Finite reflection groups and general linear groups

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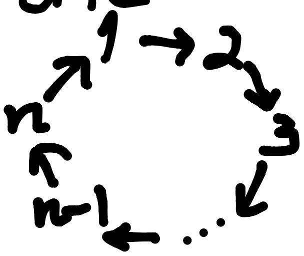
1. Three counts for objects with cyclic actions
   $\binom{n}{k}, n^{n-2}, \frac{1}{n+1}\binom{2n}{n}$
2. The right q-counts
3. Reflection group versions
4. Deformation proof idea
5. $GL_n(\mathbb{F}_q)$-analogues

1. Three counts

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts

k-element subsets of $\{1,2,\dots,n\}$.

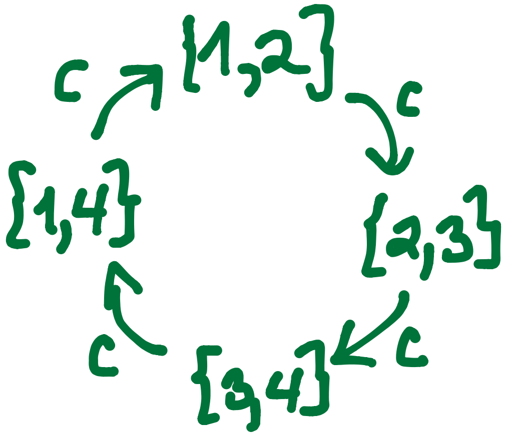
They are permuted by the n-cycle $c=(1,2,\dots,n)=$

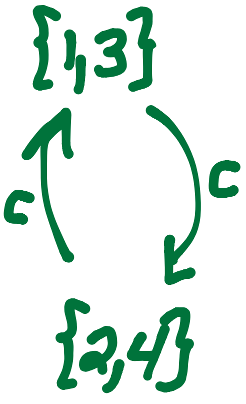


$n=4$
$k=2$

$\binom{n}{k}=\binom{4}{2}=6$ subsets,

in two c-orbits





THEOREM (Hurwitz 1891)

n^{n-2} counts factorizations

$c = (1, 2, \dots, n) = t_1 t_2 \cdots t_{n-1}$

of the n-cycle into

$n-1$ transpositions $t_k = (i, j)$

They are permuted by an operation ψ of order $(n-1) \cdot n$:

$$t_1 t_2 \cdots t_{n-2} t_{n-1} \xmapsto{\psi} c t_{n-1} c^{-1} \cdot t_1 t_2 \cdots t_{n-2}$$

$$\xmapsto{\psi^{n-1}} c t_1 c^{-1} \cdot c t_2 c^{-1} \cdot \ldots \cdot c t_{n-1} c^{-1}$$

$n=4$ $\quad n^{n-2}=4^2=16$ factorizations,

in two ψ-orbits

ψ has order $(n-1)n = 3\cdot 4 = 12$

$c = (1,2,3,4)$

$= (12)(23)(34)$

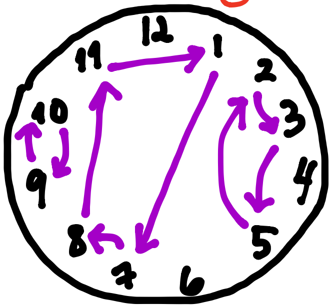
$$\begin{tikzcd}
 & (12)(23)(34) \arrow[dr, "\psi"] & \\
(23)(34)(14) \arrow[ur, "\psi"] & & (14)(12)(23) \arrow[dl, "\psi"] \\
 & (34)(14)(12) \arrow[ul, "\psi"] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (13)(34)(12) \arrow[r, "\psi"] & (23)(13)(34) \arrow[dr, "\psi"] & \\
(34)(12)(24) \arrow[ur, "\psi"] & & & (14)(23)(13) \arrow[d, "\psi"] \\
(12)(24)(23) \arrow[u, "\psi"] & & & (24)(14)(23) \arrow[d, "\psi"] \\
(24)(23)(14) \arrow[u, "\psi"] & & & (34)(24)(14) \arrow[d, "\psi"] \\
(23)(14)(13) \arrow[u, "\psi"] & & & (12)(34)(24) \arrow[dl, "\psi"] \\
 & (14)(13)(12) \arrow[ul, "\psi"] & (13)(12)(34) \arrow[l, "\psi"] &
\end{tikzcd}$$

THEOREM (Kreweras 1972 Biane 1997)

The Catalan number $\frac{1}{n+1}\binom{2n}{n}$ counts the permutations w that can be factored $w=t_1t_2\cdots t_k$ by a prefix of one of the factorizations $c=t_1t_2\cdots t_kt_{k+1}\cdots t_{n-1}$ of $c=(1,2,\ldots,n)$ into $n-1$ transpositions.

(equivalently, non-crossing set partitions[1] of $\{1,2,\ldots,n\}$)



They are permuted by $w \overset{\varphi}{\longmapsto} cwc^{-1}$

(= rotation of noncrossing partitions)

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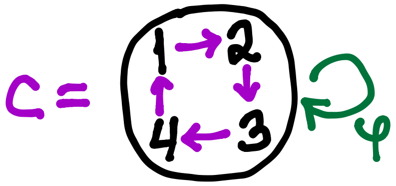
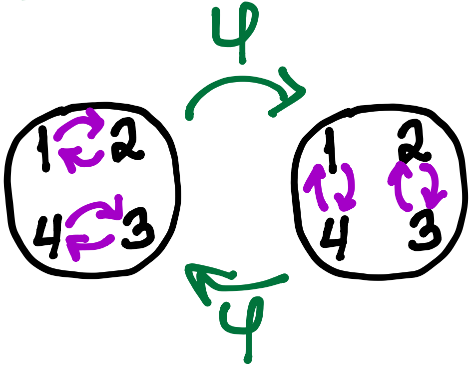
[1] See Stanley, "Catalan Numbers" #159

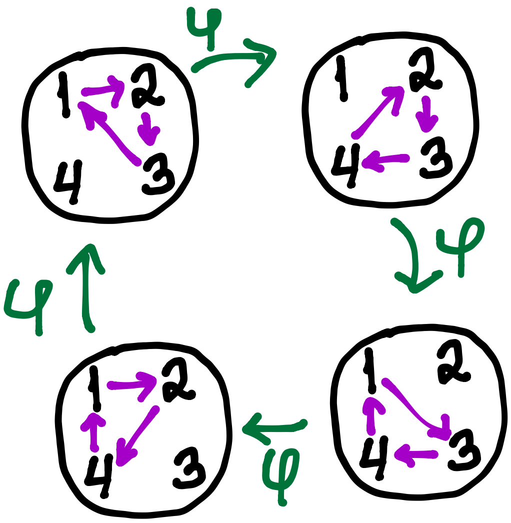
n=4

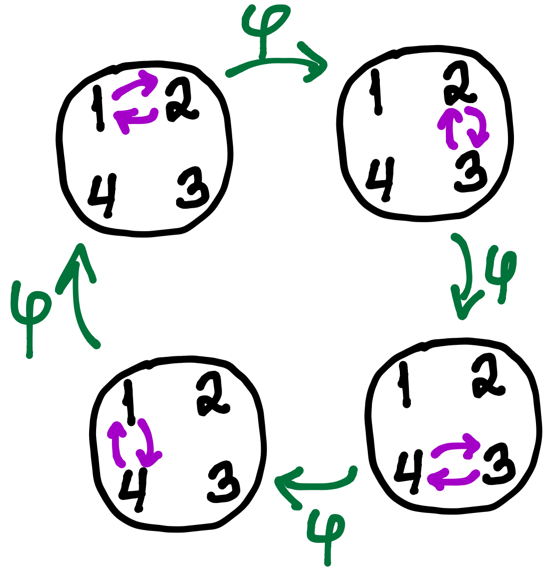
$$\frac{1}{n+1}\binom{2n}{n}=\frac{1}{5}\binom{8}{4}=14$$

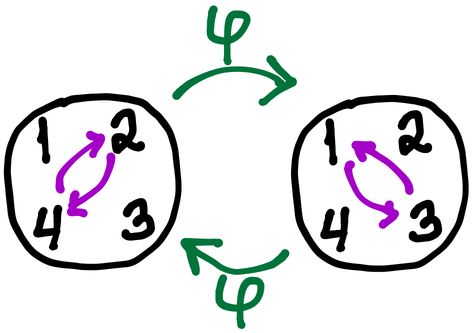
such w, in 6 φ-orbits

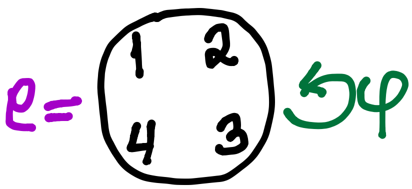


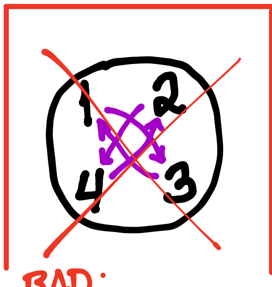




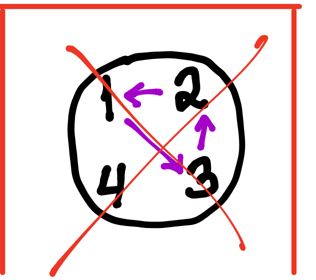








BAD;
CROSSING



BAD;
NOT CLOCKWISE

2. The right q-counts

$$\binom{n}{k} \overset{q=1}{\leftsquigarrow} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

q-binomial (Euler, Gauss)

$$n^{n-2} \overset{q=1}{\leftsquigarrow} [n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}}$$

$$\frac{1}{n+1}\binom{2n}{n} \overset{q=1}{\leftsquigarrow} \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

MacMahon's q-Catalan (1915)

where

$$[n]_q := 1+q+q^2+\dots+q^{n-1} \qquad \overset{q=1}{\rightsquigarrow} n$$

$$[n]!_q := [n]_q[n-1]_q\cdots[2]_q[1]_q \qquad \overset{q=1}{\rightsquigarrow} n!$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q\,[n-k]!_q} \qquad \overset{q=1}{\rightsquigarrow} \binom{n}{k}$$

$$6=\binom{4}{2} \xleftarrow{q=1} \begin{bmatrix}4\\2\end{bmatrix}_q = \frac{[4]_q[3]_q\cancel{[2]_q}\cancel{[1]_q}}{[2]_q[1]_q\cdot\cancel{[2]_q}\cancel{[1]_q}}$$

$$=\frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)\,(1)}$$

$$=(1+q^2)(1+q+q^2)$$

$$16=4^2 \xleftarrow{q=1} [4]_{q^2}[4]_{q^3}$$

$$=(1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$$

$$14=\frac{1}{5}\binom{8}{4} \xleftarrow{q=1} \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q$$

$$=\frac{1}{\cancel{[5]_q}}\,\frac{[8]_q[7]_q[6]_q\cancel{[5]_q}}{[4]_q[3]_q[2]_q\cancel{[1]_q}}$$

$$=(1-q+q^2)(1+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)$$

What's right about these q-counts?

First, they predict the cyclic orbit structures.

DEF'N: (R.-Stanton-White 2004) Say that a finite set X with the action of a cyclic group $\underset{\{e,g,g^2,\ldots,g^{N-1}\}\cong\mathbb{Z}/N\mathbb{Z}}{\overset{\shortparallel}{C}} \circlearrowright X$

and a polynomial $X(q)\in\mathbb{Z}[q]$ exhibit a cyclic sieving phenomenon (CSP) if

$\forall d\in\mathbb{Z}$

$$\#\{x\in X : g^d(x)=x\} = \left[X(q)\right]_{q=\zeta^d}$$

where $\zeta := e^{\frac{2\pi i}{N}}$

THEOREM

(RSW 2004)

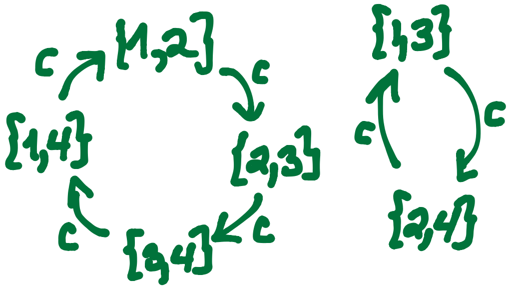
$$\left\{\begin{array}{l} X = \text{k-element subsets of } \{1,2,\ldots,n\} \\ \circlearrowright \\ C = \langle \overset{\overset{c}{\|}}{(1,2,\ldots,n)} \rangle \cong \mathbb{Z}/n\mathbb{Z} \\ X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \end{array}\right.$$

exhibit a CSP.

$n=4$

$k=2$

$\zeta = e^{\frac{2\pi i}{4}} = i$



$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = (1+q^2)(1+q+q^2)$$

$\overset{q=\zeta^0=1}{\rightsquigarrow}$ 6

$\overset{q=\zeta^2=-1}{\rightsquigarrow}$ 2

$\overset{q=\zeta^1=i}{\rightsquigarrow}$ 0

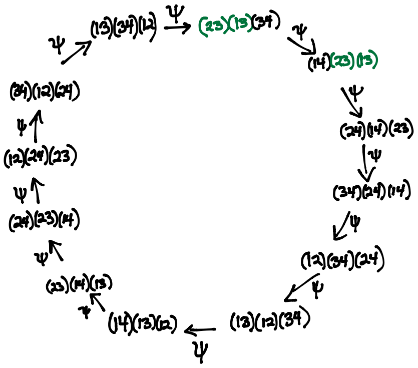
THEOREM (Douvropoulos 2017; Conj. by N. Williams 2013)

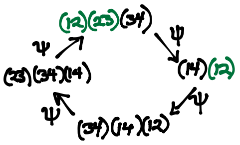
$$\left\{\begin{array}{l} X = \text{factorizations } c = t_1 t_2 \cdots t_{n-1} \text{ of} \\ \quad \text{n-cycle } c \text{ into } n-1 \text{ transpositions} \\ \circlearrowright \\ C = \langle \Psi \rangle \cong \mathbb{Z}/(n-1)n\mathbb{Z} \\ X(q) = [n]_{q^2}[n]_{q^3} \cdots [n]_{q^{n-1}} \end{array}\right.$$

exhibit a CSP.

$n=4$

$\zeta = e^{\frac{2\pi i}{12}}$





$$X(q) = [4]_{q^2}[4]_{q^3} = (1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$$

$$q = \zeta^0 = 1 \rightsquigarrow 16$$

$$q = \zeta^4 = e^{\frac{2\pi i}{3}} \rightsquigarrow 4$$

$$q = \zeta^1 = e^{\frac{2\pi i}{12}} \rightsquigarrow 0$$

$$q = \zeta^2 = e^{\frac{2\pi i}{6}} \rightsquigarrow 0$$

$$q = \zeta^3 = e^{\frac{2\pi i}{4}} = i \rightsquigarrow 0$$

$$q = \zeta^6 = e^{\frac{2\pi i}{2}} = -1 \rightsquigarrow 0$$

THEOREM

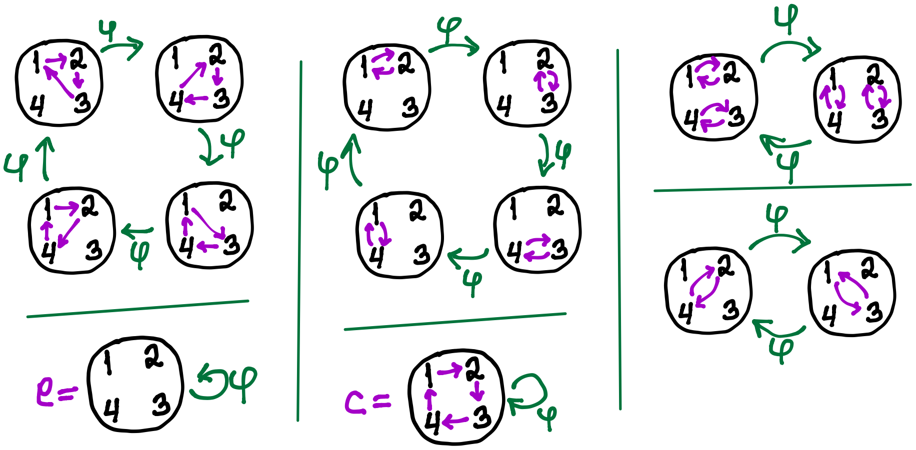
(RSW 2004)

$$\left\{\begin{array}{l} X = \text{permutations } w \text{ factored } w=t_1t_2\cdots t_k \text{ as prefixes of factorizations } c=t_1t_2\cdots t_k\cdots t_{n-1} \\ \circlearrowleft \\ C = \langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z} \\ X(q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \end{array}\right.$$

exhibit a CSP.

$n=4$

$\zeta = e^{\frac{2\pi i}{4}} = i$

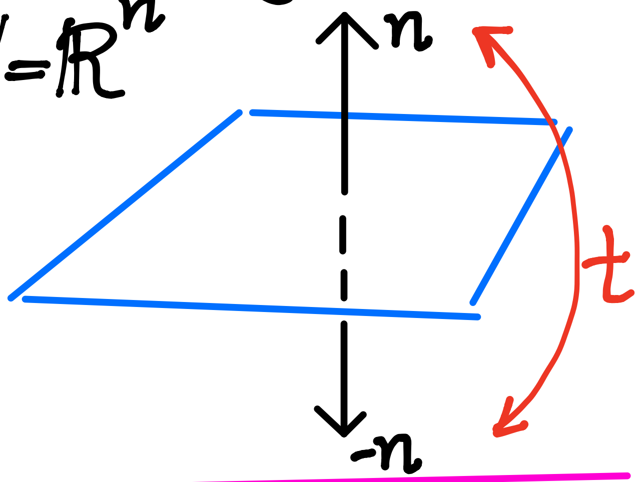


$$X(q) = \frac{1}{[5]_q}\begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = (1-q+q^2)(1+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)$$

$\overset{q=\zeta^0=1}{\rightsquigarrow} 14$ $\overset{q=\zeta^2=-1}{\rightsquigarrow} 6$ $\overset{q=\zeta^1=i}{\rightsquigarrow} 2$

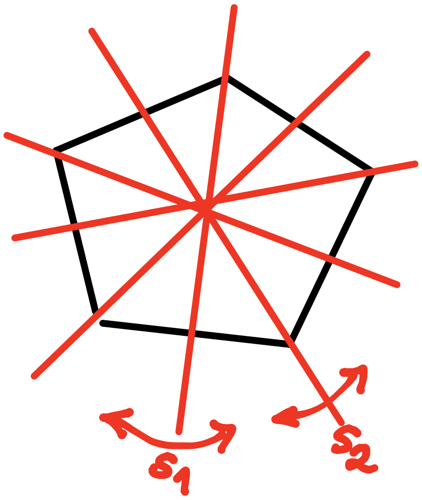
3. Reflection group versions

W a finite group generated by reflections t acting linearly on $V = \mathbb{R}^n$



EXAMPLE W = dihedral group of order 2m

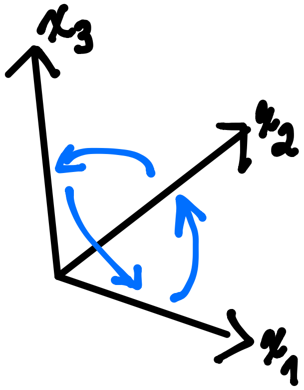
= linear symmetries of regular m-sided polygon



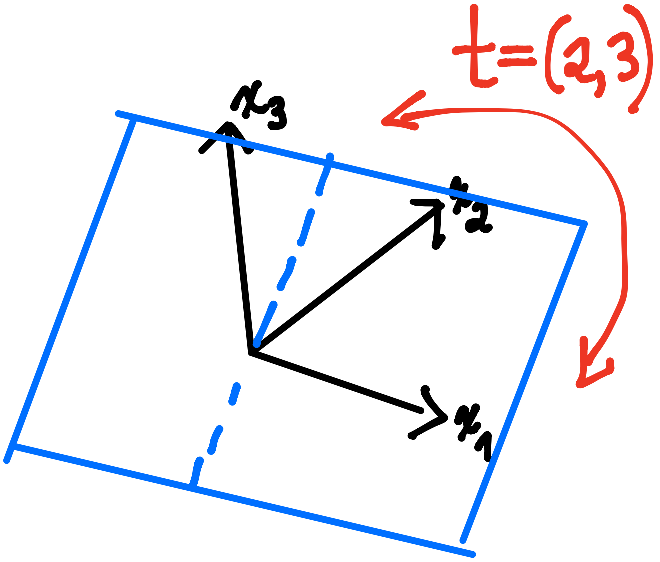
EXAMPLE

$W = \mathfrak{S}_n$ = symmetric group on n letters, acting via permutation matrices, permuting coordinates in $V = \mathbb{R}^n$

$V = \mathbb{R}^3$



$w = (1,2,3)$



transpositions $t = (i,j)$ are reflections through hyperplane $x_i = x_j$

W also acts on polynomials

$$\mathbb{R}[x_1,\dots,x_n] =: \mathbb{R}[\underline{x}]$$

via linear substitutions of the variables

$$w(x_j) = \sum_i w_{ij} x_i$$

The subalgebra of W-invariant polynomials

$$\mathbb{R}[\underline{x}]^W := \{ f(\underline{x}) \in \mathbb{R}[\underline{x}] : f(w\underline{x}) = f(\underline{x}) \ \forall w \in W \}$$

turns out surprisingly simple:

THEOREM (Shephard-Todd, Chevalley 1955) For reflection groups W,

$$\mathbb{R}[\underline{x}]^W = \mathbb{R}[f_1,\dots,f_n]$$

is a polynomial subalgebra

One can choose homogeneous $f_1, f_2, \dots, f_n$ with $\mathbb{R}[x]^W = \mathbb{R}[f_1, f_2, \dots, f_n]$ with degrees $d_1 \le d_2 \le \dots \le d_n$.

Then $h := d_n = \max\{d_i\}$ is called the Coxeter number of W

e.g.

THEOREM (Newton) When $W = \mathfrak{S}_n$ permutes variables in $\mathbb{R}[x] = \mathbb{R}[x_1, \dots, x_n]$,

$$\begin{array}{lccccccl} \mathbb{R}[x]^W = \mathbb{R}[& e_1 & , & e_2 & , \dots , & e_n &] & \\ & \| & & \| & & \| & & \\ & \sum_{i=1}^{n} x_i & & \sum_{1 \le i < j \le n} x_i x_j & & x_1 x_2 \cdots x_n & & \\ \text{with degrees} & 1 & & 2 & \cdots & n & & \\ & \| & & \| & & \| & & \\ & d_1 & \le & d_2 & \le \dots \le & d_n & =: h & \end{array}$$

elementary symmetric polynomials

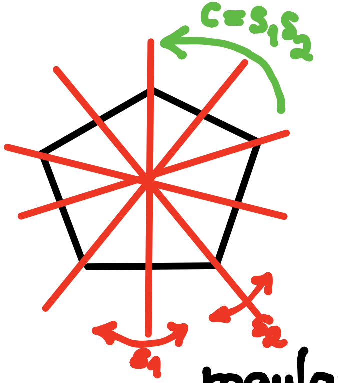
so $W = \mathfrak{S}_n$ has Coxeter number $h = n$

What is special about n-cycles $c=(1,2,\ldots,n)$?

THEOREM (Coxeter 1948) There is a W-conjugacy class of elements (called Coxeter elements) having multiplicative order $h=d_n$, represented by $c=s_1s_2\cdots s_n$, where $S=\{s_1,s_2,\ldots,s_n\}$ are a choice of Coxeter generators for W:

$$W=\langle S : s_i^2=e=(s_is_j)^{m_{ij}}\rangle \text{ with } m_{ij}\in\{2,3,\ldots\}$$

W = dihedral group has $h=m$ and Coxeter element



regular m-gon

$c=s_1s_2$
$=$ rotation through $\frac{2\pi}{m}$

$W=\mathfrak{S}_n$ = symmetric group

$$S=\{\underset{(1,2)}{\overset{}{s_1}},\underset{(2,3)}{s_2},\ldots,\underset{(n-1,n)}{s_{n-1}}\}$$

$h=d_n=n$

and Coxeter element

$c=(1,2)(2,3)\cdots(n-1,n)$

$=(1,2,3,\ldots,n)$ n-cycle

The q-counts for reflection groups W:

$$\binom{n}{k} \overset{q=1}{\leftsquigarrow} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \underset{W'=\mathfrak{S}_k \times \mathfrak{S}_{n-k}}{\overset{W=\mathfrak{S}_n}{\leftsquigarrow}} \quad \prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^{W'}]_q}$$

$$= \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

$$n^{n-2} \overset{q=1}{\leftsquigarrow} [n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}} \quad \overset{W=\mathfrak{S}_n}{\leftsquigarrow} \quad \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$$

$$\frac{1}{n+1}\binom{2n}{n} \overset{q=1}{\leftsquigarrow} \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \quad \overset{W=\mathfrak{S}_n}{\leftsquigarrow} \quad \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

W-q-Catalan number

Why are they polynomials, in $\mathbb{Z}[q]$?

What do they mean?

They are all Hilbert series

$$\mathrm{Hilb}(A, q) := \sum_{i=0}^{\infty} \dim_{\mathbb{R}}(A_i)\, q^i$$

for various graded rings

$$A = \bigoplus_{i=0}^{\infty} A_i$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \qquad \begin{array}{c} W = \mathfrak{S}_n \\ \leftsquigarrow \\ W' = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{array} \qquad \prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^{W'}]_q}$$

$$\prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^{W'}]_q} = \mathrm{Hilb}\left(\mathbb{R}[f_1^{W'}, \dots, f_n^{W'}] \big/ (f_1, \dots, f_n), q \right)$$

where

$$\begin{array}{l} \mathbb{R}[\underline{x}]^{W} = \mathbb{R}[f_1, \dots, f_n] \\ \quad \cap \\ \mathbb{R}[\underline{x}]^{W'} = \mathbb{R}[f_1^{W'}, \dots, f_n^{W'}] \end{array}$$

for a reflection subgroup $W' \subset W$

$$[n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}} \overset{W=\mathfrak{S}_n}{\leftsquigarrow} \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$$

$$\prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q} = \mathrm{Hilb}\left(\frac{\mathbb{R}[f_1,\dots,f_{n-1}]}{(\alpha_2(\underline{f}),\dots,\alpha_n(\underline{f}))} \;, q \right)$$

where the W-discriminant in $\mathbb{R}[\underline{x}]^W = \mathbb{R}[f_1,\dots,f_n]$

is expressed $\Delta_W^2 = f_n^{n} + \alpha_2(\underline{f})\, f_n^{n-2} + \alpha_3(\underline{f})\, f_n^{n-3} + \dots + \alpha_n(\underline{f})$

$$\text{if } \Delta_W := \prod_{\substack{\text{reflection}\\ \text{hyperplanes}\\ H \text{ for } W}} \ell_H(x_1,\dots,x_n) \qquad \left(\overset{\text{if } W=\mathfrak{S}_n}{=} \prod_{1\le i<j\le n} (x_i - x_j) \right)$$

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q \xleftarrow{W=\mathfrak{S}_n} \prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q}$$

$$\prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q} = \mathrm{Hilb}\left(\left(\mathbb{R}[\underline{x}]\big/(\theta_1,\dots,\theta_n)\right)^W, q\right)$$

where $\theta_1,\dots,\theta_n$ in $\mathbb{R}[\underline{x}]$

- each have same degree $h+1$
- form a system of parameters for $\mathbb{R}[\underline{x}]$
- have the map $x_i \longmapsto \theta_i$ W-equivariant

Existence of such magical $\theta_1,\dots,\theta_n$ provided by rep'n theory of rational Cherednik algebras (Gordon, Berest-Etingof-Ginzburg 2002)

4. Deformation proof idea

(for CSP with $X \circlearrowleft C$ and $X(q)$)

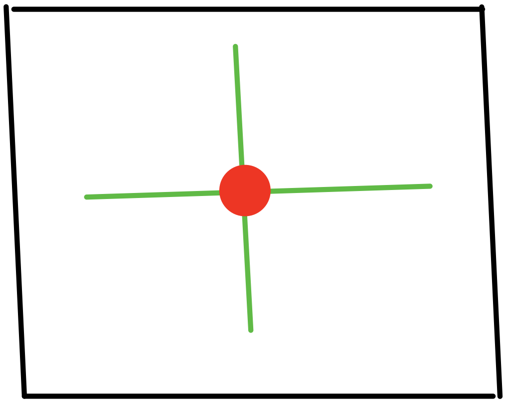
Let $X(q) = \mathrm{Hilb}(A,q)$ for graded ring

$$A = \mathbb{C}[x_1,\dots,x_n] / \underbrace{(h_1,\dots,h_n)}_{\text{homogeneous ideal } I}$$

= coordinate ring for the fat point $h_1(\underline{x}) = \dots = h_n(\underline{x}) = 0$ at the origin in $\mathbb{C}^n$

$\mathbb{R}^2$

$(\subset \mathbb{C}^2)$



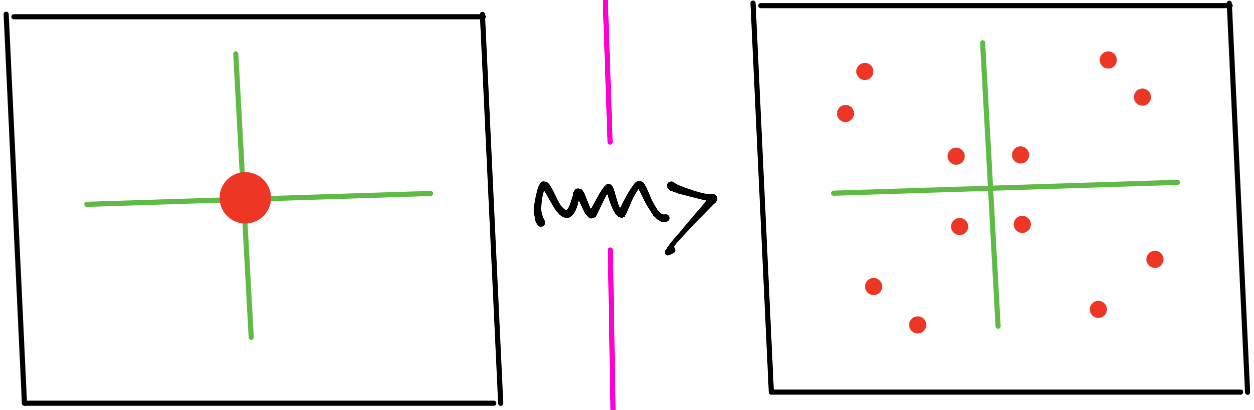
$$I=(h_1,\ldots,h_n) \overset{\text{Deform}}{\rightsquigarrow} J=(h'_1,\ldots,h'_n)$$

homogeneous

inhomogeneous

$h_1(\underline{x})=\ldots=h_n(\underline{x})=0$

$h'_1(\underline{x})=\ldots=h'_n(\underline{x})=0$



fat point of multiplicity $X(1)=\#X$, with coordinate ring $A=\mathbb{R}[\underline{x}]/I$

$$\circlearrowleft \quad g(x_i)=\zeta x_i$$

$$\underset{\langle g\rangle}{\underset{\|}{C}}=\mathbb{Z}/N\mathbb{Z}$$

$\#X$ reduced points with coordinate ring $\mathbb{R}[\underline{x}]/J$

$$\circlearrowleft \quad g(x_i)=\zeta x_i$$

$$C=\mathbb{Z}/N\mathbb{Z}$$

permuting as in $C \circlearrowleft X$

This would prove the CSP:

$$\begin{tikzcd}
\#\{x\in X : g^d(x)=x\} \arrow[r, equal, dashed, "?"'] \arrow[dd, equal] & {[X(q)]_{q=\zeta^d}} \arrow[d, equal] \\
& \sum_i \dim(A_i)\cdot(\zeta^d)^i \arrow[d, equal] \\
\text{Trace of } g^d \text{ acting on } \mathbb{R}[x]/J \arrow[r, equal] & \text{Trace of } g^d \text{ acting on } \underbrace{\mathbb{R}[x]/I}_{A}
\end{tikzcd}$$

$\mathbb{R}[x]/I$ and $\mathbb{R}[\underline{x}]/J$ agree up to a filtration

How does this work in our examples?

THEOREM (RSW 2004) Given a Coxeter element c in W, and reflection subgroup $W' \subset W$,

$$\begin{cases} X = W/W' \\ \circlearrowleft \\ C = \langle c \rangle \cong \mathbb{Z}/h\mathbb{Z} \quad \text{via } c(wW') = cwW' \\ X(q) = \displaystyle\prod_{i=1}^{n} \frac{[d_i]_q}{[d_i^{W'}]_q} \end{cases}$$

exhibits a CSP.

Proof. Deform

$$\mathbb{C}[f_1^{W'}, \ldots, f_n^{W'}] / (f_1, \ldots, f_n) \longleftarrow I$$

$$\rightsquigarrow \quad \mathbb{C}[f_1^{W'}, \ldots, f_n^{W'}] / (f_1 - f_1(v), \ldots, f_n - f_n(v)) \longleftarrow J$$

where $v \in V = \mathbb{C}^n$ is an eigenvector for c avoiding all the reflecting hyperplanes for W. $\square$

THEOREM (Douvropoulos 2017; Conj. by N. Williams 2013)

$$\left\{\begin{array}{l} X = \text{factorizations } c = t_1 t_2 \cdots t_n \text{ of a Coxeter element } c \text{ into } n \text{ reflections} \\ \circlearrowright \\ C = \langle \psi \rangle \cong \mathbb{Z}/nh\mathbb{Z} \\ X(q) = \displaystyle\prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q} \end{array}\right.$$

exhibit a CSP.

Proof. Deform

$$\mathbb{C}[f_1, f_2, \ldots, f_{n-1}] \Big/ \overset{I}{\left(\alpha_2(\underline{f}), \ldots, \alpha_n(\underline{f})\right)}$$

$$\rightsquigarrow \mathbb{C}[f_1, f_2, \ldots, f_{n-1}] \Big/ \overset{J}{\left(\alpha_2(\underline{f}) - c_2, \ldots, \alpha_n(\underline{f}) - c_n\right)}$$

for particular choices of $c_2, \ldots, c_n$, making heavy use of Bessis's 2007 results on Lyashko-Looijenga morphism. $\square$

CONJECTURE (Armstrong-R.-Rhoades 2012)

One can explain a known CSP for

$$\left\{\begin{array}{l} X = w\in W \text{ factored } w=t_1t_2\cdots t_k \text{ as prefixes of factorizations } c=t_1t_2\cdots t_k\cdots t_n \text{ of a Coxeter element } c \\ \circlearrowleft \\ C=\langle \overset{w\mapsto cwc^{-1}}{\varphi}\rangle \cong \mathbb{Z}/h\mathbb{Z} \\ X(q) = \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q} \end{array}\right.$$

W-noncrossing partitions

via this deformation:

$$\left(\mathbb{C}[\underline{x}] \Big/ \underbrace{(\theta_1,\ldots,\theta_n)}_{I}\right)^W$$

$$\rightsquigarrow \left(\mathbb{C}[\underline{x}] \Big/ \underbrace{(\theta_1-x_1,\ldots,\theta-x_n)}_{J}\right)^W$$

5. $GL_n(\mathbb{F}_q)$-analogues

$$W = \mathfrak{S}_n \overset{\text{“}q=1\text{”}}{\leftsquigarrow} GL_n(\mathbb{F}_q) =: G$$

symmetric group

finite general linear group

$$\begin{tikzcd} k\text{-element subsets of } \{1,2,\ldots,n\} \arrow[d, equal] & k\text{-dimensional } \mathbb{F}_q\text{-linear subspaces of } (\mathbb{F}_q)^n \arrow[l, squiggly, "\text{“}q=1\text{”}"'] \arrow[d, equal] \\ \mathfrak{S}_n/\mathfrak{S}_k \times \mathfrak{S}_{n-k} & G/P \end{tikzcd}$$

where

$P =$ maximal parabolic subgroup

$$\left\{ \begin{array}{c} \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \\ \underbrace{\quad}_{k}\;\underbrace{\quad}_{n-k} \end{array} \in G \right\}$$

n-cycle $c=(1,2,\dots,n)$ in $W=\mathfrak{S}_n$

$$\begin{tikzcd} c & c_q \arrow[l, squiggly, "\text{“}q=1\text{”}"'] \end{tikzcd}$$

Singer cycle c_q in $G=GL_n(\mathbb{F}_q)$

any generator for $\mathbb{F}_{q^n}^{\times} \hookrightarrow GL_n(\mathbb{F}_q)$

e.g. $c_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ in $GL_4(\mathbb{F}_2)$

Coxeter number $h=n$ for $W=\mathfrak{S}_n$

$$\begin{tikzcd} h=n & h=q^n-1 \arrow[l, squiggly, "\text{“}q=1\text{”}"'] \end{tikzcd}$$

Coxeter number $h=q^n-1$ for $W=GL_n(\mathbb{F}_q)$

THEOREM
(Newton)

$$\mathbb{R}[x_1,\dots,x_n]^{\mathfrak{S}_n} = \mathbb{R}[e_1,\dots,e_n]$$

where

$$\prod_{i=1}^{n}(t+x_i) = t^n + \sum_{k=1}^{n} e_{n-k}(\underline{x})\, t^k$$

$$\begin{array}{cccc} e_1, & e_2, & \dots, & e_n \\ 1 & 2 & & n \\ & & & \shortparallel \\ & & & h \end{array}$$

"q=1" ⇜

THEOREM
(L.E. Dickson 1911)

$$\mathbb{F}_q[x_1,\dots,x_n]^{GL_n(\mathbb{F}_q)} = \mathbb{F}_q[D_0,D_1,\dots,D_{n-1}]$$

where

$$\prod_{\substack{\text{linear forms}\\ \ell(\underline{x})\in(\mathbb{F}_q^n)^*}}\left(t+\ell(\underline{x})\right) = t^{q^n} + \sum_{k=0}^{n-1} D_k(\underline{x})\, t^{q^k}$$

$$\begin{array}{cccc} D_0, & D_1, & \dots, & D_{n-1} \\ q^n-1 & q^n-q & & q^n-q^{n-1} \\ \shortparallel & & & \\ h & & & \end{array}$$

THEOREM (RSW 2004) The k-subsets CSP has a $GL_n(\mathbb{F}_q)$-analogue

$$X = k\text{-dimensional subspaces of } (\mathbb{F}_q)^n = G/P$$

$$\circlearrowleft$$

Singer cycle

$$C = \langle c_q \rangle \cong \mathbb{Z}/(q^n-1)\mathbb{Z}$$

$$\boxed{n!_{q,t} = \prod_{i=0}^{n-1} (1 - t^{q^n - q^i})}$$

$$X(t) = \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} := \frac{n!_{q,t}}{k!_{q,t}\,(n-k)_{q,t^{q^k}}}$$

q,t-binomial

$$= \mathrm{Hilb}\left(\mathbb{F}_q[\underline{x}]^P \big/ (D_0, D_1, \ldots, D_n), t \right)$$

… and it can be proven and generalized via deformation (Broer-R.-Smith-Webb 2008)

What about the n^{n-2} factorizations of n-cycle $c = t_1 t_2 \cdots t_{n-1}$?

THEOREM (Lewis-R.-Stanton 2013) Singer cycles c_q in $GL_n(\mathbb{F}_q)$ have $(q^n - 1)^{n-1}$ factorizations $c_q = t_1 t_2 \cdots t_n$ into reflections t_i.

Here a reflection t means t has fixed space of codimension 1 (but maybe $\det(t) \neq -1$, maybe t is not semisimple!)

QUESTIONS

- Is there a deformation proof of this $(q^n-1)^{n-1}$ count?

- Is there a $GL_n(\mathbb{F}_q)$-analogue of the Williams/Douvropoulos CSP with $X(q) = \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$?

- The proof (Frobenius's method) suggests maybe there should be an Okounkov-Vershik approach to $GL_n(\mathbb{F}_q)$-characters?

The W-q-Catalan number

$$\prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q} = \mathrm{Hilb}\left(\left(\mathbb{R}[\underline{x}] \big/ (\theta_1,\ldots,\theta_n) \right)^{W}, q \right)$$

has an obvious $GL_n(\mathbb{F}_q)$-analogue:

$$X(t) := \mathrm{Hilb}\left(\left(\mathbb{F}_q[x_1,\ldots,x_n] \big/ (x_1^{q^n},\ldots,x_n^{q^n}) \right)^{GL_n(\mathbb{F}_q)}, t \right)$$

CONJECTURE (Lewis-R-Stanton 2014) More explicitly, the above $X(t)$ equals

$$\sum_{k=0}^{n} t^{(n-k)(q^n-q^k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q,t} .$$

q,t-binomials!

This conjecture (and a CSP) strongly suggest there should be a deformation proof of this form:

$$\left(\mathbb{F}_q[\underline{x}] \Big/ \underset{\nwarrow I}{\left(x_1^{q^n}, \ldots, x_n^{q^n}\right)}\right)^{GL_n(\mathbb{F}_q)}$$

$$\rightsquigarrow \left(\mathbb{F}_q[\underline{x}] \Big/ \underset{\nwarrow J}{\left(x_1^{q^n}-x_1, \ldots, x_n^{q^n}-x_n\right)}\right)^{GL_n(\mathbb{F}_q)}$$

involving $X = \left(\mathbb{F}_{q^n}\right)^n \subset \left(\overline{\mathbb{F}}_q\right)^n$

We haven't found it (yet).

Thank you
for your
attention!