(1)

Hodge theory & matroids

Ref: Adiprasito, Huh & Katz "Hodge thy for comb. geometries"

1. Unimodality, log-concavity
2. Rota-Heron-Welsh conj.
3. Mason conj.
4. Huh, ~~Lenz~~/Brylawski, ~~reductions~~ & a better Conj.
5. Truncation reduction ← 1st lecture?
6. Getting into the Hodge story

⋮

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1. Given $(a_0, a_1, \ldots, a_r) \in \{1,2,3,\ldots\}$

say they are <u>unimodal</u> if $a_0 \le a_1 \le \ldots \le a_\ell \ge \ldots \ge a_{r-1} \ge a_r$ for some ℓ

<u>log-concave</u> if $a_i^2 \ge a_{i-1} a_{i+1}$ for $i = 1, 2, \ldots, r-1$

$$\left(\underset{a_i > 0}{\Rightarrow} \frac{a_i}{a_{i-1}} \ge \frac{a_{i+1}}{a_i} \Rightarrow \text{unimodal} \right)$$

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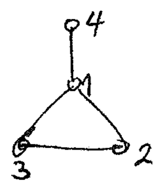
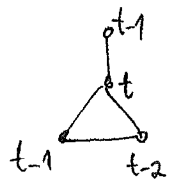
2. For a graph, $G =$ e.g.



, the <u>chromatic polynomial</u>

$\chi_G(t) := \#\{\text{proper vertex-colorings with } t \text{ colors}\}$



$= t(t-1)^2(t-2)$ in this case
$= t^4 - 4t^3 + 5t^2 - 2t$

<u>is</u> always a <u>polynomial</u> in t

<u>CONJ</u> (Read, 1968; Hoggar, 1974) Its (signless) coefficients are unimodal / log-concave
e.g. $(1,4,5,2)$ here

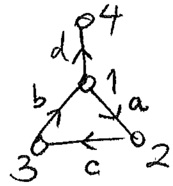
<u>CONJ</u> (Rota-Heron, 1971, 1972; Welsh, 1976) Same holds more generally for <u>characteristic polynomials</u>

$\chi_M(t)$ of any <u>matroid</u> M

$$:= \sum_{\text{flats } F \text{ of } M} \mu(\emptyset, F)\, t^{r(M)-r(F)}$$

(2)

Recall a <u>graph</u> $\overset{(V,E)}{\overset{\shortparallel}{G}}$ ⇝ a collection of $\overset{\text{(column)}}{\underline{\text{vectors}}}$ $\{e_i - e_j\}_{\{i,j\}\in E} \subset k^E$ ⇝ a <u>matroid</u> M



$V=\{1,2,3,4\}$
$E=\{a,b,c,d\}$

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} +1 & -1 & 0 & +1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array}$$

in a vector space over a field k

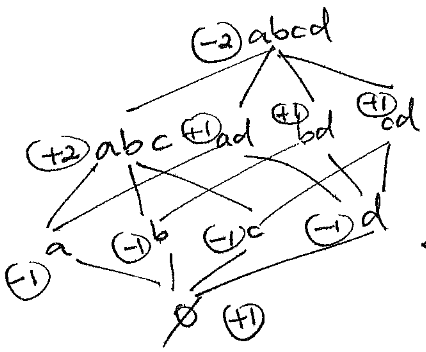
Specified as the subsets

<u>Indep. sets</u> $\mathcal{I} = \{ I \subset E : I$ indexes a <u>lin.</u> <u>indep.</u> subset of the vectors $\}$

$= \{ \emptyset,$ ①
$a, b, c, d,$ ④
$ab, ac, bc, ad, bd, cd,$ ⑥
$abd, acd, bcd \}$ ③

<u>AXIOMS:</u>
- $\emptyset \in \mathcal{I} \subset 2^E$
- $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$
- $I, J \in \mathcal{I}, |I| > |J| \Rightarrow \exists\, i \in I$ with $J \cup \{i\} \in \mathcal{I}$

Matroids M have their lattice of <u>flats</u> F := subsets $F \subset E$ indexing vectors closed under lin. span



Möbius <u>function</u> values $\mu(\emptyset, F)$ circled

$$\chi_M(t) = t^3 - 4t^2 + 5t^1 - 2 = \sum_{\text{flats } F}{}' \mu(\emptyset, F)\, t^{r(M)-r(F)}$$

In general, graph G with matroid M has $\chi_G(t) = t^{\#\text{ conn. comps of } G}\, \chi_M(t)$

easy Möbius inversion argument

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3. <u>CONJ</u> (Mason 1972) If a_k = # indep. sets of size k in matroid M, then $(a_0, a_1, \ldots, a_r)$ is log-concave

e.g. above M $(1, 4, 6, 3)$

$4^2 \geq 1\cdot 6$ ✓ $6^2 \geq 4\cdot 3$ ✓

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4. In fact, there is a better conjecture/theorem…

A matroid M has $\chi_M(t) = (t-1)\,\underbrace{\overline{\chi}_M(t)}_{\text{reduced characteristic polynomial}}$

$$:= \sum_{\substack{\text{flats } F \text{ of } M \\ \text{with } e \notin F}}{}' \mu(\emptyset, F)\, t^{r(M)-1-r(F)}$$

for <u>any</u> choice of $e \in E$

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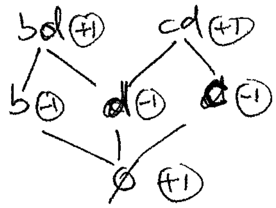
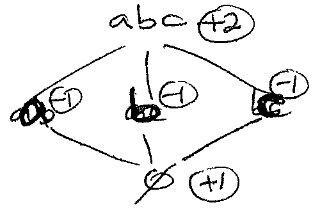
e.g. M above, choosing $e=a$ | choosing $e=d$



$+2$

$-3t$

t^2



$+2$

$-3t$

t^2

$$\chi_M(t) = t^3-4t^2+5t-2 = (t-1)\underbrace{(t^2-3t+2)}_{\overline{\chi}_M(t)}$$

PROP (Brylawski 1977, Lenz 2013) There is a matroid construction $M \longmapsto M\times e := \left(M^* \cup \{\text{generic } e^*\}\right)^*$ such that $\overline{\chi}_{M\times e}(t)$ has signless coefficient sequence $(a_0, a_1, \ldots, a_r)$ where $a_k = \#\{\text{indep. sets in } M \text{ of size } k\}$

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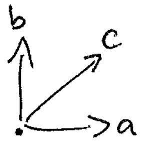
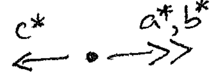
e.g. (1st) $M=$ 

$$= \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

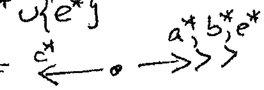
has $\mathcal{I} = \{\varnothing,$ (1)
a, b, c (3)
$ab, ac, bc\}$ (3)

(2nd) $M^* =$ 

$$= \begin{array}{c} \begin{array}{ccc} a^* & b^* & c^* \end{array} \\ \begin{bmatrix} +1 & +1 & -1 \end{bmatrix} \end{array}$$

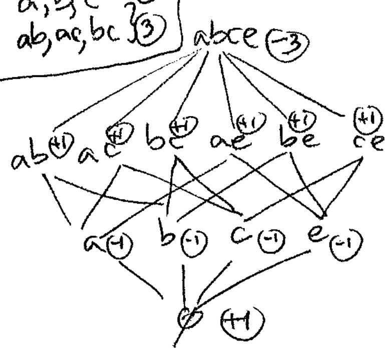
$\text{rowspace}(M^*) = \text{rowspace}(M)^\perp$
$= \text{nullspace}(M)$

(3rd) $M^* \cup \{e^*\}$

$=$ 

$$= \begin{array}{c} \begin{array}{cccc} a^* & b^* & c^* & e^* \end{array} \\ \begin{bmatrix} +1 & +1 & -1 & +1 \end{bmatrix} \end{array}$$

(4th) $M\times e = \left(M^* \cup \{e^*\}\right)^*$



-3

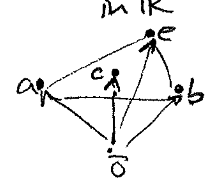
$+6t$

$-4t^2$

t^3

$$= \begin{array}{c} \begin{array}{cccc} a & b & c & e \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$=$ 4 generic vectors in $\mathbb{R}^3$



$$\chi_{M\times e}(t) = t^3-4t^2+6t-3$$
$$= (t-1)\underbrace{(t^2-3t+3)}_{\overline{\chi}_{M\times e}(t)}, \text{ signless coeffs } (1,3,3)$$

(proof of PROP is easy using $\chi_M(t)$, $\sum_{\substack{\text{indep} \\ \text{sets } I}} t^{\#I}$ both being Tutte polynomial evaluations)

$$\begin{array}{ccc} & T_M(x,y) & \\ {\scriptstyle x=1-t,\ y=0} \swarrow & & \searrow {\scriptstyle x=1+t,\ y=1} \\ \pm\chi_M(t) & & \sum_{I\in\mathcal{I}} t^{r(M)-\#I} \end{array}$$

(4)

Things started really happening a few years ago...

THM (Huh 2012) For matroids M representable by vectors in characteristic zero, $\overline{\chi}_M(t)$ has log-concave signless coeff. seq.

($\Rightarrow$ same for $\chi_M(t)$, since (easy) PROP: $(a_0, a_1, \ldots, a_{r-1})$ log-concave $\Rightarrow$ coeffs of $(1+t)(a_0 + a_1 t + \cdots + a_{r-1} t^{r-1})$ log-concave)

THM (Huh & Katz 2012) Same for matroids M rep'd over any field.

THM (Lenz 2013) Therefore Mason's Conj. holds for M rep'd over any field.

THM (Adiprasito, Huh, Katz 2015) ~~Any~~ matroid M has $\overline{\chi}_M(t)$ with log-concave signless coeff. seq.

How did they do it? Somehow mimicking a calculation in a Chow ring for the wonderful compactification of the hyperplane arrangement complement $k^n \setminus \bigcup_{H \in \mathcal{A}} H$, when M is realized over k...

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5. The truncation reduction:

One only needs to show $\boxed{a_{r-3} a_{r-1} \le a_{r-2}^2}$ at the end of the signless coeff. seq. $(a_0, a_1, \ldots, \underbrace{a_{r-3}, a_{r-2}, a_{r-1}})$ for $\overline{\chi}_M(t)$,

because the matroid operation $M \longmapsto \underset{\substack{\text{truncation of } M \text{ to 1 lower rank} \\ (\Leftrightarrow \text{generic projection of vectors})}}{tr(M)}$

has signless coeff. seq. $(a_0, a_1, \ldots, a_{r-3}, a_{r-2})$ for $\overline{\chi}_{tr(M)}(t)$.

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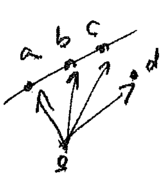
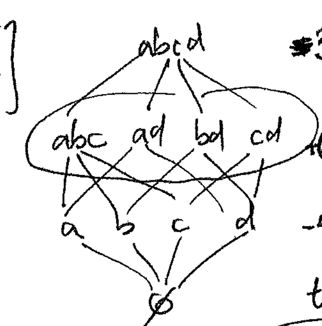
e.g.

$$M = \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} +1 & -1 & 0 & 1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -1 & 0 \end{bmatrix} \end{array}$$





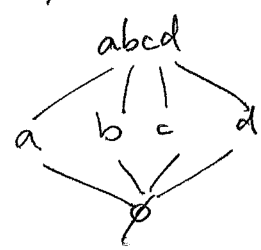
$+3 \rightsquigarrow$ $tr(M)$

$+6t$

$-4t^2$

t^3

$t^3 - 4t^2 + 6t - 3$
$= (t-1)\underbrace{(t^2 - 3t + 3)}_{\overline{\chi}_M(t)}$ $(1,3,3)$



$+3$

$-4t$

t^2

$t^2 - 4t + 3$
$= (t-1)\underbrace{(t-3)}_{\overline{\chi}_{tr(M)}(t)}$ $(1,3)$

(5) OUTLINE for more of the Hodge story ...

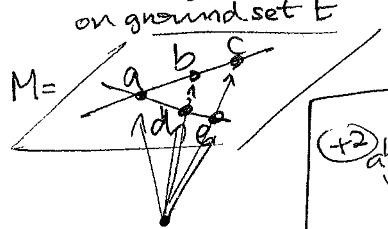
6. An amazing ring
7. Its Hodge properties
8. How they help

Recall a matroid M of rank r on ground set E $\rightsquigarrow$ lattice of flats $\rightsquigarrow$ reduced characteristic polynomial

$\overline{\chi}_M(t) = a_0 - a_1 t + \dots \pm a_{r-1} t^{r-1}$

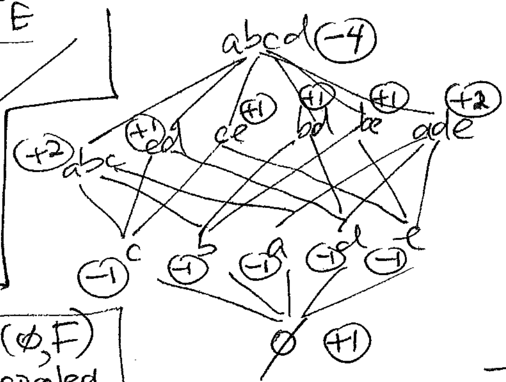
signless coefficients $(a_0, a_1, \dots, a_{r-1})$

e.g. $M=$



$E = \{a,b,c,d,e\}$

$r=3$



$\mu(\emptyset, F)$ circled

-4

$+8t$

$-5t^2$

t^3

$\chi_M(t) = t^3 - 5t^2 + 8t - 4 = (t-1)\underbrace{(t^2 - 4t + 4)}_{\overline{\chi}_M(t)}$

$\Rightarrow (a_0, a_1, a_2) = (1,4,4)$

THM (A-H-K 2015) $(a_0, a_1, \dots, a_{r-2})$ is log-concave: $a_k^2 \geq a_{k-1} a_{k+1}$

Truncation reduction means we only need to show $a_{r-2}^2 \geq a_{r-3} a_{r-1}$.

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6. $A(M) :=$ "Chow ring of M" (Feichtner & Yuzvinsky 2004 .. of De Concini & Procesi 1995) cohomology of wonderful compactification ..

$$:= \underbrace{\left(\mathbb{Z}[x_F]_{F \text{ a nonempty proper flat of } M} \Big/ \left(x_F x_G : \begin{array}{l} F \not\subset G, \\ G \not\subset F \end{array} \right) \right)}_{\text{Stanley-Reisner ring for proper part of lattice of flats}} \Big/ \left(\alpha_i - \alpha_j : \begin{array}{l} i \neq j \\ \text{in } E \end{array} \right)$$

where $\alpha_i := \sum_{\text{flats } F: \, i \in F} x_F$

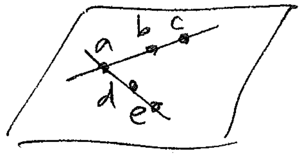
PROP: $A(M) = \bigoplus_{i=0}^{r-1} A^i(M) = A^0(M) \oplus A^1(M) \oplus \dots \oplus A^{r-1}(M)$ with $A^0(M) = \mathbb{Z}$, $A^{r-1}(M) \xrightarrow[\cong]{\deg} \mathbb{Z}$

$\deg(x_{F_1} x_{F_2} \cdots x_{F_{r-1}}) = 1$ $\forall$ maximal flags $F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{r-1}$

PROP: The elements $\alpha := \alpha_i$ for any $i \in E$

$\beta := \sum_{\text{flats } F: \, F \not\ni i} x_F = \sum_{\text{flats } F} x_F - \alpha_i$ in $A^1(M)$

satisfy $a_i = \deg(\beta^i \alpha^{r-1-i})$

(6) <u>EXAMPLE:</u> $M =$  above has

$$A(M) = \mathbb{Z}[x_a, x_b, x_c, x_d, x_e, x_{abc}, x_{ade}, x_{bd}, x_{be}, x_{cd}, x_{de}] \Big/ \left(\begin{array}{ccc} x_a x_b, & x_{abc} x_{ade}, & \alpha_a - \alpha_b \\ x_a x_c, & x_{abc} x_{bd}, & \alpha_a - \alpha_c \\ x_a x_d, & x_{bd} x_{be}, & \alpha_a - \alpha_d \\ x_a x_e, & \vdots & \alpha_a - \alpha_e \\ x_b x_c, & & \\ x_b x_d, & & \\ x_b x_e, & & \\ x_c x_d, & & \\ x_c x_e, & & \\ x_d x_e, & & \end{array} \right)$$

Stanley-Reisner relations

$$\alpha_a - \alpha_e = x_a + x_{abc} + \cancel{x_{ade}} - (x_e + x_{be} + x_{ce} + \cancel{x_{ade}}) = x_a + x_{abc} - x_e - x_{be} - x_{ce}$$

$$\cong \underset{A^0}{\mathbb{Z}} \oplus \underset{A^1}{\mathbb{Z}^7} \oplus \underset{A^2}{\mathbb{Z}}$$

checked in Macaulay2

Can check, e.g.

$$x_a x_{ade} = x_e x_{ade}$$

using $x_{abc}(\alpha_a - \alpha_e) = 0$

$$\parallel$$

$$x_{ade}(x_a + \cancel{x_{abc}} - x_e - \cancel{x_{be}} - \cancel{x_{ce}})$$

$$\parallel$$

$$x_a x_{ade} - x_e x_{ade}$$

Can check

$$\deg(x_{bd}^2) = -1$$
$$\deg(x_a^2) = -1$$
$$\deg(x_{abc}^2) = -1$$
$$\deg(x_b^2) = -2$$

e.g. do this via

$$\deg(x_b^2) = \deg(x_b(x_b - \alpha_\beta + \alpha_c))$$
$$= \deg(x_b(\cancel{x_b} - (\cancel{x_b} + x_{bd} + x_{be} + \cancel{x_{abc}}) + (\cancel{x_c} + \cancel{x_{cd}} + \cancel{x_{ce}} + \cancel{x_{abc}})))$$
$$= \deg(-x_b x_{bd} - x_b x_{be}) = -2$$

Then $\alpha := \alpha_a = x_a + x_{abc} + x_{ade}$

$\beta := x_b + x_c + x_d + x_e + x_{bd} + x_{be} + x_{cd} + x_{ce}$

have $\alpha^2 = x_a^2 + x_{abc}^2 + x_{ade}^2 + 2(x_a x_{abc} + x_a x_{ade} + \cancel{x_{abc} x_{ade}}) \xmapsto{\deg} -1 -1 -1 + 2(1+1) = 1 = a_0$ ✓

$$\alpha\beta = (x_a + x_{abc} + x_{ade})(x_b + x_c + x_d + x_e + x_{bd} + x_{be} + x_{cd} + x_{ce})$$
$$= x_b x_{abc} + x_c x_{abc} + x_d x_{ade} + x_e x_{ade} \xmapsto{\deg} +1+1+1+1 = 4 = a_1 ✓$$

$$\beta^2 = x_b^2 + x_c^2 + x_d^2 + x_e^2 + x_{bd}^2 + x_{be}^2 + x_{cd}^2 + x_{ce}^2 + 2(x_b x_{bd} + x_b x_{be} + x_c x_{cd} + x_c x_{ce} + x_d x_{bd} + x_d x_{cd} + x_e x_{be} + x_e x_{ce})$$

$$\xmapsto{\deg} (-2)+(-2)+(-2)+(-2) -1 -1 -1 -1 + 2(8) = 4 = a_2 ✓$$

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7. The Hodge properties: (the majority of the work in A-H-K 2015):

- THM (Poincaré duality over $\mathbb{Z}$) The bilinear pairing

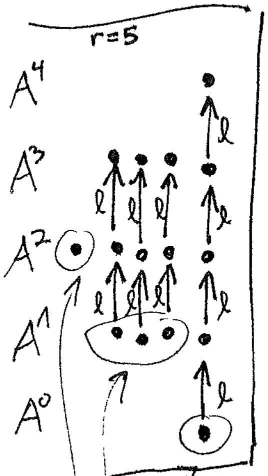
$$A^k(M) \times A^{r-1-k}(M) \longrightarrow \mathbb{Z}$$
$$(\ x\ ,\ y\ ) \longmapsto \deg(x \cdot y)$$

is nondegenerate, i.e. it induces an iso. $A^{r-1-k}(M) \cong \mathrm{Hom}_{\mathbb{Z}}(A^k(M), \mathbb{Z})$

- THM (Hard Lefschetz property) When one extends scalars in $A(M)$ to $\mathbb{R}$, there are elements ℓ in $A^1(M)$ with a property called ampleness that makes this map an isomorphism for $k \leq \frac{r-1}{2}$:

$$A^k(M) \xrightarrow{\ \sim\ } A^{r-1-k}(M)$$
$$x \longmapsto \ell^{r-1-2k} \cdot x$$

specifically: $\ell = \sum_{\substack{\emptyset \neq F \subsetneq E \\ \text{flats}}} c_F \cdot x_F$ where c_F is the restriction of a strictly submodular function $c: 2^E \to \mathbb{R}$, $c_{F \cup G} + c_{F \cap G} < c_F + c_G$



primitive parts circled

- THM (Hodge-Riemann relations) For ample $\ell \in A^1(M)$, when one restricts the quadratic form on $A^k(M)$ defined by $Q(x) := \deg(x \cdot \ell^{r-1-2k} \cdot x)$ to the primitive part of $A^k(M)$ $:= \ker(x \mapsto x \cdot \ell^{r-2k})$, it becomes $\begin{cases} \text{pos. definite if } k \text{ even} \\ \text{neg. definite if } k \text{ odd} \end{cases}$

8. How does this help?

We want $(a_{r-3}, a_{r-2}, a_{r-1})$, with $a_{r-3} = \deg(\alpha^2\beta^{r-3})$, $a_{r-2} = \deg(\alpha\beta^{r-2})$, $a_{r-1} = \deg(\beta^{r-1})$

to satisfy $a_{r-2}^2 \underset{(*)}{\geq} a_{r-3}\, a_{r-1}$

PROP: $\beta \in A^1(M)$ is not ample, but is a limit $\lim_{t \to 0} \beta_t = \beta$ with β_t ample (β is nef)

Now we can show $(*)$ holds with strict inequality for α, β_t using the Hodge-Riemann relations for the ample $\beta_t \in A^1(M)$:

Since α is not a multiple of β_t in $A^1(M)$, the 2-plane $\mathbb{R}\alpha + \mathbb{R}\beta_t$ inside $A^1(M)$ has the quadratic form $Q(x) = \deg(x \cdot \beta_t^{r-3} \cdot x)$ restricted to it indefinite (it contains the line $\mathbb{R}\beta_t$ where it is pos. def., plus some primitive part of $A^1(M)$ where it is neg. def.)

orthogonal direct sum!

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$$\deg(x\,\beta_t^{r-3}\,y) = \mathbb{B}(x,y)$$

Consequently if we write down the Gram matrix for the bilinear form associated to Q with respect to the choice of basis α, β of this plane

$$\begin{array}{c|c}
 & \begin{array}{cc} \qquad\alpha\qquad & \qquad\beta\qquad \end{array} \\
\begin{array}{c} \alpha \\[4em] \beta \end{array} &
\begin{bmatrix}
\underset{\mathbb{B}(\alpha,\alpha) = \deg(\alpha^2\beta_t^{r-3})}{Q(\alpha)} & \mathbb{B}(\alpha,\beta_t) = \deg(\alpha\beta_t^{r-2}) \\[2em]
\mathbb{B}(\beta_t,\alpha) = \deg(\alpha\beta_t^{r-2}) & \underset{\mathbb{B}(\beta_t,\beta_t) = \deg(\beta_t^{r-1})}{Q(\beta_t)} \longleftarrow \text{positive}
\end{bmatrix}
\end{array}$$

then it should have negative determinant, that is

$$\deg(\alpha\beta_t^{r-2})^2 > \deg(\alpha^2\beta_t^{r-3})\deg(\beta_t^{r-1}),$$

as desired.

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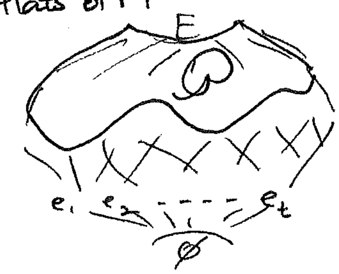
9. How do they prove the Hodge properties for $A(M)$?

They embed them within a larger family of rings, to do induction:

$A(M, \mathcal{P})$ for $\mathcal{P}$ an order filter within the proper part of the lattice flats of M



$\mathcal{P} = \emptyset$ $\rightsquigarrow$ $A(M, \emptyset) \cong \mathbb{Z}[x]/(x^r)$ independent of M, trivially satisfying the Hodge properties

$\mathcal{P} = \{\text{all proper non-}\emptyset\text{ flats}\}$ $\rightsquigarrow$ $A(M, \mathcal{P}) = A(M)$

Each time they enlarge $\mathcal{P}$ by one flat F (of rank ≥ 2), say $\mathcal{P}_+ = \mathcal{P}_- \sqcup \{F\}$, they show an explicit iso.

$$A^i(M, \mathcal{P}_-) \oplus A^{i-p}(\underbrace{M/F}_{\text{contraction of } M \text{ on } F}) \xrightarrow{\sim} A^i(M, \mathcal{P}_+)$$

orthogonal direct sum with respect to the quadratic forms, respecting the primitive parts, etc. (lots of details and work...)

Induction both on $\#\mathcal{P}$ and on rank(M) ∇