A glimpse of Minnesota Combinatorics

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~reiner/combinatorics_group.html

The combinatorialists

Gregg Musiker
Andrew Odlyzko
Pasha Pylyavskyy
Vic Reiner
Dennis Stanton

Ben Brubaker
Peter Webb

- Several postdocs, grad students
- Weekly seminar and grad seminar
- Intro grad sequence Math 8268–8269 alternates yearly with ...
- 2-semester Topics sequence Math 8680

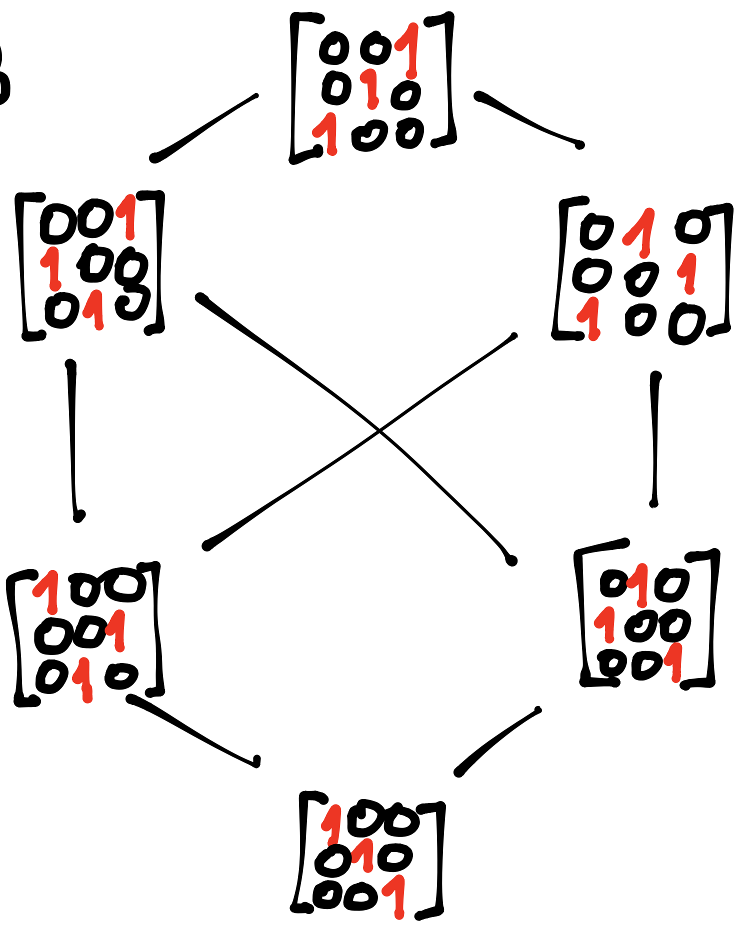
Topics of interest

- enumeration
- algebra
  - representation theory
  - commutative algebra
  - number theory
- geometry & topology
- probability & statistical physics
- analysis & special functions

A sample topic of interest

Permutations of $\{1, 2, \ldots, n\}$

$n = 3$

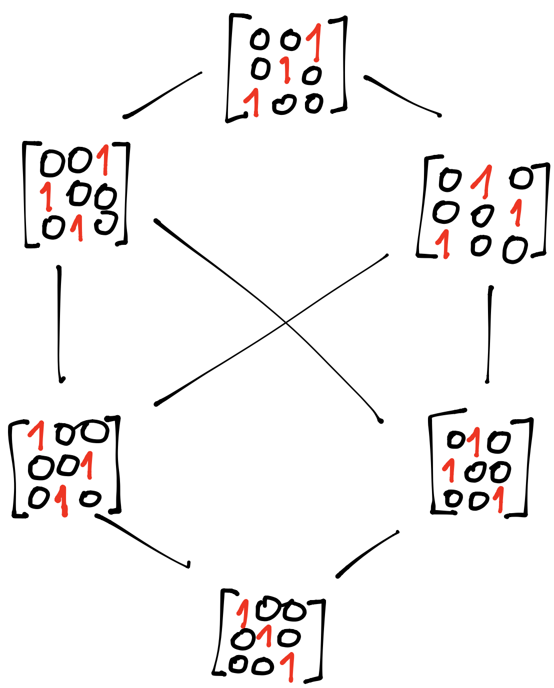


How many? $n!$

$$\sum_{\substack{\text{permutations}\\ \sigma \text{ of } \{1,2,\dots,n\}}} q^{\#\{\text{inversions } i<j \text{ with } \sigma(i)>\sigma(j)\}} = [n]!_q$$

$$= [n]_q[n-1]_q\cdots[3]_q[2]_q[1]_q$$

where $[n]_q = 1+q+q^2+\dots+q^{n-1}$



$$q^3 + 2q^2 + 2q^1 + q^0 = (1+q)(1+q+q^2)$$

$$= [2]_q[3]_q = [3]!_q$$

More generally,

$n \times n$ alternating sign matrices (ASMs)

$$\begin{bmatrix} 0 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix}$$

- entries $+1, 0, -1$
- $+/-$ signs alternate in rows, columns
- row, column sums all $+1$

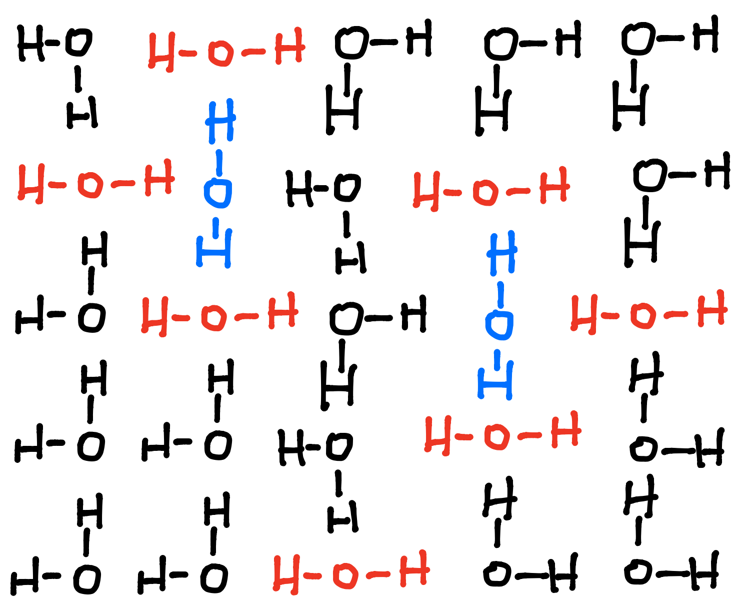
They generalize permutations

$n=3$

$$
\begin{tikzcd}
 & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \arrow[dr, dash] & & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \arrow[dl, dash] \\
 & \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \arrow[dr, dash] & & \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \arrow[dl, dash] \\
 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &
\end{tikzcd}
$$

... and biject with square ice configurations

$$\begin{bmatrix} 0 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix}$$



How many $n \times n$ ASMs?

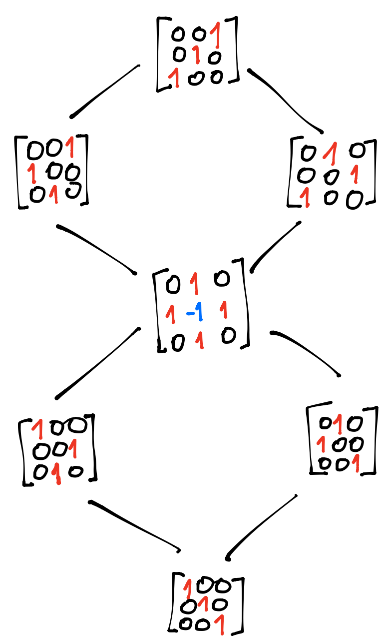
Mills, Robbins, Rumsey conjectured (1983)

THEOREM (Zeilberger 1996
Kuperberg 1996)

Exactly $$\frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!}$$

$n=3$:

$$\frac{1!\,4!\,7!}{3!\,4!\,5!} = \frac{7\cdot 6}{3\cdot 2} = 7$$



The q-analogue does something amazing and mysterious ...

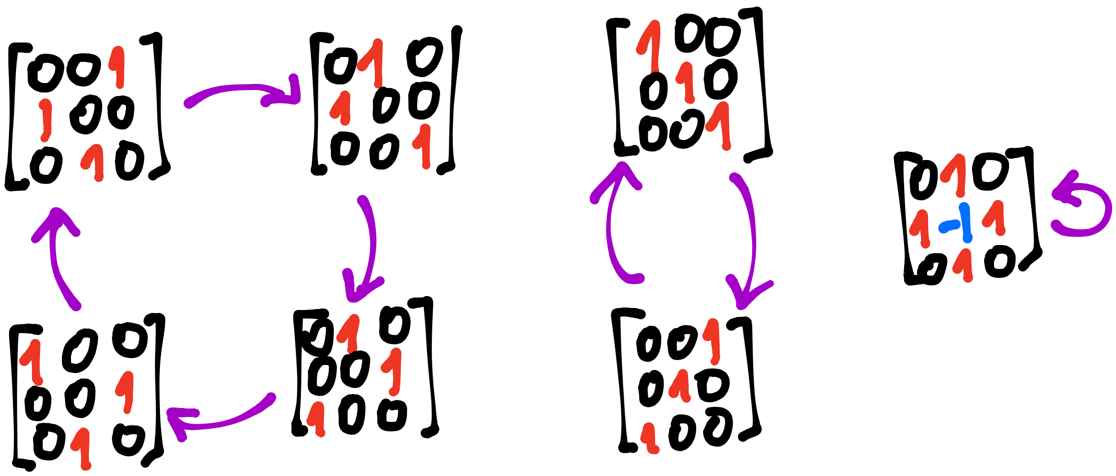
THEOREM (Stanton 2006)

For $d = 0, 1, 2, 3$,

$$\left.\frac{[1]!_q\,[4]!_q\,[7]!_q \cdots [3n-2]!_q}{[n]!_q\,[n+1]!_q \cdots [2n-1]!_q}\right|_{q = \left(e^{\frac{2\pi i}{4}}\right)^d = i^d}$$

counts $n \times n$ ASMs fixed by d^{th} power of 90° rotation

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

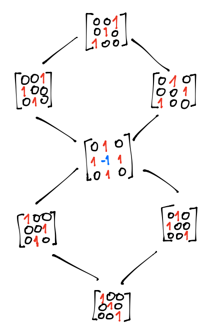


$$\frac{[1]!_q\,[4]!_q\,[7]!_q}{[3]!_q\,[4]!_q\,[5]!_q}$$

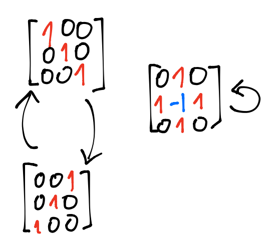
$$= \frac{[7]_q\,[6]_q}{[3]_q\,[2]_q}$$

$$= 1+q^2+q^3+q^4+q^5+q^6+q^8$$

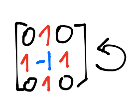
$q=1$ ⟼ 7



$q=-1$ ⟼ 3



$q=\pm i$ ⟼ 1



Statistical sums over ASMs arise in

- work of Brubaker and collaborators on formulas for Whittaker functions of Eisenstein series

- Mills, Robbins, Rumsey's λ-determinant, based on Dodgson's condensation formula (1866)...

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = \frac{\det[a]\det[d] - \det[b]\det[c]}{\det[\,]}$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \frac{\det\begin{bmatrix} a & b \\ d & e \end{bmatrix}\det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - \det\begin{bmatrix} b & c \\ e & f \end{bmatrix}\det\begin{bmatrix} d & e \\ g & h \end{bmatrix}}{\det[e]}$$

THEOREM (Desnanot-Jacobi 1819)

$$\det A = \frac{\det A^{NW} \det A^{SE} - \det A^{NE} \det A^{SW}}{\det A^{middle}}$$

where $A = \begin{bmatrix} \boxed{A^{NW}} & \vdots \\ \cdots & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdots & \cdot \\ \vdots & \boxed{A^{middle}} & \vdots \\ \cdot & \cdots & \cdot \end{bmatrix}$

λ-determinant replaces

$$\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad - bc$$

$\lambda = -1$

with $$\det_\lambda \begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad + \lambda bc$$

$$\det A = \sum_{\substack{\text{permutations} \\ \sigma \text{ of } \{1,2,\ldots,n\}}} (-1)^{\#\text{inversions}(\sigma)} \prod_{i=1} a_{i,\sigma(i)}$$

THEOREM (Robbins-Rumsey 1986)

$$\det_\lambda A = \sum_{\substack{n\times n \\ \text{ASMs } M}} \lambda^{\#\text{inversions}(M)} (1+\lambda^{-1})^{\#(-1)\text{'s in } M} \prod_{i,j} a_{ij}^{M_{ij}}$$

$$\det_{\lambda} A = \sum_{\substack{n\times n \text{ ASMs} \\ M}} \lambda^{\#\text{inversions}(M)} \left(1+\lambda^{-1}\right)^{\#(-1)\text{'s in } M} \prod_{i,j} a_{ij}^{M_{ij}}$$

$$\begin{array}{rl|c}
 & & M \\
\det_{\lambda}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = & aei & \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} \\
 & + \lambda\, afh & \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix} \\
 & + \lambda\, bdi & \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix} \\
 & + \lambda^{2}\, bfg & \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix} \\
 & + \lambda^{2}\, cdh & \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix} \\
 & + \lambda^{3}\, ceg & \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix} \\
 & + \lambda^{3}(1+\lambda^{-1})\; bde^{-1}fh & \begin{bmatrix} 0&1&0\\1&-1&1\\0&1&0 \end{bmatrix}
\end{array}$$

Rewritten Desnanot-Jacobi identity

$$\det A^{NW} \det A^{SE} = \det A^{middle} \det A + \det A^{NE} \det A^{SW}$$

arises also in theory of

- cluster algebras (Musiker, Pylyavskyy)
- total positivity of matrices (Pylyavskyy)
- frieze patterns & quiver representations (Musiker, Webb)

Thanks for your attention

(and see you at lunch!)