Monoids Seminar 4/12/2022
Chapter 6 The Grothendieck Ring

§6.1 is generalities on the ...

Grothendieck ring $G_0(\Bbbk M)$ for $\Bbbk$ a field, M a finite monoid

$:\overset{\text{DEF'N}}{=}$ free $\mathbb{Z}$-module on basis $\{[V] : \text{isomorphism classes of fin dim'l } \Bbbk M\text{-mods}\}$

$\Big/ \operatorname{span}_{\mathbb{Z}} \{[V] - ([U]+[W]) : 0 \to U \to V \to W \to 0 \text{ a s.e.s. of } \Bbbk M\text{-mods}\}$

and multiplication via

$$[V]\cdot[W] := [V \otimes_{\Bbbk} W]$$

$$m(v \otimes w) := m(v) \otimes m(w)$$

PROP 6.1: This multiplication is

- well-defined
- associative, commutative
- makes $G_0(\Bbbk M)$ a ring with 1

proof: Well-defined since

$$0 \to U_1 \to U_2 \to U_3 \to 0$$ short exact (so $[U_2]=[U_1]+[U_3]$)

$$\Downarrow V \otimes_{\Bbbk} (-)$$

$$0 \to V\otimes_{\Bbbk} U_1 \to V \otimes_{\Bbbk} U_2 \to V \otimes_{\Bbbk} U_3 \to 0$$ also short exact

(so $[V\otimes U_2]=[V\otimes U_1]+[V\otimes U_3]$)

because exactness is about $\Bbbk$-modules (vector spaces) and V is a free $\Bbbk$-module.

Associative, commutative because $\otimes$ is, up to iso.

$1 = [\Bbbk]$ = (class of) trivial module where every $m \in M$ acts as 1 on $\Bbbk$

since $\Bbbk \otimes V \cong V$ is a $\Bbbk M$-mod isomorphism $\blacksquare$

$1 \otimes v \mapsto v$

REMARK

Every element $\sum_{i=1}^{t} z_i [V_i]$ in $G_0(\Bbbk M)$

$$= \sum_{i : z_i > 0} z_i [V_i] + \sum_{i : z_i < 0} z_i [V_i]$$

$$= [V] - [W] \text{ for fin. dim'l } V, W$$

Letting $\mathrm{Irr}_{\Bbbk}(M) := \left\{ \begin{array}{l} \text{isomorphism classes} \\ \text{of simple } \Bbbk M\text{-mods } S \end{array} \right\}$

then recall that Jordan-Hölder Theorem says

$\forall$ simples $S \in \mathrm{Irr}_{\Bbbk}(M)$, every $\Bbbk$-mod V has

a well-defined composition multiplicity

$[V : S] :=$ # of composition factors $V_i / V_{i-1} \cong S$

in any composition series

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{t-1} \subset V_t = V$$

PROP 6.2: $0\to U\to V\to W\to 0$ s.e.s.

$$\Rightarrow [V:S]=[U:S]+[W:S]\quad \forall S\in \mathrm{Irr}_k(M)$$

Proof: Replace U with a submodule of V and W with V/U (up to isomorphisms), and then pick a comp. series

$$0=\underbrace{V_0\subset V_1\subset\cdots\subset}_{\text{a comp. series for } U} \underset{\overset{\shortparallel}{U}}{V_r}\ \underbrace{\subset\cdots\subset V_{t-1}\subset V_t}_{\text{lifting a comp. series for } V/U}=V$$

$$\Rightarrow [V:S]=[U:S]+[V/U:S] \qquad \blacksquare$$

COROLLARY: (PROP 6.3) The map
$$G_0(\Bbbk M) \longrightarrow \mathbb{Z}^{\mathrm{Irr}_{\Bbbk}(M)}$$
$$[V] \longmapsto \big([V:S]\big)_{S \in \mathrm{Irr}_{\Bbbk}(M)}$$
is an abelian group isomorphism (no ring statement)

and $[V] \overset{(*)}{=} \sum_{S \in \mathrm{Irr}_{\Bbbk}(M)} [V:S] \cdot [S]$ in $G_0(\Bbbk M)$.

proof: It's a well-defined map by Jordan-Hölder and PROP 6.2.

It's surjective since $[S] \longmapsto e_S = S^{th}$ standard basis element on right.

It's injective if we believe $(*)$, which is easy to prove by induction on comp. length of V:

$$0 = \underbrace{V_0 \subset V_1 \subset \cdots \subset V_{t-2} \subset V_{t-1}}_{[V_{t-1}] = \sum_S [V_{t-1}:S] \cdot [S] \text{ by induction.}} \subset V_t = V \quad \blacksquare$$

REMARK: Contrast $G_0(\Bbbk M)$ with the more subtle representation ring

$$R_{\Bbbk}(M) \overset{\text{DEF}}{:=} \frac{\text{free } \mathbb{Z}\text{-module on basis } \left\{ [V] : \begin{array}{l}\text{isomorphism classes of}\\ \text{fin dim'l } \Bbbk M\text{-mods}\end{array} \right\}}{\operatorname{span}_{\mathbb{Z}}\left\{ [V \oplus W] - ([V]+[W]) \right\}}$$

and similar multiplication $[V]\cdot[W] = [V \otimes_{\Bbbk} W]$.

It has a ring surjection

$$\begin{array}{ccc} R_{\Bbbk}(M) & \twoheadrightarrow & G_0(\Bbbk M) \\ [V] & \longmapsto & [V] \end{array}$$

and one has

$$R_{\Bbbk}(M) \cong \mathbb{Z}^{\left\{ \begin{array}{l}\text{iso. classes of}\\ \text{indecomposable } \Bbbk M\text{-mods}\end{array} \right\}}$$

a trickier set, often infinite

using the Krull-Schmidt Thm.

§6.2 The restriction isomorphism

To understand more about the ring $G_0(\Bbbk M)$, we recast some of the Chap. 5 results.

Recall for an idempotent $e \in M$

we had
$$\begin{array}{ccccc} A & \supset & eAe & \supset & G_e \\ \shortparallel & & \text{ring} & & \shortparallel \\ \Bbbk M & & & & (eAe)^\times \end{array}$$
group of units

and a restriction map

$$\begin{array}{ccccc} A\text{-mods} & \longrightarrow & eAe\text{-mods} & \longrightarrow & \Bbbk G_e\text{-mods} \\ V & \longmapsto & eV & \longmapsto & eV =: \mathrm{Res}_e(V) \end{array}$$

PROP 6.4: The map
$$\begin{array}{ccc} G_0(\Bbbk M) & \xrightarrow{\mathrm{Res}_e} & G_0(\Bbbk G_e) \\ [V] & \longmapsto & [eV] \end{array}$$

induces a well-defined ring homomorphism.

PROP 6.4: The map $G_0(\Bbbk M) \xrightarrow{Res_e} G_0(\Bbbk G_e)$
$[V] \longmapsto [eV]$

induces a well-defined ring homomorphism.

proof: $Res_e(-)$ is exact (PROP. 4.2):

$$0 \to U \to V \to W \to 0 \quad \text{s.e.s}$$
$$\Downarrow$$
$$0 \to eU \to eV \to eW \to 0 \quad \text{s.e.s}$$

because $eV \cong \mathrm{Hom}_A(Ae, V)$
$\varphi(e) \longleftarrow\!\shortmid\ \varphi$

a projective A-module

So it induces a well-defined abelian gp. homom.

To see it respects multiplication, check that inside $V \otimes W$ one has $e(V \otimes W) = eV \otimes eW$:

$\subseteq$: By def'n, $e(v \otimes w) = ev \otimes ew \in eV \otimes eW$

$\supseteq$: $ev \otimes ew = e(ev \otimes ew) \in e(V \otimes W)$

Lastly note it maps $1_{G_0(\Bbbk M)} \longmapsto 1_{G_0(\Bbbk G_e)}$:

$$Res_e([\Bbbk]_{\Bbbk M}) = [e\Bbbk] = [\Bbbk]_{\Bbbk G_e} \qquad \blacksquare$$

GOAL: Compile these $\mathrm{Res}_e(-)$ maps into a direct sum isomorphism $G_0(\Bbbk M) \to \bigoplus_i G_0(\Bbbk G_{e_i})$

Recall for a $\Bbbk M$-simple S that an idempotent $e \in M$ was called an apex for S if $eS \neq 0$ but every $m \in I_e := eMe \setminus G_e$ has $mS = 0$.

always $\Bbbk G_e$-simple if e is an apex for S

PROP 5.4 If e is an apex for S, then

(i) $\{m \in M : mS = 0\} = I(e) \quad \left(:= \{m \in M : e \notin MmM\} \right)$

$e \notin MmM \Leftrightarrow MeM \not\subseteq MmM$

(ii) Another idempotent $f \in M$ is an apex for this same $S \iff e \underset{\mathcal{J}}{\sim} f$

i.e. $MeM = MfM$

THEOREM 5.5 M a finite monoid, $\Bbbk$ a field

(i) $\exists$ a bijection

$$\left\{ \begin{array}{l} S \in \mathrm{Irr}_{\Bbbk}(M) \\ \text{with apex } e \end{array} \right\} \rightleftarrows \mathrm{Irr}(\Bbbk G_e)$$

$$S \xmapsto{\mathrm{Res}_e} eS$$

$$\mathrm{soc}(\mathrm{Coind}_{G_e}(V)) =: V^{\#} \longleftarrow V$$

$$[\mathrm{top}(\mathrm{Ind}_{G_e}(V)) = \ldots =]$$

(ii) Every $\Bbbk M$-simple S has an apex e, unique up to $\mathcal{J}$-equivalence (by PROP 5.4 (ii))

((iii) $\forall$ $\Bbbk G_e$-simples V, every comp. factor of $\mathrm{Coind}_{G_e}(V)$ or $\mathrm{Ind}_{G_e}(V)$ has apex f with $MeM \subseteq MfM$,

and $V^{\#}$ is the unique comp. factor with apex e of $\mathrm{Coind}_{G_e}(V)$ or $\mathrm{Ind}_{G_e}(V)$, appearing with multiplicity 1.)

DON'T NEED – But helpful for Exercise 6.1?

So now we order a set of idempotents

$$e_1, e_2, \ldots, e_s$$

representing the (regular) $\mathcal{J}$-classes of M, insisting that if $Me_jM \subseteq Me_iM$ then $j \leq i$.

Then put a consistent total order on

$$\bigsqcup_{i=1}^{s} \mathrm{Irr}(\Bbbk G_{e_i})$$

, and use the bijection from THM 5.5 to give the same order on $\mathrm{Irr}_{\Bbbk}(M)$.

THM 6.5 With this set-up, one has a ring isomorphism

$$G_0(\Bbbk M) \xrightarrow{\mathrm{Res}} \prod_{i=1}^{s} G_0(\Bbbk G_{e_i})$$

$$[V] \longmapsto ([e_1V], \ldots, [e_sV])$$

which is lower unitriangularly expressed in our ordered $\mathbb{Z}$-bases for $G_0(\Bbbk M)$

and $\prod_{i=1}^{s} G_0(\Bbbk G_{e_i})$.

proof: We saw each $\mathrm{Res}_e : [V] \mapsto [eV]$ is a ring map, so this Res is one also.

If $W^{\#} \xleftrightarrow{\text{THM 5.5}} W$

$\mathrm{Irr}(\Bbbk M) \ni W^{\#}$ with apex e_j, $W \in \mathrm{Irr}(\Bbbk G_{e_j})$

then we know $e_j W^{\#} = W$

and PROP 5.4 (i) said $I(e_j) := \{m \in M : mW^{\#} = 0\}$

$= \{m \in M : MeM \not\subseteq MmM\}$

so $e_i W^{\#} = 0$ unless $Me_jM \subseteq Me_iM$

$\Rightarrow j \leq i$

Hence

$$\mathrm{Res}([W^{\#}]) = \sum_{i=1}^{s} [e_i W^{\#}]$$

$$= \underbrace{[e_j W^{\#}]}_{[W]} + \sum_{i : i > j} \sum_{V \in \mathrm{Irr}(\Bbbk G_{e_i})} [e_i W^{\#} : V] \cdot [V]$$

unitriangular

$\blacksquare$

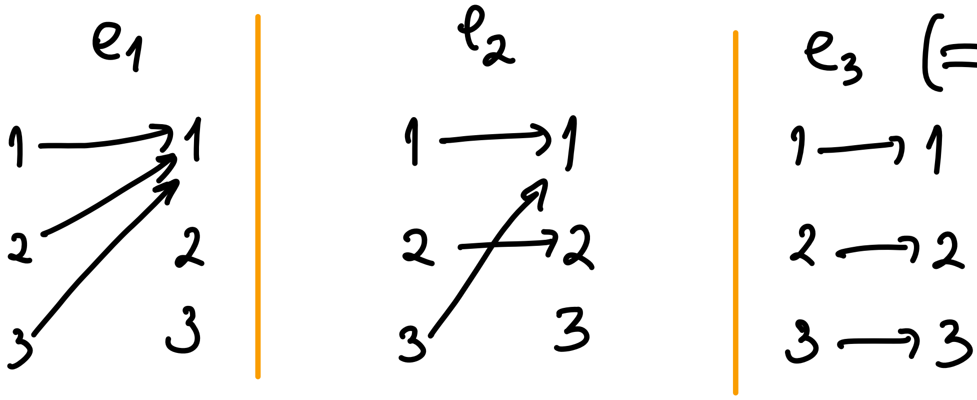
EXAMPLE 6.7

T_n = full transformation monoid

$= \{\text{all maps } f : [n] \to [n]\}$

$\mathcal{J}$-class of f is determined by $\# \operatorname{im}(f)$, and all $\mathcal{J}$-classes regular, rep'd by idempotents

e_1 | e_2 | e_3 $(=1)$



with $G_{e_r} \cong S_r$ = symmetric group on r letters

$\quad\, \shortparallel$

$(e_r T_n e_r)^\times$

having $\Bbbk G_{e_r}$-simples (for $\operatorname{char}(\Bbbk) = 0$)

given by Specht modules $\{\mathcal{S}_\lambda\}_{\lambda \vdash r}$

The matrix L expressing our map

$$G_0(\Bbbk T_3) \longrightarrow \prod_{i=1}^{3} G_0(\Bbbk G_{e_i})$$
$$= G_0(\Bbbk S_1) \times G_0(\Bbbk S_2) \times G_0(\Bbbk S_3)$$

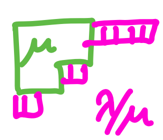
is

$$L = \begin{array}{c} \\ \mathbb{S}_{(1)} \\ \mathbb{S}_{(1,1)} \\ \mathbb{S}_{(2)} \\ \mathbb{S}_{(1,1,1)} \\ \mathbb{S}_{(2,1)} \\ \mathbb{S}_{(3)} \end{array}
\begin{array}{c} \begin{array}{cccccc} \mathbb{S}^{\#}_{(1)} & \mathbb{S}^{\#}_{(1,1)} & \mathbb{S}^{\#}_{(2)} & \mathbb{S}^{\#}_{(1,1,1)} & \mathbb{S}^{\#}_{(2,1)} & \mathbb{S}^{\#}_{(3)} \end{array} \\
\left[\begin{array}{c|cc|ccc}
1 & & & & & \\ \hline
0 & 1 & & & & \\
1 & 0 & 1 & & & \\ \hline
0 & 0 & 0 & 1 & & \\
0 & 1 & 1 & 0 & 1 & \\
1 & 0 & 1 & 0 & 0 & 1
\end{array}\right] \end{array}$$

$$L = \begin{array}{c|c} & \begin{matrix} \mathcal{S}^{\#}_{(1)} & \mathcal{S}^{\#}_{(1,1)} & \mathcal{S}^{\#}_{(2)} & \mathcal{S}^{\#}_{(1,1,1)} & \mathcal{S}^{\#}_{(2,1)} & \mathcal{S}^{\#}_{(3)} \end{matrix} \\ \begin{matrix} \mathcal{S}_{(1)} \\ \mathcal{S}_{(1,1)} \\ \mathcal{S}_{(2)} \\ \mathcal{S}_{(1,1,1)} \\ \mathcal{S}_{(2,1)} \\ \mathcal{S}_{(3)} \end{matrix} & \left[\begin{array}{c|cc|ccc} 1 & & & & & \\ \hline 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ \hline 0 & 0 & 0 & 1 & & \\ 0 & 1 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \end{array}$$

where

$$L_{\lambda\mu} = \begin{cases} 1 & \text{if } \mu = (1^r) \text{ and } \lambda = (m-r+1, 1^{r-1}) \text{ for } m \geq r \\ & \left(\text{because THM 5.9 said } \mathcal{S}^{\#}_{(1^r)} = \Lambda^{r-1}(V) \text{ where } V = \{\underline{x} \in \Bbbk^n : \sum_i x_i = 0\} \text{ with apex } e_r \text{ and } e_m \Lambda^{r-1}(V) = \mathcal{S}_{(m-r+1,1^{r-1})} \text{ for } m \geq r\right) \\ 1 & \text{if } \mu \neq (1^r) \text{ and } \lambda/\mu \text{ is a horizontal strip} \\ & \left(\text{because COR 5.13 said for } 0 \leq r \leq n,\ \mu \neq (1^r):\ \mathcal{S}^{\#}_{\mu} = \Bbbk \underbrace{I_{n,r}}_{\substack{\text{injective} \\ \text{maps } [r] \to [n]}} \otimes_{\Bbbk S_r} \mathcal{S}_\mu \cong \mathcal{S}_\mu \boxtimes \mathbb{1}_{S_{n-r}} \text{ as } S_n\text{-reps (induction product)}, \text{ and } e_m \mathcal{S}^{\#}_{\mu} \cong \mathcal{S}_\mu \boxtimes \mathbb{1}_{S_{m-r}} \text{ as } S_m\text{-reps}\right) \\ 0 & \text{else} \end{cases}$$



Back to more generalities ...

§6.3 The triangular Grothendieck ring

DEF'N:

$$G_0^{\triangledown}(\Bbbk M) := \mathrm{span}_{\mathbb{Z}}\{[S] : S \in \mathrm{Irr}_{\Bbbk}(M),\ \dim_{\Bbbk} S = 1\}$$

a $\mathbb{Z}$-submodule of $G_0(\Bbbk M)$,

but also a subalgebra, since $\dim_{\Bbbk} S = \dim_{\Bbbk} T = 1$

$\Rightarrow \dim_{\Bbbk} S \otimes T = 1$

It's call the triangular Grothendieck ring because of ...

LEMMA 6.8 A $\Bbbk M$-mod V has

$[V] \in G_0^{\triangle}(\Bbbk M) \iff V$ is triangularizable ("$\triangle$-ble"):

$\exists$ a basis $\{v_i\}$ for V in which every $m \in M$ acts triangularly:

$$m = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_n \end{array} & \begin{bmatrix} * & & * & \\ & * & & \\ & & \ddots & \\ 0 & & & * \end{bmatrix} \end{array}$$

proof: If V is Δ-ble by basis $\{v_i\}$, then $V_i := \mathrm{span}_{\Bbbk}\{v_1, v_2, \dots, v_i\}$ gives a $\Bbbk M$-mod filtration $0 = V_0 \subset V_1 \subset \dots \subset V_{n-1} \subset V_n = V$ with factors V_i/V_{i-1} that are 1-dim'l, so simple, and hence $[V] \in G_0^{\Delta}(\Bbbk M)$.

Conversely, if $[V] \in G_0^{\Delta}(\Bbbk M)$ then any comp. series for V gives a triangularizing basis $\blacksquare$

COROLLARY 6.9: V, W Δ-able $\Rightarrow$ $V \otimes_{\Bbbk} W$ Δ-able.
(and $\Bbbk$ is Δ-able).

proof: V, W Δ-able

$\iff [V], [W] \in G_0^{\Delta}(\Bbbk M)$

$\Rightarrow [V][W] = [V \otimes W] \in G_0^{\Delta}(\Bbbk M)$

$\iff V \otimes W$ Δ-able. $\blacksquare$

§6.4 The Grothendieck group (not ring) of projectives

Recall a projective $\Bbbk M$-mod P is one with a lifting property for surjections:
$$\begin{tikzcd} & & P \arrow[d] \arrow[dll, dotted, "\exists"'] & \\ W \arrow[rr] & & V \arrow[r] & 0 \end{tikzcd}$$

Equivalently P is a direct summand of a free $\Bbbk M$-module: $(\Bbbk M)^I = P \oplus P'$ for some P'

DEF'N: $K_0(\Bbbk M) :=$

$$\frac{\text{free } \mathbb{Z}\text{-module on basis } \{[P] : \text{isomorphism classes of fin dim'l projective } \Bbbk M\text{-mods}\}}{\mathrm{span}_{\mathbb{Z}}\{[P \oplus P'] - ([P] + [P'])\}}$$

Since P projective and $P = P_1 \oplus P_2$ imply P_1, P_2 projective, $K_0(\Bbbk M)$ will be $\mathbb{Z}$-spanned by $[P_i]$ for indecomposable projectives P_i.

But the Krull-Schmidt Theorem says that in any decomposition $P = \bigoplus_{i=1}^{t} P_i$ into indecomposable kM-mods P_i, the multiset of isomorphism types of the P_i is the same. So this shows

$$K_0(kM) \cong \mathbb{Z}^{\{\text{iso. classes of proj. indecomposables } P_i\}}$$

Also, as before, every element in $K_0(kM)$ can be written as $[P]-[Q]$ with P, Q projectives and one can write the $+$ operation as

$$([P]-[Q])+([P']-[Q']) = [P\oplus P']-[Q\oplus Q'].$$

REMARK: P, Q projective $\not\Rightarrow$ $P\otimes Q$ projective in general, so there is no ring structure on $K_0(kM)$ coming from $\otimes$ in general.

Since direct sums $P \oplus Q$ also give a s.e.s.

$$0 \to P \to P \oplus Q \to Q \to 0,$$

one has a well-defined $\mathbb{Z}$-module map

"Cartan"

$$\begin{array}{ccc} K_0(\Bbbk M) & \xrightarrow{C} & G_0(\Bbbk M) \\ [P] & \longmapsto & [P] = \sum\limits_{S \in \mathrm{Irr}_{\Bbbk}(M)} [P:S] \cdot [S] \end{array}$$

Recalling that there is a bijection

$$\left\{ \begin{array}{l} \text{iso. classes of} \\ \text{indecomp. proj.} \\ \Bbbk M\text{-mods } P_i \end{array} \right\} \rightleftarrows \left\{ \begin{array}{l} \text{iso. classes of} \\ \text{simple } \Bbbk M\text{-mods } S_i \end{array} \right\}$$

$$P_i \longmapsto \text{top}(P_i) := P_i / \mathrm{rad}(P_i)$$

$$\text{proj. cover } P(S_i) \longleftarrow S_i$$

one has corresponding $\mathbb{Z}$-bases for $K_0(\Bbbk M), G_0(\Bbbk M)$ and the matrix for C above in those bases is called the Cartan matrix.

(a method for computing it comes in § 7.5)

Since $0 \to U \to V \to W \to 0$ s.e.s.

$$\Downarrow \mathrm{Hom}_{kM}(P,-) \text{ for } P \text{ projective}$$

$$0 \to \mathrm{Hom}_{kM}(P,U) \to \mathrm{Hom}_{kM}(P,V) \to \mathrm{Hom}_{kM}(P,W) \to 0 \text{ s.e.s},$$

get well-defined $\langle [P], - \rangle$ functional on $G_0(kM)$:

$$\langle [P], [V] \rangle := \dim_k \mathrm{Hom}_{kM}(P,V).$$

And since $\langle [P \oplus Q], [V] \rangle = \dim_k \underbrace{\mathrm{Hom}_{kM}(P \oplus Q, V)}_{\cong \mathrm{Hom}(P,V) \oplus \mathrm{Hom}(Q,V)}$

$$= \langle [P],[V] \rangle + \langle [Q],[V] \rangle,$$

it becomes a well-defined $\mathbb{Z}$-bilinear pairing

$$K_0(kM) \times G_0(kM) \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{Z}.$$

$$([P], [V]) \longmapsto \langle [P],[V] \rangle$$

One can reformulate it for a proj. indecomposable $P_i = Af_i$ for some idempotent $f_i \in A = \Bbbk M$:

$$\langle [P_i], [V] \rangle = \dim_{\Bbbk} \mathrm{Hom}_A(P_i, V)$$
$$= \dim_{\Bbbk} \mathrm{Hom}_A(Af_i, V)$$
$$= \dim_{\Bbbk}(f_i V)$$
$$= [V : S_i] \text{ where } S_i = \mathrm{top}(P_i)$$

if $\Bbbk$ is algebraically closed; requires work, see Webb Chap. 7

In particular, $\langle [P_i], [S_j] \rangle = [P_i : S_j] = \delta_{ij}$

so $\{[P_i]\}_{i=1,\dots,r}$ and $\{[S_j]\}_{j=1,\dots,r}$ are dual $\mathbb{Z}$-bases for $K_0(\Bbbk M)$ and $G_0(\Bbbk M)$ with respect to $\langle -, - \rangle$.