New combinatorics from the invariant theory of reflection groups

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Outline

I. New combinatorics:
the cyclic sieving phenomenon
with three examples.

II. Reflection groups

III. Generalizations of the three examples.

IV. How invariant theory helps.

I. The cyclic sieving phenomenon (CSP) (–, Stanton, and White 2004)

Given

- a finite set X, and
- a polynomial $X(q) \in \mathbb{Z}[q]$, and
- a cyclic group C permuting X,

the triple $(X, X(q), C)$ exhibits the CSP

if for

- any c in C and
- any root-of-unity $\omega \in \mathbb{C}^{\times}$ of the same order

one has

$$|X^c| = [X(q)]_{q=\omega}.$$

In examples,
most often $X(q) \in \mathbb{N}[q]$, and
sometimes $X(q)$ is a generating function
for X of the form

$$X(q) = \sum_{x \in X} q^{s(x)}.$$

Special case when $C = \mathbb{Z}_2$:
Stembridge's $q = -1$ phenomenon (1994):

$$[X(q)]_{q=-1} = |X^c|$$

for some involution $c : X \to X$

Example 1

Let

$$X := k\text{-subsets of } \{1, 2, \ldots, n\}$$

$$X(q) := q\text{-binomial coefficient}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

where

$$[n]!_q := [n]_q \cdots [2]_q [1]_q$$

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

$$C := \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$$

cyclically permuting $\{1, 2, \ldots, n\}$

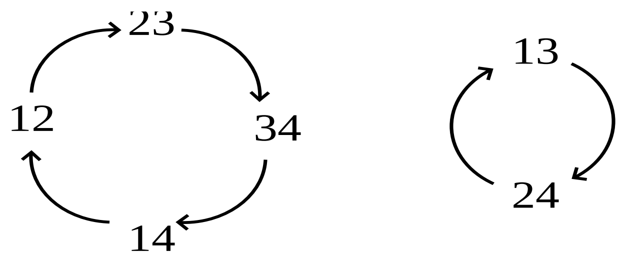
and thus permuting k-subsets .

Example 1 (continued)

For $n = 4, k = 2$, the set

$$X = \{12, 13, 14, 23, 24, 34\}$$

carries this action of $C = \mathbb{Z}_4$:



$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]_q [3]_q}{[2]_q} = 1 + q + 2q^2 + q^3 + q^4$$

evaluates at 4^{th}-roots of unity as

$$X(\omega) = \begin{cases} 6 & \text{if } \omega = 1 \\ 2 & \text{if } \omega = -1 \\ 0 & \text{if } \omega = \pm i \end{cases}$$

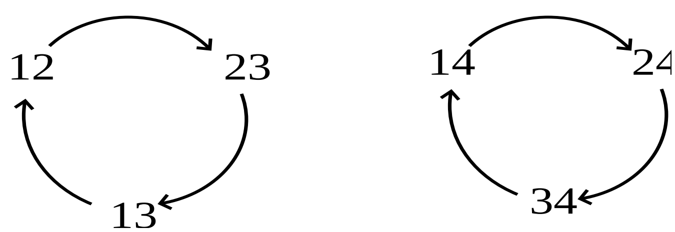
matching the fixed-point cardinalities $|X^c|$ for elements c in C of the same orders.

Example 1 (continued)

Same set X and
same polynomial $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ work with a
different cyclic group $C = \mathbb{Z}_{n-1}$,
generated by the $(n-1)$-cycle

$$c = (1\ 2\ \cdots\ n-2\ n-1)(n).$$

For $n = 4, k = 2$, one has this action of $C = \mathbb{Z}_3$:



$$\text{and } X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1 + q + 2q^2 + q^3 + q^4$$

evaluates at 3^{rd}-roots of unity as

$$X(\omega) = \begin{cases} 6 & \text{if } \omega = 1 \\ 0 & \text{if } \omega = e^{\frac{\pm 2\pi i}{3}}. \end{cases}$$

Example 1 (continued)

WARNING:
Same set X and
same polynomial $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$
generally fails for cyclic actions $C = \langle c \rangle$
unless c acts on $\{1, 2, \ldots, n\}$ as a power of
an n-cycle or $(n-1)$-cycle.

In fact, for other roots of unity ω,
one will generally have $[X(q)]_{q=\omega} \notin \mathbb{N}$.

Foreshadowing...
Powers of n-cycles and $(n-1)$-cycles
are exactly the regular elements
of the symmetric group $W = \mathfrak{S}_n$.

Remark: The $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ in Example 1 had many extra features:

- It's a simple generating function for X:
$$X(q) = \sum_{x \in X} q^{s(x)}$$
where $s(x) = \left(\sum_{i \in x} i\right) - \binom{k}{2}$ for a k-subset x of $\{1, 2, \ldots, n\}$.

- It has a product formula, making $[X(q)]_{q=\omega}$ easy to evaluate (useful for brute force proofs of CSP's).

- It has meaning for $q = p^k$ a prime power (counting k-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$).

- It's the Hilbert series of some naturally occurring graded ring (from invariant theory).

- $X(q)$ is the character of a naturally occurring representation (of $sl_2(\mathbb{C})$ on $\wedge^k \mathbb{C}^n$).

Examples 2, 3

For both of these examples, let

$$X(q) := \mathrm{Cat}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

the q-Catalan number (Fürlinger-Hofbauer 1985).

$$\begin{aligned} \mathrm{Cat}_1(q) &= 1 \\ \mathrm{Cat}_2(q) &= 1 + q^2 \\ \mathrm{Cat}_3(q) &= 1 + q^2 + q^3 + q^4 + q^6 \\ \mathrm{Cat}_4(q) &= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 \\ &\quad + q^7 + 2q^8 + q^9 + q^{10} + q^{12} \end{aligned}$$

There are plenty* of sets X counted by the Catalan numbers

$$X(1) = \mathrm{Cat}_n = \frac{1}{n+1} \binom{2n}{n},$$

many with natural cyclic group C actions.
We'll consider two such sets X...

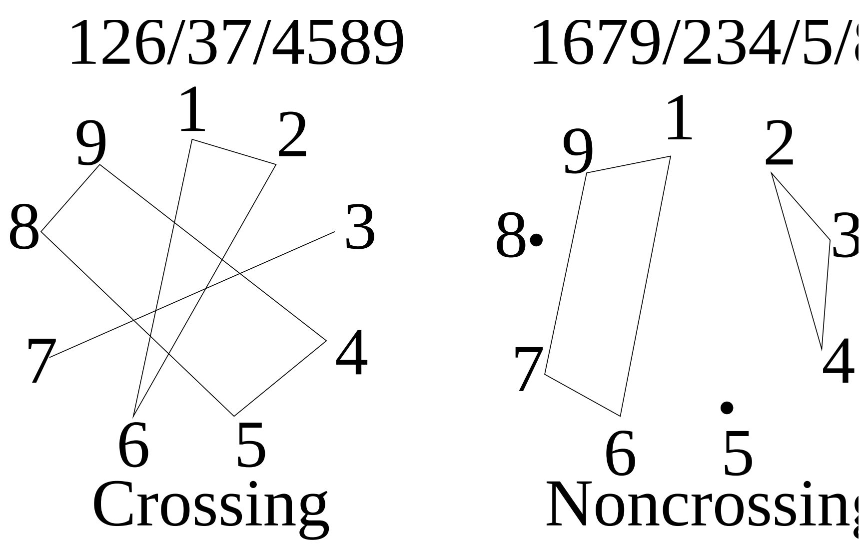
*At least 142 on December 20, 2006, according to Richard Stanley

Example 2

$$X = NC(n)$$
$$:= \text{noncrossing partitions}$$
$$\text{of the set } \{1, 2, \ldots, n\},$$

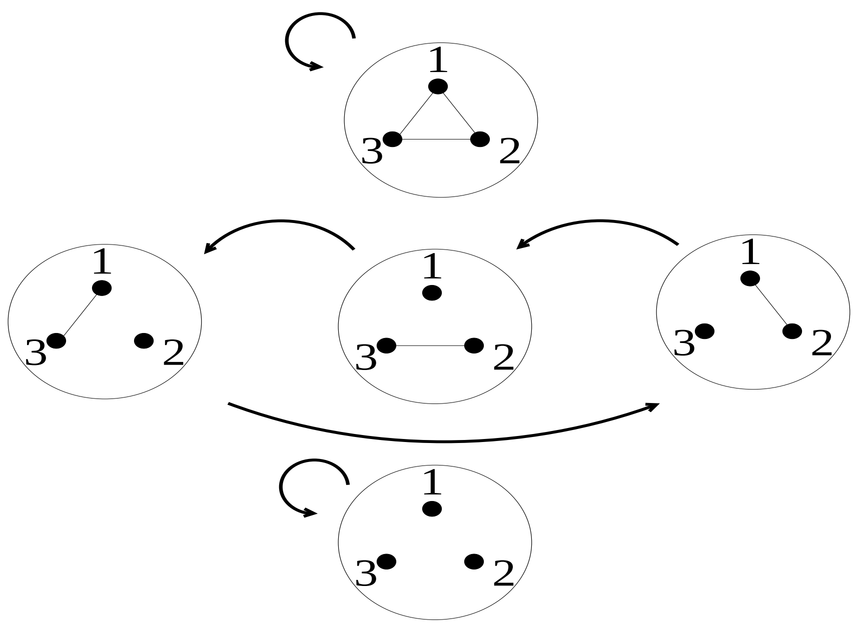
$$X(q) = \mathrm{Cat}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

$C = \mathbb{Z}_n$ cyclically rotating $\{1, 2, \ldots, n\}$.



Example 2 (continued)

$n = 3$: the action of $C = \mathbb{Z}_3$ on $X = NC(3)$



Meanwhile

$$X(q) = \text{Cat}_3(q) = 1 + q^2 + q^3 + q^4 + q^6$$

evaluates at 3^{rd}-roots of unity as

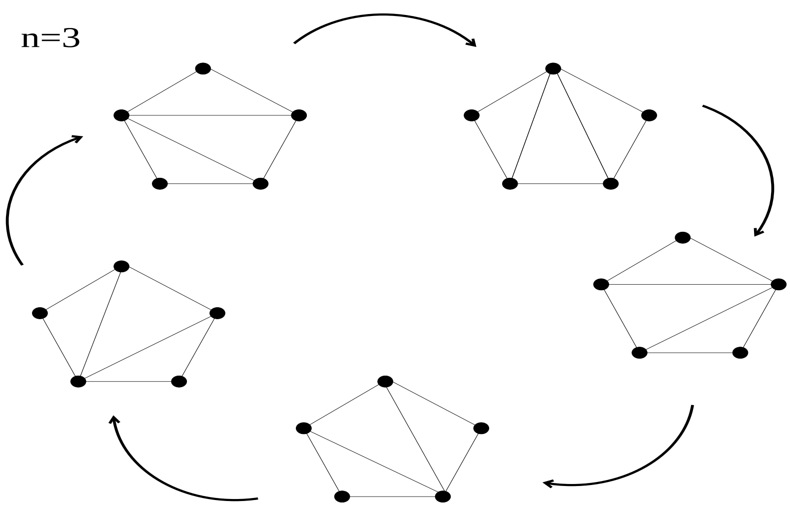
$$X(\omega) = \begin{cases} 5 & \text{if } \omega = 1 \\ 2 & \text{if } \omega = e^{\pm\frac{2\pi i}{3}} \end{cases}$$

Example 3

X = triangulations of a convex $(n+2)$-gon,

$$X(q) = \text{Cat}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

$C = \mathbb{Z}_{n+2}$ via rotation.



$$X(q) = \text{Cat}_3(q) = 1 + q^2 + q^3 + q^4 + q^6$$

evaluates at 5^{th}-roots of unity as

$$X(\omega) = \begin{cases} 5 & \text{if } \omega = 1 \\ 0 & \text{if } \omega = (e^{\frac{2\pi i}{5}})^j \text{ for } j = 1, 2, 3, 4 \end{cases}$$

II. Reflection groups

Examples 1,2,3 all generalize in some way to reflection groups.

Q: What's a reflection group?

A: Not totally clear, but here's the definition we'll use ...

Definition. Let $\mathbb{F}$ be any field,
V an n-dimensional vector space over $\mathbb{F}$.
A reflection group is a finite subgroup

$$W \subset GL(V)(\cong GL_n(\mathbb{F}))$$

for which the W-action on the symmetric algebra

$$S := \mathrm{Sym}(V^*)(\cong \mathbb{F}[x_1, \ldots, x_n])$$

has invariant subring S^W a polynomial algebra.

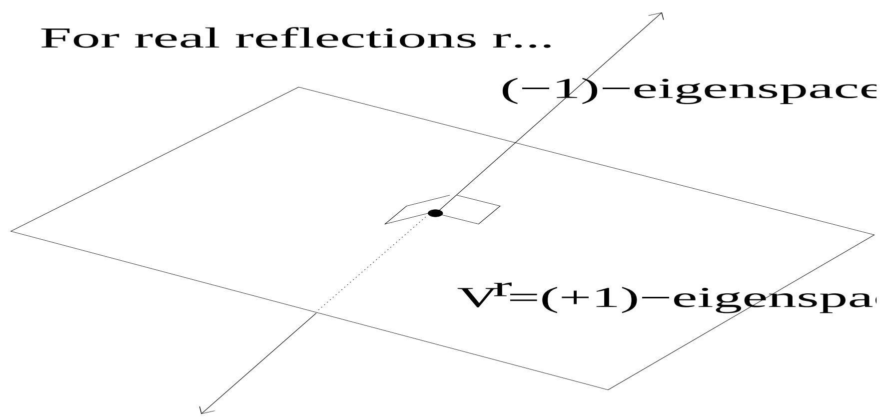
$$S^W = \mathbb{F}[f_1, \ldots, f_n].$$

Q: Where was the word “reflection” in that definition?
A: It’s implicit via a theorem of Serre...

Theorem(Serre 1967)
For finite subgroups $W \subset GL_n(\mathbb{F})$,
S^W polynomial implies
W is generated by reflections.

But you have to interpret “reflection” broadly when working over an arbitrary field $\mathbb{F}$...

Here a reflection means any element r of $GL(V)$ with fixed subspace V^r is of codimension 1, i.e., V^r is a hyperplane.



Warning: one allows
• diagonalizable reflections whose non-unit eigenvalue is a root of unity in $\mathbb{F}^\times$ not necessarily -1,

• non-diagonalizable reflections, called transvections, in positive characteristic.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Motivating precursor in characteristic zero:

Theorem
(Chevalley 1955, Shephard and Todd 1954)
For finite subgroups $W \subset GL_n(\mathbb{F})$,
with $\mathbb{F}$ of characteristic zero,
S^W polynomial if and only if
W is generated by reflections.

Shephard and Todd classified
all finite complex reflection groups,
and used this to prove the theorem,
along with much amazing numerology
of the fundamental degrees

$$d_1 \leq d_2 \leq \cdots \leq d_n$$

for any choice of homogeneous
basic invariants $f_1, \ldots, f_n$ generating S^W.

Chevalley proved the theorem uniformly.

The Shephard-Todd classification has
one infinite family $G(d, e, n)$
for positive integer d, e, n with e dividing d, and
34 exceptional groups

$G(d, e, n) = n \times n$ matrices with
– exactly one nonzero entry in each row/column,
– required to be d^{th} roots-of-unity,
– whose product is a $\frac{d}{e}^{th}$ root-of-unity.

The family $G(d, e, n)$ contains the
- symmetric groups $\mathfrak{S}_n$ (type A_{n-1}) as $G(1, 1, n)$,
- Weyl groups of type B_n/C_nas $G(2, 1, n)$,

(and wreath products $\mathbb{Z}_d \wr \mathfrak{S}_n$ as $G(d, 1, n)$)
- Weyl groups of type D_n as $G(2, 2, n)$
- dihedral groups of order $2m$ as $G(m, m, 2)$

Some taxonomy of reflection groups

~~Finite groups~~
~~generated by reflections~~

|

Finite linear groups with
polynomial invariants
$(e.g., \ GL_n(\mathbb{F}_q))$
(Example 1?)

|

Complex reflection groups
$(e.g., \ G(d,e,n))$
(Example 1)

|

Well-generated complex reflection groups
= those generated by $\dim_{\mathbb{C}} V$ reflections
$(e.g., \ G(d,1,n), G(e,e,n))$
(Example 2)

|

Finite Coxeter (=real reflection) groups
(types $B_n/C_n, D_n, E_6, E_7, E_8, F_4, H_3, H_4, I_2(m)$)
(Example 3)

|

symmetric groups $\mathfrak{S}_n$
$(= G(1,1,n) =$ type $A_{n-1})$

Example: symmetric groups $W = \mathfrak{S}_n$

Fundamental theorem of symmetric functions:

$$\begin{aligned} S^W &= \mathbb{F}[x_1, \ldots, x_n]^{\mathfrak{S}_n} \\ &= \mathbb{F}[e_1, \ldots, e_n]. \end{aligned}$$

where e_i are elementary symmetric functions

$$\begin{aligned} e_1 &= x_1 + \cdots + x_n \\ e_2 &= x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n \\ &\vdots \\ e_n &= x_1 x_2 \cdots x_n \end{aligned}$$

Fundamental degrees:

1,2,3,. . . ,n

Example: $W = G(d, e, n)$.

Fundamental theorem of symmetric functions implies

$$\begin{aligned} S^W &= \mathbb{F}[x_1, \ldots, x_n]^{G(d,e,n)} \\ &= \mathbb{F}[e_1(\mathbf{x}^d), \ldots, e_{n-1}(\mathbf{x}^d), e_n(\mathbf{x})^{\frac{d}{e}}] \end{aligned}$$

Fundamental degrees:
$d, 2d, 3d \ldots, (n-2)d, (n-1)d$, and $n\frac{d}{e}$

Example: finite general linear groups

$$W = GL_n(\mathbb{F}_q)$$

Dickson's theorem (1911):

$$\begin{aligned} S^W &= \mathbb{F}_q[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_q)} \\ &= \mathbb{F}_q[D_{n,0},\ D_{n,1},\ \ldots, D_{n,n-1}] \end{aligned}$$

where

$$D_{n,k} = \sum_{k\text{-subspaces } U \subset V^*} \prod_{\ell \notin U} \ell(\mathbf{x})$$

are called Dickson polynomials.

For example, $n = 2, q = 2$

$$\begin{aligned} &\mathbb{F}_2[x, y]^{GL_2(\mathbb{F}_2)} \\ &= \mathbb{F}_2[\, xy(x + y)\, ,\, x^2 + xy + y^2\,] \\ &= \mathbb{F}_2[D_{2,0}, D_{2,1}] \end{aligned}$$

Fundamental degrees:

$q^n - q^{n-1},\ q^n - q^{n-2}, \ldots, q^n - q,\ q^n - 1.$

III. Generalizing the examples

Generalizing Examples 1,2, requires Springer's (1972) notion of a regular element in a reflection group W:

an element c having an eigenvector

$$v \in \overline{V} := V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$
$$c(v) = \omega v$$

which is regular in the sense that it is fixed by no reflections for W.

Call $\omega \in \overline{\mathbb{F}}^{\times}$ a regular eigenvalue for c; it will always be a root of unity, of the same order as c.

Example: $W = \text{Sym}_n$ has regular elements

- the n-cycle

$$c = (1\ 2\ \cdots\ n-1\ n)$$

regular eigenvector $v = (1, \omega, \omega^2, \ldots, \omega^{n-1})$, where ω is any primitive n^{th} root-of-unity,

- the $(n-1)$-cycle

$$c = (1\ 2\ \cdots\ n-1)(n)$$

regular eigenvector $v = (1, \zeta, \zeta^2, \ldots, \zeta^{n-1}, 0)$, where ζ is any primitive $(n-1)^{st}$ root-of-unity,

- their conjugates,

- their powers,

and no other regular elements.

Recall Example 1: one has a CSP for

$$X = k\text{-subsets of } \{1, 2, \ldots, n\}$$
$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$
$$C = \langle c \rangle \text{ for } c \text{ an } n\text{-cycle or } (n-1)\text{-cycle}$$

Theorem 1: (–,Stanton,White 2004)
Let $W \subset GL_n(\mathbb{C})$ be a finite reflection group, and $c \in W$ be any regular element. Then

$$X = \text{ any set with transitive } W\text{-action, say } X = W/W'$$
$$X(q) = \frac{\mathrm{Hilb}(S^{W'}, q)}{\mathrm{Hilb}(S^W, q)}$$
$$C := \langle c \rangle \text{ translating the cosets } wW'$$

gives a triple $(X, X(q), C)$ exhibiting the CSP.

Conjecture: $\mathbb{F} = \mathbb{C}$ was unnecessary;
One needs no assumption on the field $\mathbb{F}$, just S^W polynomial.

For Example 1,

$$\begin{aligned} X &= k\text{-subsets of } \{1,2,\ldots,n\} \\ &= \mathfrak{S}_n/(\mathfrak{S}_k \times \mathfrak{S}_{n-k}) \\ &= W/W' \end{aligned}$$

with C-action by translating cosets;
this agrees with cycling $\{1,2,\ldots,n\}$.

Meanwhile

$$\begin{aligned} S^{W'} = \mathbb{F}[&e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k), \\ &e_1(x_{k+1},\ldots,x_n),\ldots,e_{n-k}(x_{k+1},\ldots,x_n)] \end{aligned}$$

so

$$\begin{aligned} X(q) &= \frac{\mathrm{Hilb}(S^{W'},q)}{\mathrm{Hilb}(S^{W},q)} \\ &= \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)\cdots(1-q^k)\cdot(1-q)\cdots(1-q^{n-k})} \\ &= \begin{bmatrix} n \\ k \end{bmatrix}_q . \end{aligned}$$

Example 2 generalizes to well-generated complex reflection groups W, where there is a notion (Bessis 2001,2004) of W-noncrossing partitions...

Numerology shows that when W is well-generated, for the Coxeter number

$$h := d_n = \max\{d_1, \ldots, d_n\}$$

there always exist regular elements with regular eigenvalue $e^{\frac{2\pi i}{h}}$, called Coxeter elements.

In fact, they're all conjugate in W.
So fix one and call it c.

E.g., for $W = \mathfrak{S}_n$, the Coxeter number $h = n$, and the Coxeter elements are n-cycles, so fix

$$c = (1\,2\,\cdots\,n-1\,n).$$

Define the absolute or reflection length on W (not the Coxeter group length!)

$$\ell(w) := \min\{\ell | w = r_1 \cdots r_\ell \text{ for reflections } r_i\}.$$

In fact, $\ell(w)$ is the codimension of the fixed space V^w.

Define the W-noncrossing partitions

$$NC(W) := \{w \in W : \ell(w) + \ell(w^{-1}c) = n\}.$$

Theorem (Bessis 2004, case-by-case):

$$|NC(W)| = \prod_{i=1}^{n} \frac{h + d_i}{d_i} =: W\text{-Catalan number}$$

Note that conjugation by W preserves $\ell(-)$, conjugation by c acts on $NC(W)$.

Recall Example 2: one has a CSP for

$$X = NC(n)$$
$$= \text{ noncrossing partitions of } \{1, 2, \ldots, n\},$$
$$X(q) = \mathrm{Cat}_n(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$
$$C = \mathbb{Z}_n \text{ via cyclic rotation}$$

Theorem 2:(– and Bessis, 2006) Let W be a well-generated complex reflection groups W, and c a chosen Coxeter element. Then

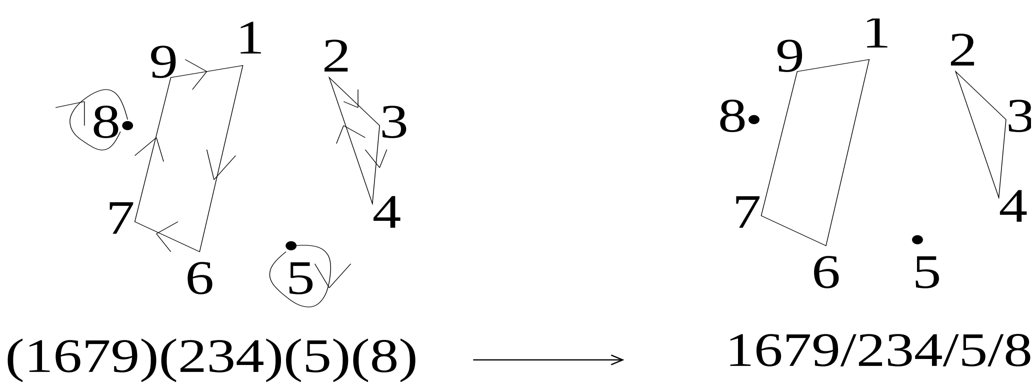
$$X = NC(W) = W\text{-noncrossing partitions}$$
$$X(q) = \mathrm{Cat}(W, q) = W\text{-}q\text{-Catalan number}$$
$$:= \prod_{i=1}^{n} \frac{[h + d_i]_q}{[d_i]_q}$$
$$C = \langle c \rangle \text{ via conjugation}$$

gives a triple $(X, X(q), C)$ exhibiting the CSP.

For $W = \mathfrak{S}_n$, and $c = (1\,2 \cdots n-1\,n)$ the map

$$\begin{array}{ccc} \text{permutations} & \longrightarrow & \text{set partitions} \\ w & \longmapsto & \text{cycles of } w \end{array}$$

restricts to a bijection $NC(W) \to NC(n)$.



Under this correspondence, the C-action by conjugation on $NC(W)$ = rotation on $NC(n)$.

Recall Example 3: one has a CSP for

$$X = \text{ triangulations of an } n\text{-gon}$$
$$X(q) = \mathsf{Cat}_n(q)$$
$$C = \mathbb{Z}_{n+2} \text{ via rotation}$$

triangulations $\rightsquigarrow$ maximal clusters
in the cluster complexes of finite type
(Fomin-Zelevinksy 2003, Fomin-Reading 2006)

rotation $\rightsquigarrow$ "deformed" Coxeter element τ

Theorem 3: (Eu and Fu 2006)
Let W be a finite real reflection group,
with Coxeter number h, and
deformed Coxeter element τ. Then

$$X = \text{ maximal } W\text{-clusters}$$
$$X(q) = \mathsf{Cat}(W, q)$$
$$C := \mathbb{Z}_{h+2} = \langle \tau \rangle$$

gives a triple $(X, X(q), C)$ exhibiting the CSP.

IV. How invariant theory helps

The proof of Theorem 3 (on W-clusters) is (currently) case-by-case.

The proof of Theorem 2 (on $NC(W)$) is partly invariant theory, but uses some facts verified (currently) case-by-case.

The proof of Theorem 1 (on W/W'-cosets) is uniform, and easy via invariant theory...

When $S^W = \mathbb{F}[f_1, \ldots, f_n]$, consider the coinvariant algebra

$$S/(S^W_+) = S/(f_1, \ldots, f_n).$$

Both Chevalley, Shephard-Todd proved, assuming $|W| \in \mathbb{F}^\times$,
one has an isomorphism of W-representations

$$S/(S^W_+) \underset{\mathbb{F}[W]-mod}{\cong} \mathbb{F}[W].$$

Springer generalized this, taking into account the action of a regular element ...

Theorem(Springer 1972)
Assume S^W is polynomial, $|W| \in \mathbb{F}^\times$, and let $C = \langle c \rangle$ for any regular element c, with regular eigenvalue ω^{-1}. Then one has an isomorphism of $W \times C$-representations

$$S/(S_+^W) \underset{\overline{\mathbb{F}}[W \times C]-mod}{\cong} \overline{\mathbb{F}}[W].$$

in which on $S/(S_+^W)$,

- W acts by linear substitutions,
- C acts by scalar substitutions

$$c(x_i) = \omega x_i$$
$$c(f) = \omega^d f \text{ if } \deg(f) = d,$$

while on $\overline{\mathbb{F}}[W]$

- W acts by left-multiplication,
- C acts by right-multiplication.

Proof of Theorem 1:

Starting with the isomorphism

$$S/(S_+^W) \underset{\overline{\mathbb{F}}[W\times C]-mod}{\cong} \overline{\mathbb{F}}[W].$$

restrict to the W'-fixed subspaces:

$$\begin{array}{ccc} (S/(S_+^W))^{W'} & \underset{\overline{\mathbb{F}}[C]-mod}{\cong} & \overline{\mathbb{F}}[W]^{W'} \\ \| & & \| \\ S^{W'}/(S_+^W) & & \overline{\mathbb{F}}[W/W'] \end{array}$$

and then equate the character/trace of $c^i \in C$ on either side:

$$\begin{array}{ccc} \left[\mathrm{Hilb}(S^{W'}/(S_+^W), q)\right]_{q=\omega^i} & = & (W/W')^{c^i} \\ \| & & \| \\ \left[\frac{\mathrm{Hilb}(S^{W'},q)}{\mathrm{Hilb}(S^W,q)}\right]_{q=\omega^i} & & |X^{c^i}| \\ \| & & \\ [X(q)]_{q=\omega^i} & & \end{array}$$

giving the CSP.

What will it take to remove the assumption that $|W| \in \mathbb{F}^\times$?

Without this assumption,
$S^{W'}$ isn't always Cohen-Macaulay

So instead of looking at

$$S^{W'}/(S^W_+) = S^{W'} \otimes_{S^W} \overline{\mathbb{F}} = \mathrm{Tor}_0^{S^W}(S^{W'}, \overline{\mathbb{F}}),$$

prove the following about all of $\mathrm{Tor}_*^{S^W}(S^{W'}, \overline{\mathbb{F}})$, and the same CSP will follow:

Conjecture When S^W is polynomial,
for any subgroup $W' \subset W$, one has a
virtual Brauer-isomorphism
of $N_W(W') \times C$-representations

$$\mathrm{Tor}_*^{S^W}(S^{W'}, \overline{\mathbb{F}}) \underset{\overline{\mathbb{F}}[N_W(W') \times C]-mod}{\sim} \overline{\mathbb{F}}[W/W'].$$

Known for $W' = 1$ (–,Stanton,Webb, 2005).
Known without C-action (–,Smith,Webb 2005).

Recap

Type A combinatorics

$\rightsquigarrow$

reflection group combinatorics

$\rightsquigarrow$

ARSENAL

-invariant theory/commutative algebra

-representation theory

-rational Cherednik and Hecke algebras?

-trace formulae?