A glimpse of Minnesota combinatorics

Vic Reiner
reiner@math.umn.edu

Outline

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1. Our combinatorics group
2. Actitivities and interests
3. Some math!

Personnel

Faculty

- Gregg Musiker (new!)
- Andrew Odlyzko
- Pavlo Pylyavskyy (new!)
- Vic Reiner
- Dennis Stanton
- Dennis White

Postdocs

- Jang Soo Kim
- Ricky Liu

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More personnel

Students

- Adil Ali
- Pat Byrnes
- Alex Csar
- Kevin Dilks
- Rob Edman
- Jia Huang
- Thomas McConville
- Alex Miller
- Nathan Williams
- ... and more

Activities

Weekly seminars:

- Combinatorics seminar
  (sometimes with subsidized dinner!)
- Student combinatorics seminar

2-semester grad course sequences:

- Intro to grad combinatorics (every other year)
- Topics in combinatorics (every other year)

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Our interests

We are interested in central topics of combinatorics such as enumeration, as well as relations of combinatorics to the landscape of modern mathematics, such as

- algebra, including
  - representation theory
  - number theory
  - commutative algebra
- geometry, including
  - discrete geometry
  - algebraic geometry
- topology
- probability
- analysis

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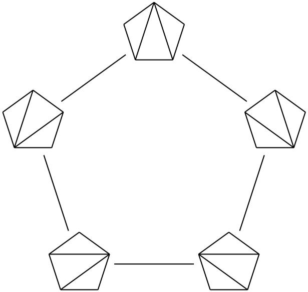
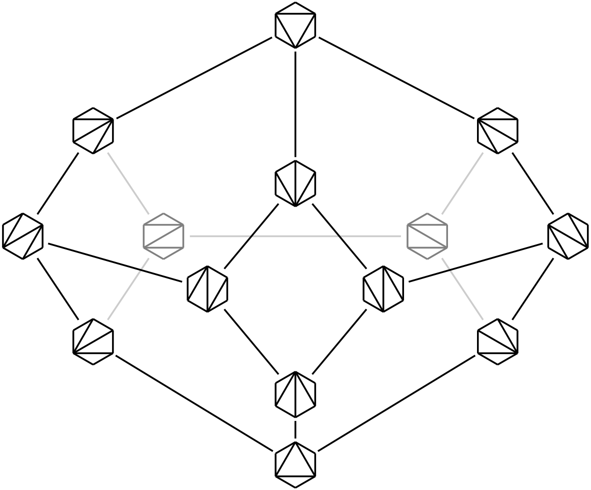
Some counting

Ever counted the triangulations of a convex polygon?
There are 5 for a pentagon...



There are 14 for a hexagon...



The numbers start $1, 1, 2, 5, 14, 42, \cdots$, and there are

$$\frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}$$

for an $(n+2)$-sided polygon, but this isn't obvious!

This is called the n^{th} Catalan number.
E.g. for $n = 4$, one has

$$\frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{70}{5} = 14.$$

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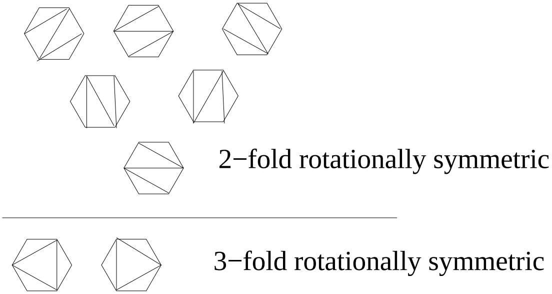
More counting

How many of them have 2-fold rotational symmetry? 3-fold rotational symmetry, etc?



There's a polynomial in q controlling this:
the q-Catalan number

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q = \frac{[2n]!_q}{[n+1]!_q \cdot [n]!_q}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{[n]!_q}{[k]!_q \cdot [n-k]!_q}$$

$$[m]!_q := [1]_q \cdot [2]_q \cdots [m-1]_q \cdot [m]_q$$

$$[m]_q := 1 + q + q^2 + \cdots q^{m-1} = \frac{1-q^m}{1-q}$$

For example, with $n = 4$ again,

$$\frac{1}{[4+1]_q}\begin{bmatrix} 2\cdot 4 \\ 4 \end{bmatrix}_q = \frac{1}{[5]_q}\cdot\frac{[8]!_q}{[4]!_q\cdot[4]!_q}$$

$$= \frac{[8]_q[7]_q[6]_q[5]_q}{[5]_q[4]_q[3]_q[2]_q}$$

$$= \frac{(1-q^8)(1-q^7)(1-q^6)(1-q^5)}{(1-q^5)(1-q^4)(1-q^3)(1-q^2)}$$

$$= 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}$$

Theorem

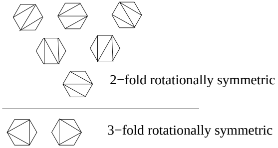
Plugging in a primitive d^{th} root-of-unity to the q-Catalan number

$$\frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

counts the d-fold rotationally symmetric triangulations of a regular $(n+2)$-sided polygon.

For example, for the hexagon,

$$\frac{1}{[4+1]_q}\begin{bmatrix}2\cdot 4\\ 4\end{bmatrix}_q = 1+q^2+q^3+2q^4+q^5+2q^6 + q^7+2q^8+q^9+q^{10}+q^{12}$$



$$= \begin{cases} 14 & \text{plugging in } q = e^{\frac{2\pi i}{1}} = +1, \\ 6 & \text{plugging in } q = e^{\frac{2\pi i}{2}} = -1, \\ 2 & \text{plugging in } q = e^{\frac{2\pi i}{3}}, \\ 0 & \text{plugging in } q = e^{\frac{2\pi i}{6}}. \end{cases}$$

What's with this q-binomial coefficient?

The q-binomial $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is full of meaning!

For example, when q is a power of a prime,
and therefore counts the size of a finite field $\mathbf{F}_q$,

the q-binomial counts k-dimensional subspaces of $V = (\mathbf{F}_q)^n$,
the points in the Grassmannian manifold/variety $Gr(k, V)$.

Check this out...

- The Catalan number

$$\frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$

also counts $\mathbf{Z}/(2n+1)\mathbf{Z}$-orbits when one cycles n element subsets of $\mathbf{Z}/(2n+1)\mathbf{Z}$ mod $2n+1$, and ...

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also counts $\mathbf{F}^{\times}_{q^{2n+1}}$-orbits when one lets $\mathbf{F}^{\times}_{q^{2n+1}}$ cycle the n-dimensional $\mathbf{F}_q$-subspaces of $\mathbf{F}_{q^{2n+1}} \cong (\mathbf{F}_q)^{2n+1}$.

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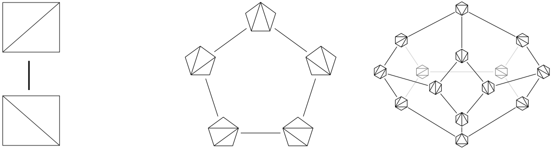
Flips between triangulations

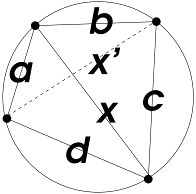
Why did we draw triangulations connected by flip edges?



It makes an interesting convex polyhedron, the associahedron, but also reflects two bits of algebra and geometry...

Ptolemy's relation

For four cocircular points, one has
Ptolemy's relation among their mutual distances:



Ptolemy:
x x' = ac + bd

So one can get rid of x', expressing it as

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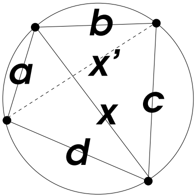

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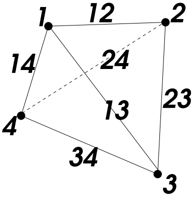
Plücker's relations

The 2×2 minors $p_{ij} := \det \begin{bmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{bmatrix}$ of a 2×4 matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

satisfy a Plücker relation:



Plücker:
$p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}$

So one can get rid of p_{24}, expressing it as

$$p_{24} = \frac{p_{12}p_{34} + p_{14}p_{23}}{p_{13}} = p_{12}p_{34}p_{13}^{-1} + p_{14}p_{23}p_{13}^{-1}$$

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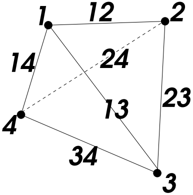

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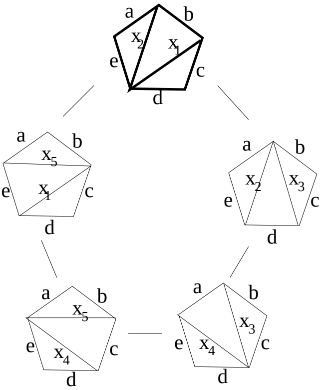
For n cocircular points,
(or 2×2 minors of a $2 \times n$ matrix),

this lets one can express any distance as a rational function in the edges of a chosen triangulation



$$x_3 = \frac{cx_2 + bd}{x_1} = cx_2x_1^{-1} + bdx_1^{-1}$$

$$x_4 = \frac{ex_3 + ad}{x_2} = \frac{e(cx_2x_1^{-1} + bdx_1^{-1}) + ad}{x_2}$$
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$$x_5 = \frac{bx_4 + ac}{x_3} = \frac{b(cex_1^{-1} + bdx_1^{-1}x_2^{-1} + adx_2^{-1}) + ac}{cx_2x_1^{-1} + bdx_1^{-1}}$$
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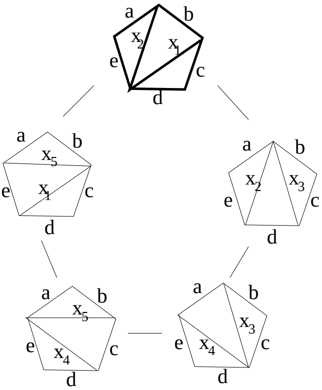

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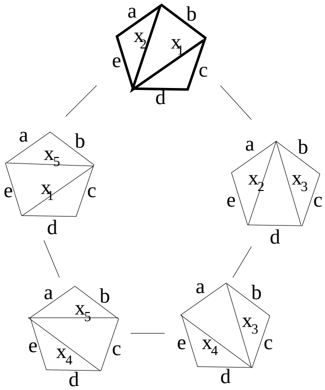

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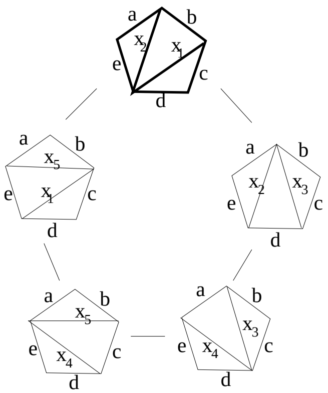

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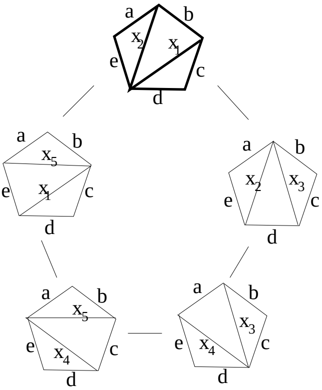

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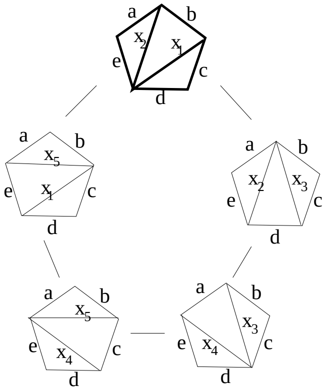

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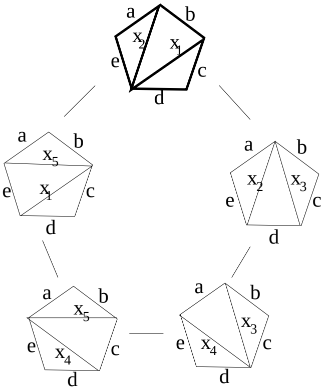

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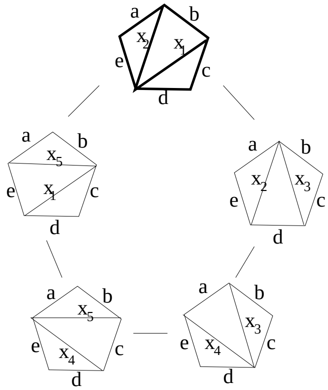

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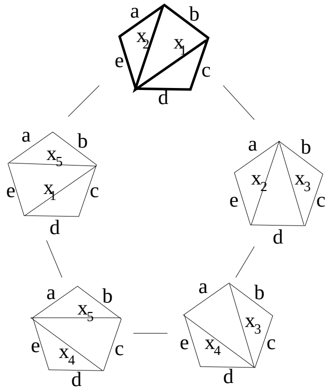

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Surprisingly, these rational functions are always

- Laurent polynomials (the Laurent phenomenon),
- with nonnnegative coefficients.

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and Plücker relations in the coordinate ring
for the Grassmannian of 2-planes
are the first examples of cluster algebras.

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- triangulations on other surfaces with boundary.
- coordinate rings of all Grassmannians.

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Thanks for listening!