L}_\Gamma$ is an avalanche-finite matrix:

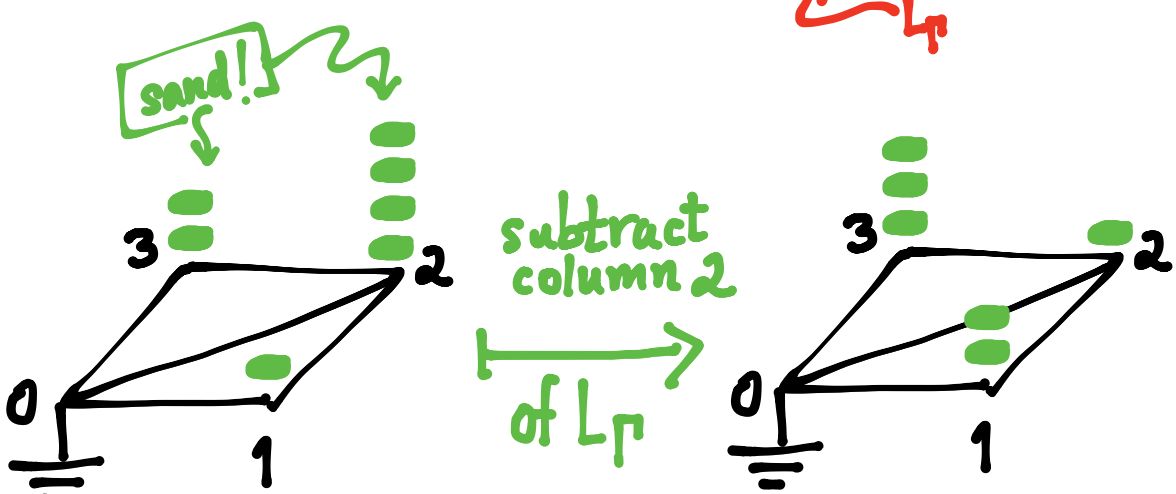
- entries in $\mathbb{Z}$
- off-diagonal entries ≤ 0
- invertible,
  with inverse entries ≥ 0

(Also known as
nonsingular M-matrices)

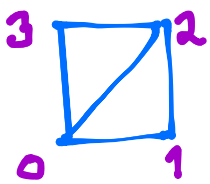
This implies every vector $x \in \mathbb{N}^\ell$ can be brought via a finite sequence of steps that subtract columns of $\tilde{L}_\Gamma$, keeping it in $\mathbb{N}^\ell$, until no such subtraction is possible; x is stable.

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array} \quad \leftarrow \tilde{L}_\Gamma$$

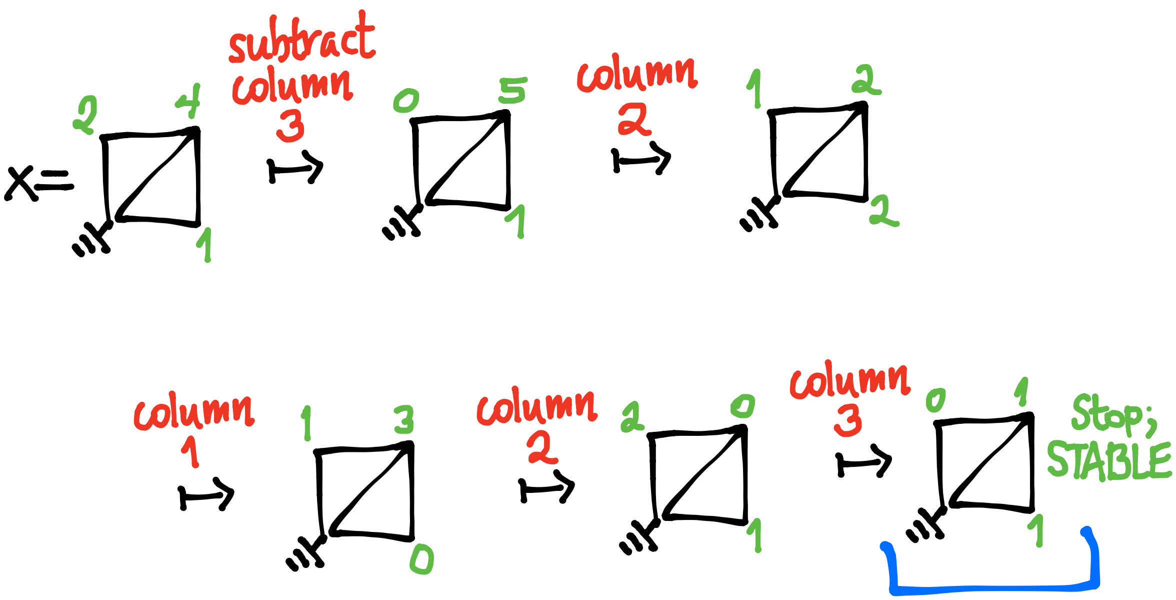


EXAMPLE

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 3 & -1 \\ 3 & -1 & 0 & -1 & 2 \end{array}$$

$\widetilde{L}_\Gamma$



The stabilization is unique, independent of choices of firings (!)

Leads to two interesting classes of coset representatives in $\mathbb{N}^\ell$ for $K(\Gamma) = \mathbb{Z}^\ell / \operatorname{im} L_\Gamma$

- critical configurations (= stable + recurrent)
- superstable configurations (= no subset of nodes can fire simultaneously keeping in $\mathbb{N}^\ell$)

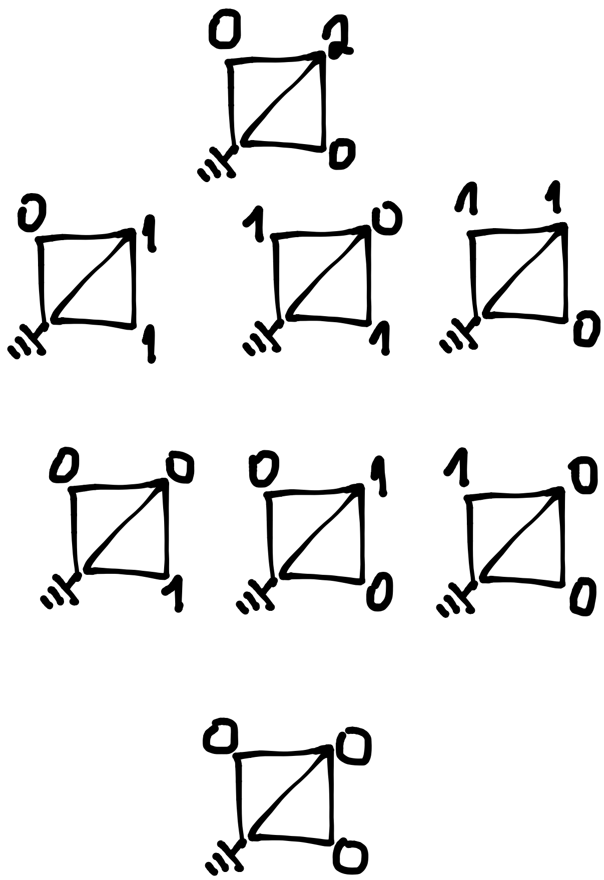
1987 Bak-Tang-Wiesenfeld

1990 Dhar

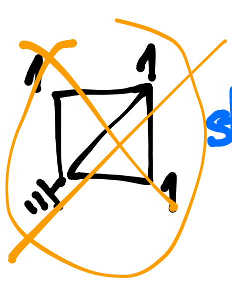
1991 Lorenzini

1993 Gabrielov

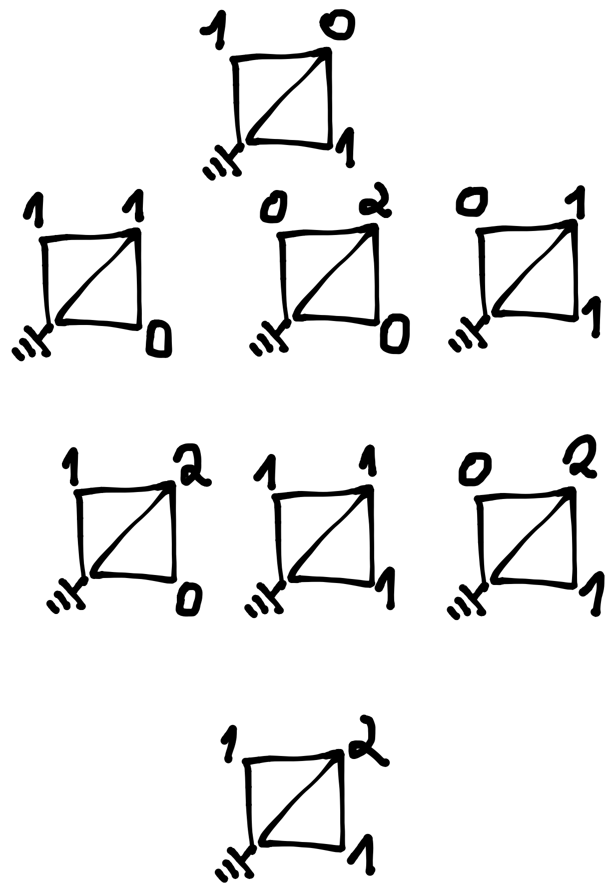
2007 Baker-Norine



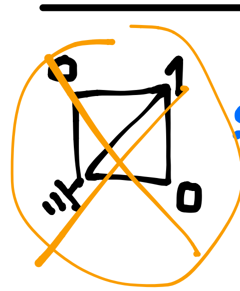
8 superstable configurations



stable, but not superstable



8 critical configurations



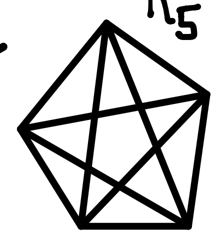
stable, but not recurrent

The exact structure of the sandpile group $K(\Gamma) = \mathbb{Z}^{\ell} / \operatorname{im} \overline{L}_{\Gamma}$ is known for very few graphs Γ, even when eigenvalues and eigenvectors and $\tau(\Gamma) = \# K(\Gamma)$ are easy.

(easy)

EXAMPLE Complete graphs K_n

K_5



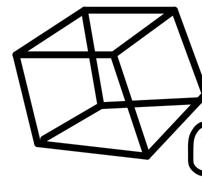
have $\tau(K_n) = n^{n-2}$

and $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$

(frustrating!)

EXAMPLE n-dimensional cubes Q_n

have L_{Q_n} eigenspaces easy



Q_3

and

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

The p-primary/p-Sylow structure of $K(Q_n)$ is known for p odd

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p \bigoplus_{k=1}^{n} (\mathbb{Z}/k\mathbb{Z})^{\binom{n}{k}}$$

but for $p=2$

$\mathrm{Syl}_2 K(Q_n)$ is an unknown mess!

Finite group representations

G a finite group has representations

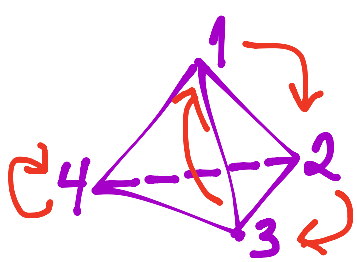
$$G \xrightarrow{\rho} GL_n(\mathbb{C})$$

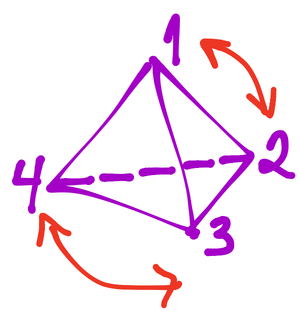
EXAMPLE

$G = \mathcal{A}_4$ = alternating group

$:=$ even permutations of $\{1,2,3,4\}$

$\cong$ rotational symmetries of tetrahedron





$$G \xrightarrow{\rho} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$$

- irreducible G-representations

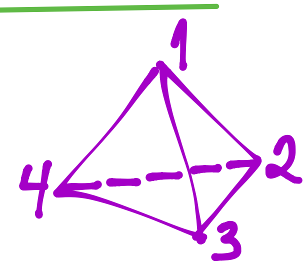
trivial G-rep $\mathbb{1}_G = S_0, S_1, S_2, \ldots, S_\ell$

- characters $\chi_0, \chi_1, \ldots, \chi_\ell$

where $\chi_\rho(g) = \mathrm{Trace}(\rho(g))$

- character table

EXAMPLE $\quad G = \mathcal{A}_4$



	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

$\omega = e^{2\pi i/3}$

DEFINITION:

Given a representation

$$G \xrightarrow{\rho} GL_n(\mathbb{C})$$

define its McKay matrix $M_\rho = (m_{ij})$ via

$$(\chi_{S_i \otimes \rho} =) \, \chi_i \, \chi_\rho = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

or equivalently

$$S_i \otimes \rho = \bigoplus_{j=0}^{\ell} S_j^{\oplus m_{ij}}$$

$$(X_{S_i \otimes \rho^{-}})\, X_i X_\rho = \sum_{j=0}^{\ell} m_{ij} X_j \quad \text{defines } M_\rho$$

Then...

- $L_\rho := nI_{\ell+1} - M_\rho$

- $\tilde{L}_\rho := L_\rho - \begin{Bmatrix} X_0 \text{ row} \\ X_0 \text{ column} \end{Bmatrix}$

- $K(\rho) := \operatorname{coker}\left(\mathbb{Z}^{\ell} \xrightarrow{\tilde{L}_\rho} \mathbb{Z}^{\ell}\right)$

sandpile group

or

$$\mathbb{Z} \oplus K(\rho) = \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_\rho} \mathbb{Z}^{\ell+1}\right)$$

EXAMPLE For $G = \mathcal{A}_4$, let's consider

$$G = \mathcal{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subseteq GL_3(\mathbb{C})$$

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\chi_\rho = \chi_3$	3	0	0	-1

$$\chi_0 \chi_\rho = \chi_1 \chi_\rho = \chi_2 \chi_\rho = \chi_\rho = 1\,\chi_3$$

$$\chi_3 \chi_\rho = 1\chi_0 + 1\chi_1 + 1\chi_2 + 2\chi_3$$

$$\rightsquigarrow \quad M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

McKay matrix

$$L_\rho = 3I_4 - M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \left[\begin{array}{c|ccc} 3 & 0 & 0 & -1 \\ \hline 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{array}\right] \end{array}$$

$$\overline{L}_\rho = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{normal form}]{\text{Smith}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$K(\rho) = \operatorname{coker} \overline{L}_\rho \cong \mathbb{Z}/3\mathbb{Z}$$

Sandpile group

Why is $\operatorname{coker} L_\rho = \mathbb{Z} \oplus \underbrace{\operatorname{coker} \overline{L}_\rho}_{K(\rho)}$?

L_ρ has

$$\bar{s} = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix} = \begin{bmatrix} 1 \\ \dim(S_1) \\ \vdots \\ \dim(S_\ell) \end{bmatrix}$$

as nullvector.

EXAMPLE

$$L_\rho \bar{s} = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM

The inclusion $\mathbb{R}\bar{s} \subseteq \ker L_\rho$ is an equality

$\Leftrightarrow$ $G \xhookrightarrow{\rho} GL_n(\mathbb{C})$ is faithful

(analogous to Γ connected)

More generally, we know the full eigenspace decomposition for M_ρ and L_ρ: their eigenbasis is columns of the character table

$$\bar{s}(g) := \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$$

because $\displaystyle\sum_{j=0}^{\ell} m_{ij} \chi_j(g) = \chi_\rho(g)\, \chi_i(g)$

implies
$$M_\rho\, \bar{s}(g) = \chi_\rho(g)\, \bar{s}(g)$$
$$L_\rho\, \bar{s}(g) = \left(n - \chi_\rho(g)\right) \bar{s}(g)$$

REMARK

$\mathbb{Z}\oplus K(\rho) = \operatorname{coker}(\mathbb{Z}^{\ell+1} \xrightarrow{L_\rho} \mathbb{Z}^{\ell+1})$

has the structure of a ring, because

$\mathbb{Z}^{\ell+1} \cong$ Grothendieck ring $R(G)$ of

virtual G-representations $[V]$

with $[V]+[W]=[V\oplus W]$

$[V]\cdot[W]=[V\otimes W]$

and

M_ρ is multiplication by $[\rho]$ in $R(G)$

L_ρ is multiplication by $n-[\rho]$ in $R(G)$

So $\mathbb{Z}\oplus K(\rho) = R(G)/(n-[\rho])$

RESULTS & EXAMPLES

When the finite group G is abelian, we are back to (directed) graph sandpile groups...

THEOREM (Benkart-Klivans-R.)
For faithful abelian group reps $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$

$$K(\rho) = K(\Gamma)$$

usual directed graph sandpile group

where Γ is a certain Cayley digraph

How do we get this Cayley digraph Γ

having $K(\rho) = K(\Gamma)$

from an abelian $G \xrightarrow{\rho} GL_n(\mathbb{C})$?

Γ = Cayley digraph

for the dual group of characters

$$\check{G} = \mathrm{Hom}(G, \mathbb{C}^\times)$$
$$= \{\chi_0, \chi_1, \ldots, \chi_\ell\}$$

with respect to generators $\{\chi_{i_1}, \ldots, \chi_{i_n}\}$,

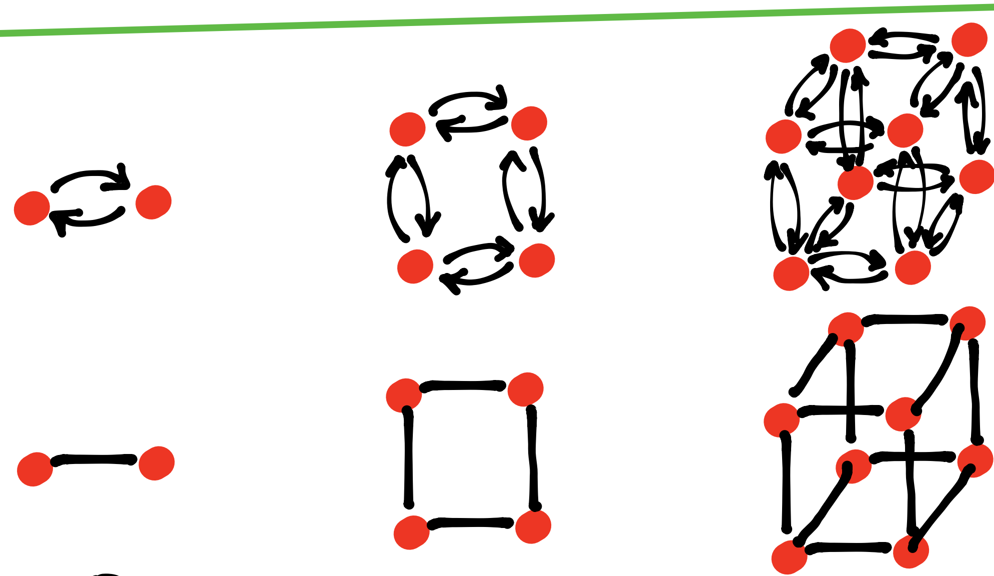
where $\chi_\rho = \chi_{i_1} + \cdots + \chi_{i_n}$.

EXAMPLE

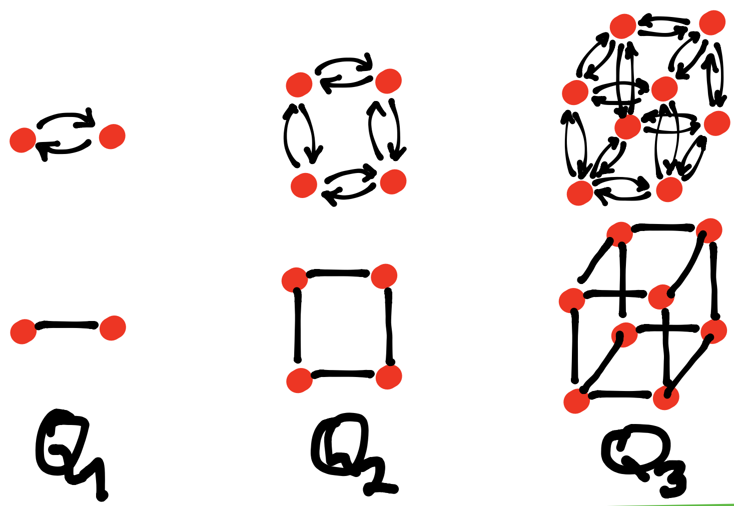
$$G = (\mathbb{Z}/2\mathbb{Z})^n \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \longmapsto \begin{bmatrix} (-1)^{\epsilon_1} & & & 0 \\ & (-1)^{\epsilon_2} & & \\ & & \ddots & \\ 0 & & & (-1)^{\epsilon_n} \end{bmatrix}$$

has $K(\rho) = K\Big(\overset{n\text{-cube}}{Q_n}\Big)$



Q_1 $\qquad$ Q_2 $\qquad$ Q_3



Bonus ring structure here:

$$\mathbb{Z} \oplus K(Q_n) = R(G)/(n-[P])$$
$$\cong \mathbb{Z}[x_1, \ldots, x_n] \Big/ \left(x_1^2 - 1, \ldots, x_n^2 - 1, \; n-(x_1+\ldots+x_n) \right)$$

$\rightsquigarrow$ bounds on e in $\mathbb{Z}/2^e\mathbb{Z}$ in $K(Q_n)$ and conjectural Gröbner basis

(REU 2016 students Anzis & Prasad)

The analogue of $\#K(\Gamma) = \tau(\Gamma) = \dfrac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell+1}$ is

THEOREM (Gaetz) For any faithful representation ρ of G,

$$\#K(\rho) = \frac{1}{\#G} \cdot \prod_{\substack{G\text{-conj. classes} \\ [g] \neq \{e\}}} (n - \chi_\rho(g))$$

EXAMPLE $G = \mathfrak{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$

had $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

	e	(123)	(132)	$(12)(34)$
χ_ρ	3	0	0	-1

$$\#K(\rho) = \frac{1}{12}(3-0)(3-0)(3-(-1)) = 3$$

COROLLARY (Goetz) If $n = \#G$, regular representation of G

$$G \xhookrightarrow{\mathrm{reg}_G} GL_n(\mathbb{C}) \quad \text{has}$$

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{\#\left(\substack{G\text{-conjugacy}\\ \text{classes}}\right) - 2}$$

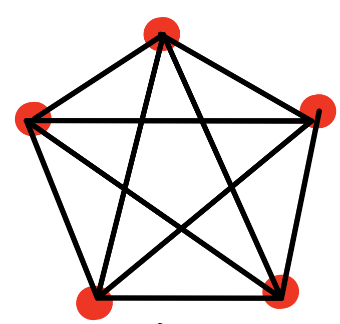
$\Big\downarrow$ G abelian

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

$\Big\downarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

complete graph



K_5

THEOREM (Benkart-Klivans-R)

For **faithful** G-reps ρ,
$\overline{L}_\rho$ is an **avalanche-finite** matrix,
so one can compute in
$K(\rho) = \mathrm{coker}(\overline{L}_\rho)$ via toppling
with **superstable** or **critical**
coset representatives in $\mathbb{N}^\ell$

EXAMPLE (continued)

$$\overline{L}_\rho = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 3 & 0 & -1 \\ x_2 & 0 & 3 & -1 \\ x_3 & -1 & -1 & 1 \end{array}$$

with $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

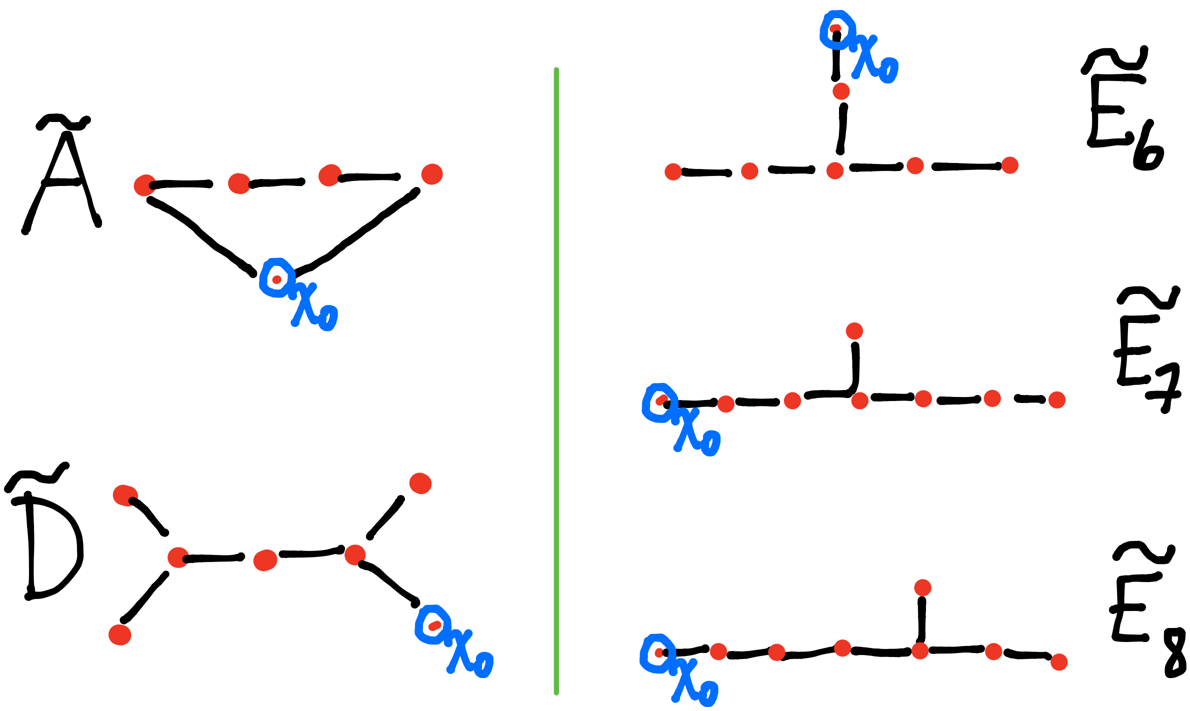
x_1	x_2	x_3		x_1	x_2	x_3
0	0	0		2	2	0
1	0	0		1	2	0
0	1	0		2	1	0
superstables				criticals		

McKay's original theorem (1980)

When $G \xhookrightarrow{\rho} SL_2(\mathbb{C})$, then

$\overline{L}_\rho$, L_ρ are the Cartan, extended Cartan

matrices for a simply-laced root system Φ



THEOREM (Benkart-Klivans-R)

In McKay's $G \overset{\rho}{\hookrightarrow} SL_2(\mathbb{C})$ setting,

$$K(\rho) \cong \mathrm{Hom}(G, \mathbb{C}^\times)$$

$$= \text{1-dim'l characters } \chi_i \text{ of } G$$

$$\left(\overset{\text{Pontrjagin dual}}{\cong} \underbrace{G^{ab}}_{\text{abelianization of } G} = G/[G,G] \right)$$

$$\left(\cong \underbrace{P(\Phi)}_{\text{weight lattice}} / \underbrace{Q(\Phi)}_{\text{root lattice}} \cong \underbrace{\pi_1 \left(\begin{array}{c}\text{adjoint form} \\ \text{of compact} \\ \text{Lie group} \\ \text{associated to } \Phi\end{array} \right)}_{\text{fundamental group of } \Phi} \right)$$

THEOREM (Benkart-Klivans-R)

More generally, when $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$

one has a surjection

$$K(\rho) \twoheadrightarrow \mathrm{Hom}(G, \mathbb{C}^\times)$$

THEOREM (Gaetz)

When $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$,

$$\begin{array}{c} \left\{ \begin{array}{c} \text{1-dimensional} \\ \text{characters } \chi_i \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{superstable} \\ \text{configurations} \\ \text{for } \overline{L}_\rho \end{array} \right\} \\ \parallel \\ \mathrm{Hom}(G, \mathbb{C}^\times) \end{array}$$

EXAMPLE Recall

$$G = \mathcal{A}_4 \xrightarrow{\rho} SO_3(\mathbb{R}) \subset SL_3(\mathbb{C})$$

had $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

Here

$$\mathrm{Hom}(G, \mathbb{C}^\times) \cong G^{ab} = \mathcal{A}_4/V_4 \cong \mathbb{Z}/3\mathbb{Z}$$

so the surjection

$$\begin{array}{ccc} K(\rho) & \twoheadrightarrow & \mathrm{Hom}(G, \mathbb{C}^\times) \\ \| & & \cong \\ \mathbb{Z}/3\mathbb{Z} & & \mathbb{Z}/3\mathbb{Z} \end{array}$$

is again an isomorphism.

$K(p)$ has led us to many interesting questions about representations of groups in positive characteristic, and of finite-dimensional Hopf algebras.

Thanks for your attention,

and thanks to the Maheshwari family!