

A glimpse of  
Minnesota  
Combinatorics

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[reiner/combinatorics\\_group.html](http://reiner/combinatorics_group.html)

# The combinatorialists

Gregg Musiker

Andrew Odlyzko

Pasha Pylyavskyy

Vic Reiner

Dennis Stanton

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Christine Berkesch

Ben Brubaker

Peter Webb

- Several postdocs,  
grad students
- Weekly seminar  
and grad seminar
- Intro grad sequence  
Math 8668-8669  
alternates yearly with ...
- 2-semester Topics sequence  
Math 8680

# Topics of interest

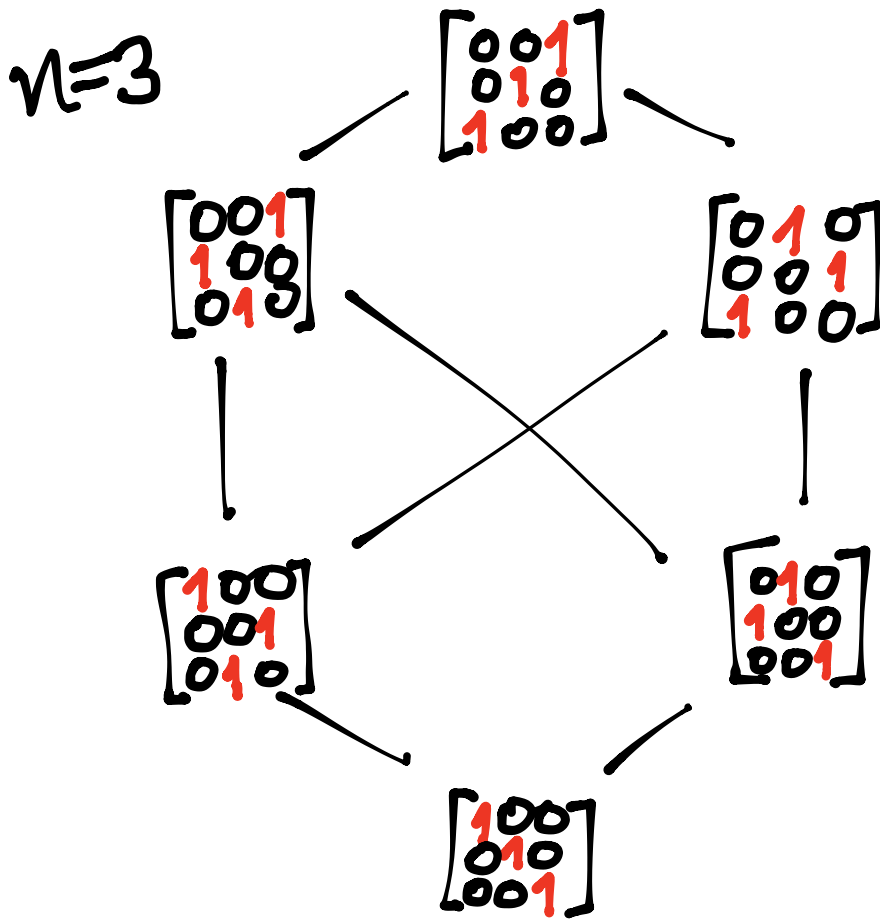
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- enumeration
- algebra
  - representation theory
  - commutative algebra
  - number theory
- geometry & topology
- probability & statistical physics
- analysis & special functions

# A sample topic of interest

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## Permutations of $\{1, 2, \dots, n\}$



How many?  $n!$

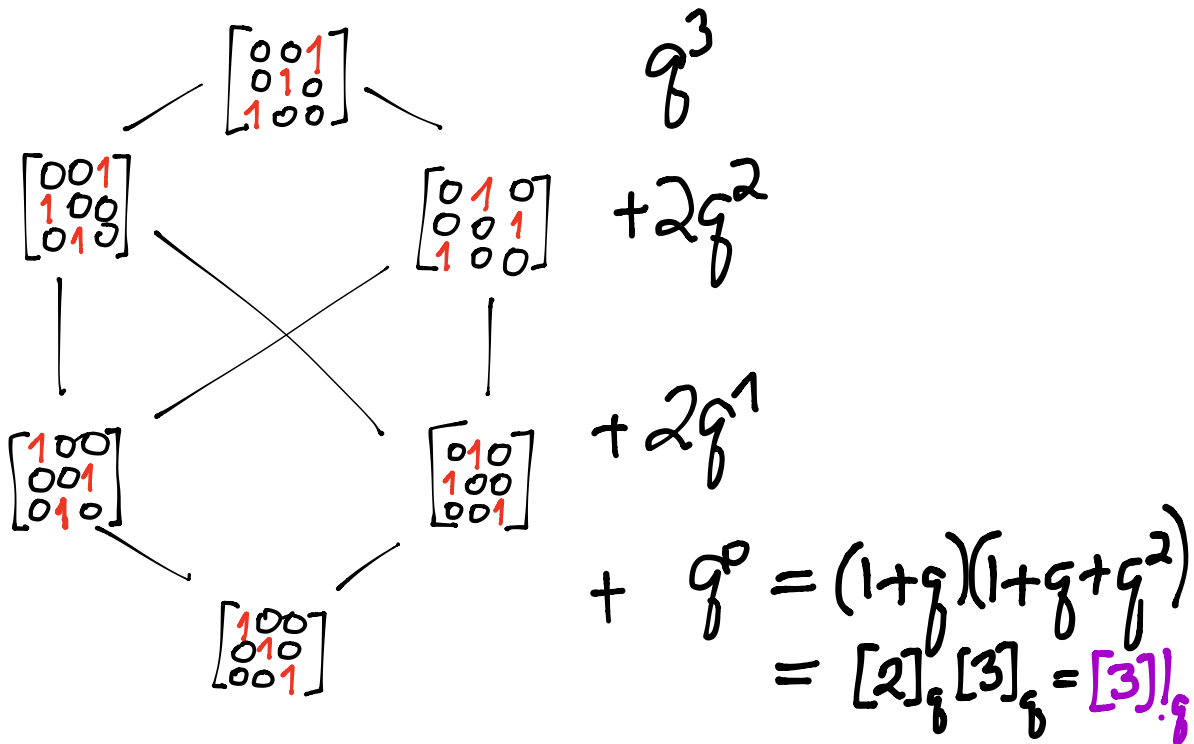
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$$\sum_{\text{permutations } \sigma \text{ of } \{1, 2, \dots, n\}} q^{\#\{\text{inversions } i < j \text{ with } \sigma(i) > \sigma(j)\}} = [n]!_q$$

$$= [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$$

where  $[n]_q = 1 + q + q^2 + \dots + q^{n-1}$

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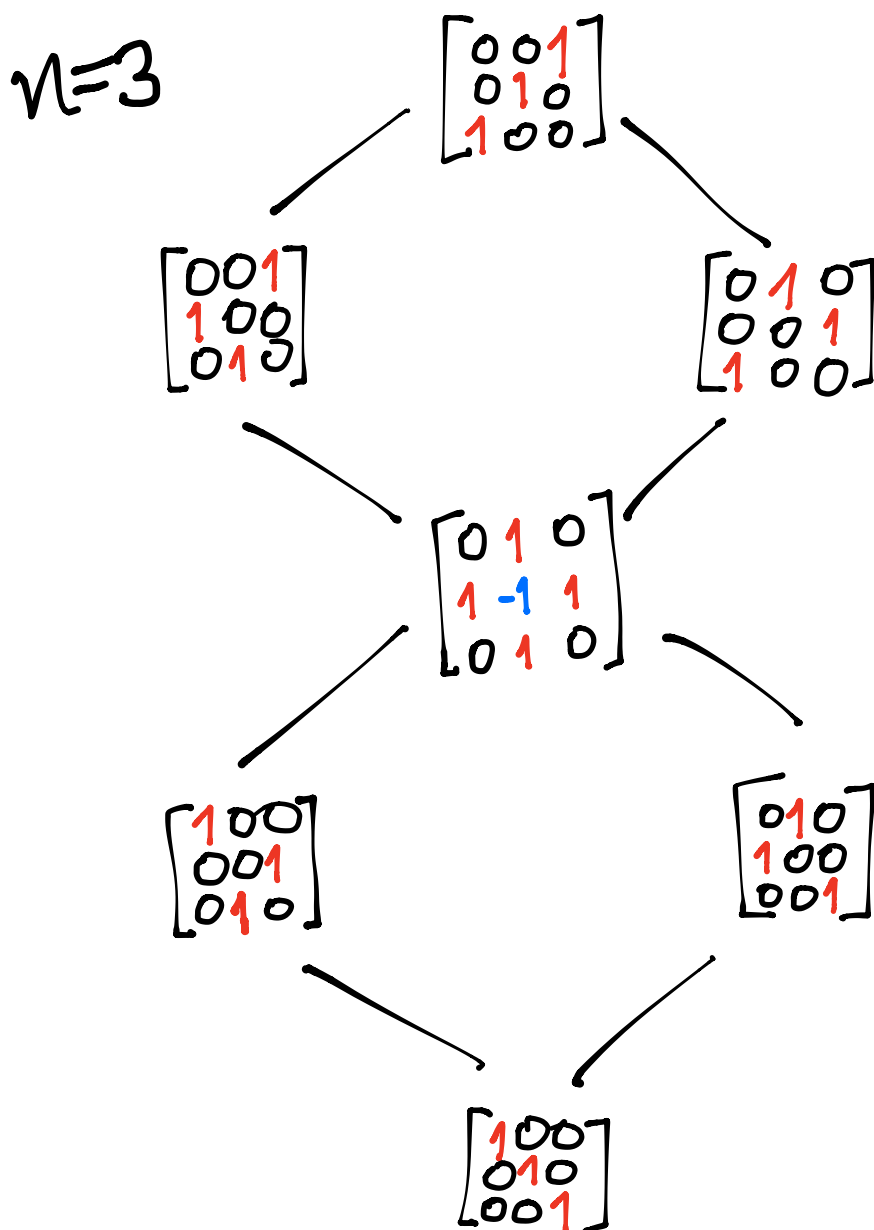


More generally,  
 $n \times n$  alternating sign matrices (ASMs)

$$\begin{bmatrix} 0 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix}$$

- entries  $+1, 0, -1$
- $+/-$  signs alternate in rows, columns
- row, column sums all  $+1$

They generalize permutations

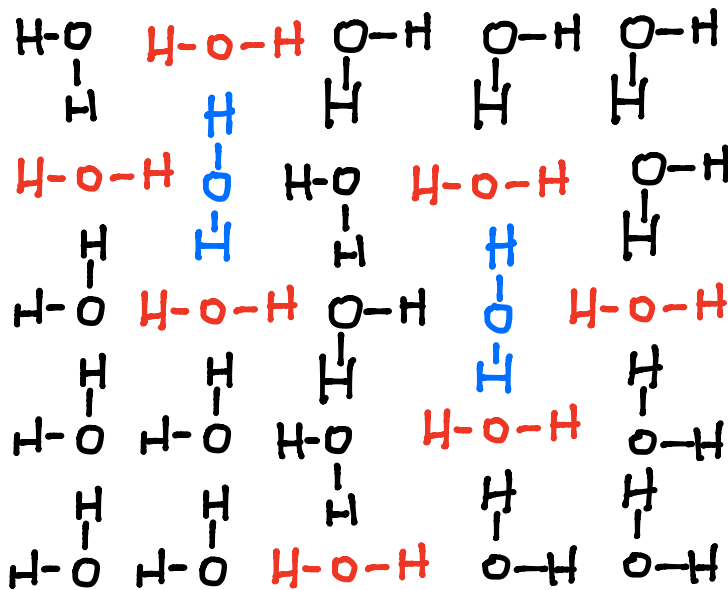




... and biject with square ice configurations

$$\begin{bmatrix} 0 & +1 & 0 & 0 & 0 \\ +1 & -1 & 0 & +1 & 0 \\ 0 & +1 & 0 & -1 & +1 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 \end{bmatrix}$$


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# How many $n \times n$ ASM's?

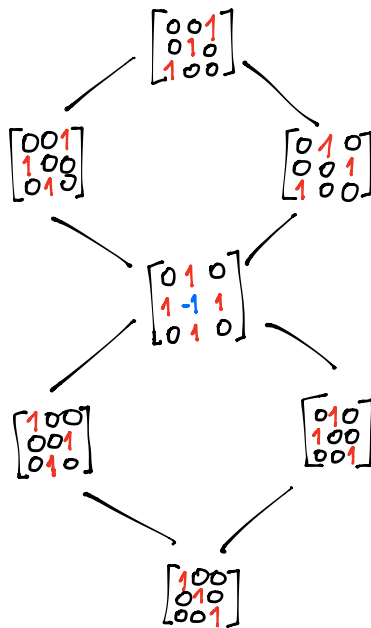
Mills, Robbins, Rumsey conjectured (1983)

**THEOREM** (Zeilberger 1996  
Kuperberg 1996)

Exactly 
$$\frac{1! 4! 7! \dots (3n-2)!}{n! (n+1)! \dots (2n-1)!}$$

$n=3$ :

$$\frac{1! 4! 7!}{3! 4! 5!} = \frac{7 \cdot 6}{3 \cdot 2} = 7$$



The  $q$ -analogue does something amazing and mysterious ...

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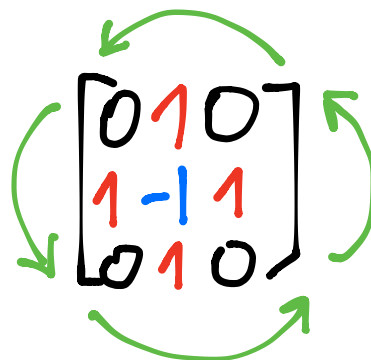
**THEOREM** (Stanton 2006)

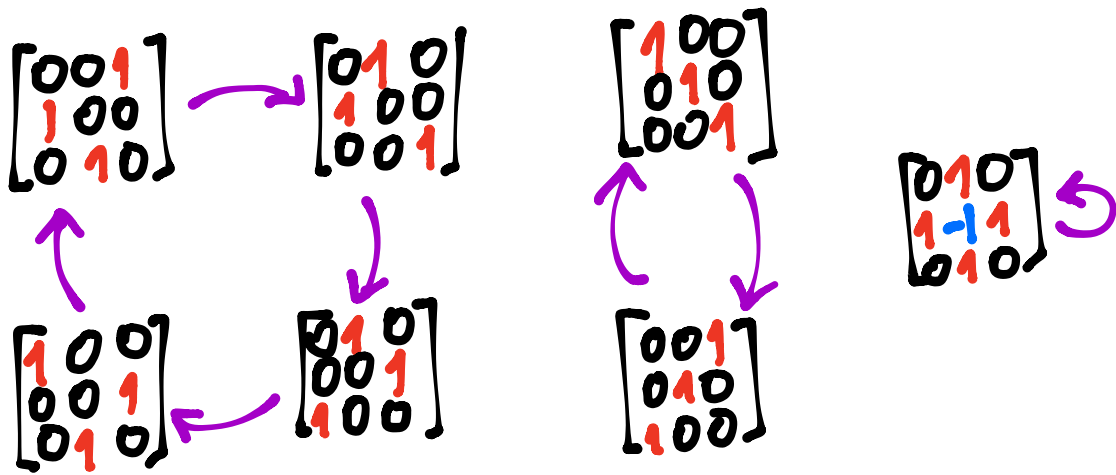
For  $d=0,1,2,3,$

$$\frac{[1]!_q [4]!_q [7]!_q \dots [3n-2]!_q}{[n]!_q [n+1]!_q \dots [2n-1]!_q} \Big|_q = \left( e^{\frac{2\pi i}{4}} \right)^d = i^d$$

counts  $n \times n$  ASMs fixed by

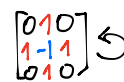
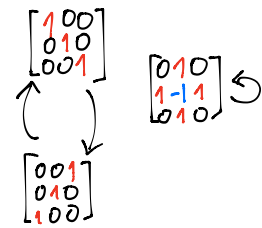
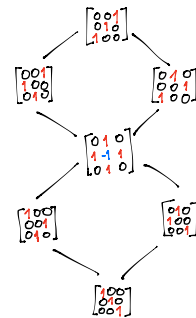
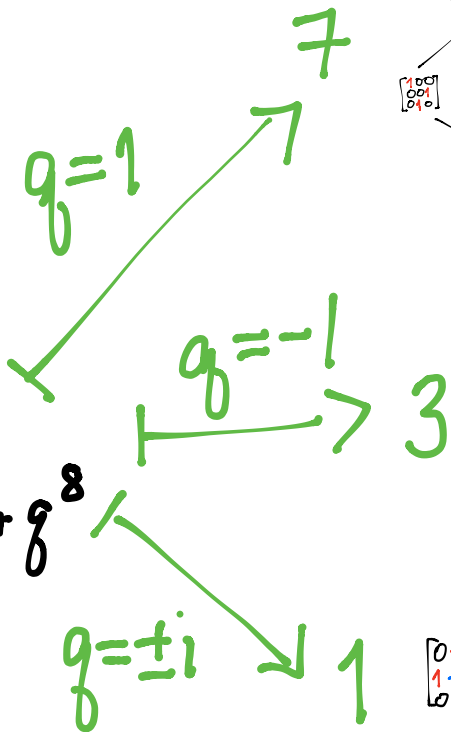
$d^{\text{th}}$  power of  $90^\circ$  rotation





$$\frac{[1]_8 [4]_8 [7]_8}{[3]_8 [4]_8 [5]_8} = \frac{[7]_8 [6]_8}{[3]_8 [2]_8}$$

$$= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8$$



Statistical sums over ASMs arise in

- work of Brubaker and collaborators on formulas for Whittaker functions of Eisenstein series
- Mills, Robbins, Rumsey's  $\lambda$ -determinant, based on Dodgson's condensation formula (1866)...

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc = \frac{\det[a] \det[d] - \det[b] \det[c]}{\det[1]}$$


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$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \frac{\det \begin{bmatrix} a & b \\ d & e \end{bmatrix} \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - \det \begin{bmatrix} bc \\ ef \end{bmatrix} \det \begin{bmatrix} de \\ gh \end{bmatrix}}{\det[e]}$$


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**THEOREM** (Desnanot - Jacobi 1819)

$$\det A = \frac{\det A^{NW} \det A^{SE} - \det A^{NE} \det A^{SW}}{\det A^{\text{middle}}}$$

where  $A = \begin{bmatrix} \begin{matrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ \dots & \dots & \dots & \dots & \dots \end{matrix} \\ \begin{matrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{matrix} \end{bmatrix} = \begin{bmatrix} \dots \\ \dots \\ \dots \\ \begin{matrix} \text{middle} \\ A \\ \dots \end{matrix} \\ \dots \\ \dots \end{bmatrix}$

$\lambda$ -determinant replaces

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

with  $\det_{\lambda} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad + \lambda bc$

$\lambda = -1$

$$\det A = \sum_{\substack{\text{permutations} \\ \sigma \text{ of } \{1, 2, \dots, n\}}} (-1)^{\# \text{inversions}(\sigma)} \prod_{i=1}^n a_{i, \sigma(i)}$$

**THEOREM** (Robbins-Rumsey 1986)

$$\det_{\lambda} A =$$

$$\sum_{\substack{n \times n \\ \text{ASM}_s}} \lambda^{\# \text{inversions}(M)} (1 + \lambda^{-1})^{\#(-1)\text{'s in } M} \prod_{i, j} a_{ij}^{M_{ij}}$$

$$\det_{\lambda} A = \sum_{\substack{n \times n \text{ ASM}_s \\ M}} \lambda^{\# \text{inversions}(M)} (1 + \lambda^{-1})^{\#(-1)'s \text{ in } M} \prod_{i,j} a_{ij}^{M_{ij}}$$

	M
$\det_{\lambda} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$+ \lambda afh$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$+ \lambda bdi$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$+ \lambda^2 bfg$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$
$+ \lambda^2 cdh$	$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$
$+ \lambda^3 ceg$	$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
$+ \lambda^3 (1 + \lambda^{-1}) bde^{-1}fh$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$



Rewritten Desnanot-Jacobi identity

$$\det A^{NW} \det A^{SE} = \det A^{\text{middle}} \det A + \det A^{NE} \det A^{SW}$$

arises also in theory of

- cluster algebras (Musiker, Pylyavskyy)
- total positivity of matrices (Pylyavskyy)
- frieze patterns & quiver representations (Musiker, Webb)

Thanks for  
your  
attention

(and see you at lunch!)