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Reflection groups and enumeration

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Abstract

The main objective of these notes is to illustrate a class of theorems that seem surprising and very general but use the same ideas that come in invariant theory and representation theory.

Keywords: representation theory, symmetric group, symmetric powers, symmetric algebra, reflection groups, Boolean algebras, Ferrer diagrams, unlabelled graphs, unimodality, generating function, irreducible representations, tensor algebra, character tables, covariant algebras, MacMahon's master theorem, Molien's theorem, Springer's theorem, cyclic sieving phenomenon (CSP).

1.1 Lecture 1: q-counting quotients of Boolean algebras

We start with some important *posets* (partially ordered sets). Let us recall that a poset can be represented by a *Hasse diagram* that depicts the graph whose vertices are the poset elements and there is an edge $\{S, T\}$ whenever $S \lessdot T$, i.e., whenever S is *covered* by T, meaning $S \leq T$ and there is no U with $S \lneq U \lneq T$.

The first example is the Boolean algebra $2^{[n]}$, where $[n] = \{1, 2, \dots, n\}$. The elements of $2^{[n]}$ are all the subsets $S \subseteq [n]$, partially ordered by inclusion: $S \leq T$ if $S \subseteq T$. The Boolean algebra with this order is a *lattice*, i.e., it is a poset such that every two elements have a unique minimal upper bound and a unique maximal lower bound. Thus this poset is also called the *Boolean lattice*.

Example 1.1 (Boolean algebra) The Boolean algebra for $n = 4$ can be represented by the Hasse diagram in Figure 1.1. If we record the rank sizes with a

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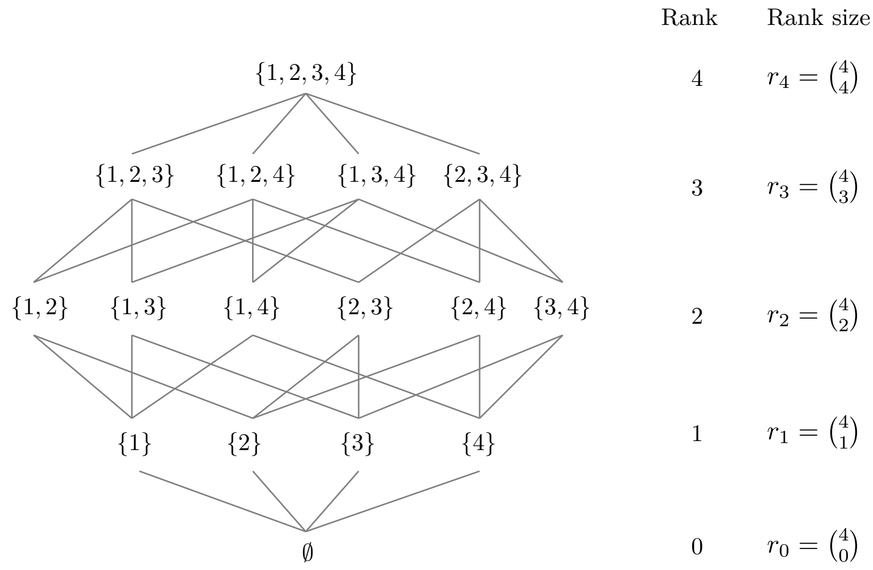


Figure 1.1 Boolean algebra $2^{[4]}$ together with the ranks and rank sizes r_k.

variable q we obtain $\binom{4}{0} + \binom{4}{1}q + \binom{4}{2}q^2 + \binom{4}{3}q^3 + \binom{4}{4}q^4 = (q+1)^4$. This is the *rank generating function* of this ranked poset $2^{[4]}$.

Proposition 1.2 *For integers n, k, with $n \geq k \geq 0$ the following properties hold:*

- *Symmetry* $\binom{n}{k} = \binom{n}{n-k}$.
- *Alternating sum* $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} + \cdots \pm \binom{n}{n} = 0$ *for* $n \geq 1$.
- *Rank generating function* $\binom{n}{0} + \binom{n}{1}q + \binom{n}{2}q^2 + \cdots + \binom{n}{n}q^n = (1+q)^n$.
- *Unimodality* $\binom{n}{0} \leq \binom{n}{1} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor}$.

We will generalize the properties of the rank sizes in Proposition 1.2 by considering subgroups of the symmetric group acting on the Boolean algebra $2^{[n]}$.

1.1.1 Three important examples

Denote by $\mathfrak{S}_n$ the symmetric group permuting $\{1, 2, \ldots, n\}$. Consider a *permutation subgroup* $G \subseteq \mathfrak{S}_n$ and the *orbit poset* $2^{[n]}/G$. The G-orbits $\mathcal{O}$ are subsets of $2^{[n]}$, with the ordering $\mathcal{O}_1 \leq \mathcal{O}_2$ if there exist $S_1 \in \mathcal{O}_1$ and $S_2 \in \mathcal{O}_2$ having $S_1 \subseteq S_2$.

Example 1.3 (black and white necklaces) Let G be the subgroup generated by an n-cycle $G = \langle(1, 2, \ldots, n)\rangle \subset \mathfrak{S}_n$. Then $G \cong \mathbb{Z}/n\mathbb{Z}$ has G-orbits $\mathcal{O}$ in bijection with black and white *necklaces* having n beads, e.g., if $n = 6$ the orbits are in correspondence with the necklaces in Figure 1.2.

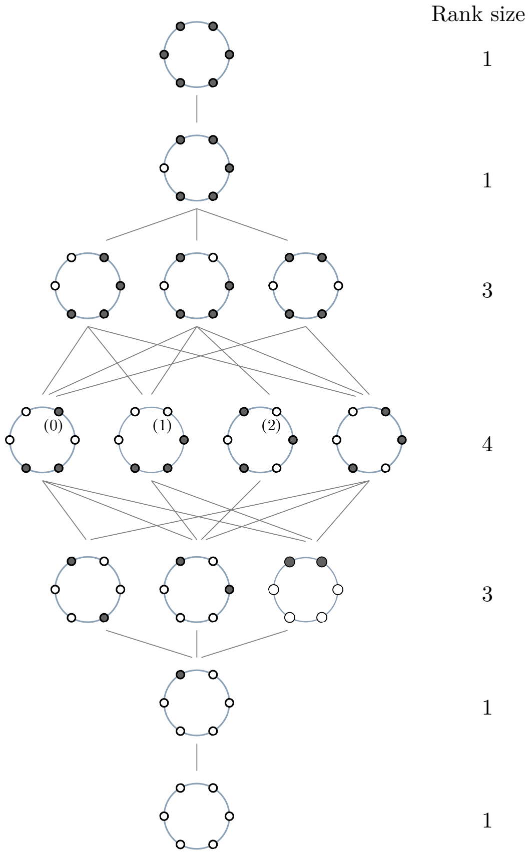
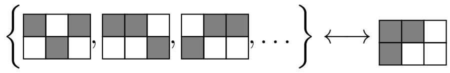
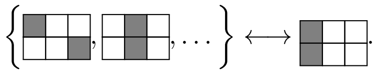


Figure 1.2 Poset $2^{[6]}/G$ with $G = \langle(1,2,\ldots,6)\rangle$. Each necklace corresponds to a G-orbit, e.g., $\mathcal{O} = \{\{1,2,5\},\{2,3,6\},\{1,3,4\},\{2,4,5\},\{3,4,6\},\{1,4,5\}\}$ and $\mathcal{O}' = \{\{1,3,5\},\{2,4,6\}\}$ correspond to the orbits labeled (0) and (2) above, respectively.

Example 1.4 (Ferrers diagrams inside a $k \times \ell$ rectangle) Consider the unit squares inside a $k \times \ell$ rectangle, permuted by the symmetric group $\mathfrak{S}_{k\ell}$. Inside $\mathfrak{S}_{k\ell}$ lies a subgroup $G = \mathfrak{S}_k[\mathfrak{S}_\ell]$, called a *wreath product*, containing a subgroup $\mathfrak{S}_\ell \times \mathfrak{S}_\ell \times \cdots \times \mathfrak{S}_\ell$ that permutes within rows arbitrarily, and also contains a subgroup $\mathfrak{S}_k$ that wholesale swaps rows. Convenient G-orbit representatives for subsets $S \subseteq [k\ell]$ are given by Ferrers diagrams inside the $k \times \ell$ rectangle.

For instance, if $k = 2$ and $\ell = 3$ then $G = \mathfrak{S}_2[\mathfrak{S}_3] \subset \mathfrak{S}_6$ contains $\mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5,6\}}$ but also $\mathfrak{S}_2 = \langle(14)(25)(36)\rangle$. The G-orbit representatives are depicted as darkened subsets of the rectangle $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array}$. E.g.,

$\mathcal{O}_1 =$ 

$\mathcal{O}_2 =$ 

The orbit poset $2^{[6]}/G$ is given in Figure 1.3 (left).

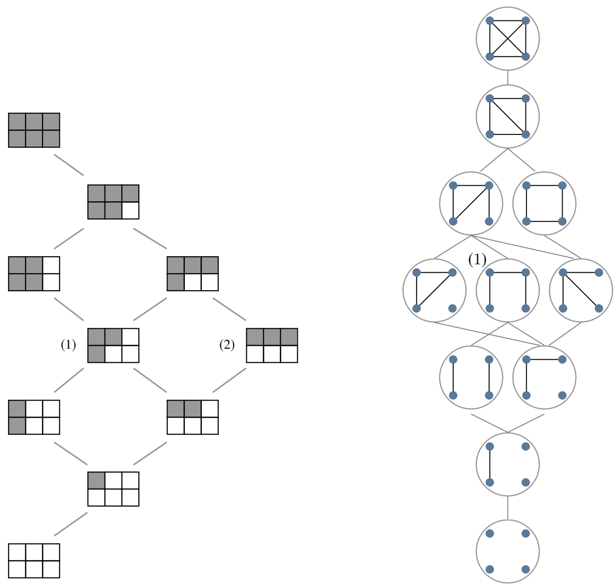
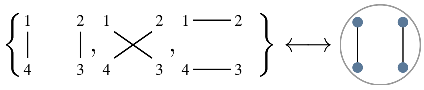


Figure 1.3 Orbit posets $2^{[6]}/G$, on the left: Ferrers diagrams inside a 2×3 box, $G = \mathfrak{S}_2[\mathfrak{S}_3]$; on the right: unlabeled simple graphs on 4 vertices, $G = \mathfrak{S}_4 \subset \mathfrak{S}_{\binom{[4]}{2}}$.

Example 1.5 (unlabelled (simple) graphs on v vertices) Let $\binom{[v]}{2}$ denote the edges of the complete graph K_v on vertices $[v] = \{1, 2, \ldots, v\}$. E.g., the complete graph K_v for $v = 5$ is illustrated in Figure 1.4. Inside the symmetric group $\mathfrak{S}_{\binom{[v]}{2}}$ that permutes these edges lies a subgroup $G = \mathfrak{S}_v$ consisting of permutations of the form $\sigma(\{i, j\}) = \{\sigma(i), \sigma(j)\}$ for $\sigma \in \mathfrak{S}_v$. Then G-orbits $\mathcal{O}$ of subsets S of edges correspond to isomorphism classes of simple graphs on n vertices. For example, when $n = 4$, here are two such G-orbits:

$\mathcal{O}_1 =$ 

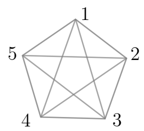
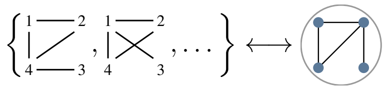
$K_5 =$ 

Figure 1.4 Complete graph on the vertices $[5] = \{1, 2, 3, 4, 5\}$.

$\mathcal{O}_2 =$ 

The orbit poset $2^{[6]}/G$ is given in Figure 1.3 (right).

Given G a subgroup of $\mathfrak{S}_n$, if $r_0, r_1, \ldots, r_n$ are the rank sizes of the orbit poset $2^{[n]}/G$, that is the number of G-orbits of k-element subsets, then we have

$$r_k = \left| \binom{[n]}{k} \,/\, G \right|.$$

Here are several of their properties, generalizing the case of binomial coefficients, that compile results of de Bruijn [2], Redfield [4], Pólya [3], and Stanley [8].

Proposition 1.6 (symmetry) *For any $0 \le k \le n$ the rank sizes satisfy $r_k = r_{n-k}$.*

Theorem 1.7 (de Bruijn 1959; alternating sum) *The number of self-complementary G-orbits, i.e., the orbits $\mathcal{O}$ such that $S \in \mathcal{O}$ if and only if $[n]\backslash S \in \mathcal{O}$, is given by $r_0 - r_1 + r_2 + \cdots \pm r_n$.*

Theorem 1.8 (Redfield 1927, Pólya 1937; generating function) *For a variable q, the generating function satisfies*

$$r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles } c \\ \text{of } \sigma}} \left(1 + q^{|c|}\right).$$

Theorem 1.9 (Stanley 1982; unimodality) *The rank sizes increase as $r_0 \le r_1 \le r_2 \le \cdots \le r_{\lfloor n/2 \rfloor}$.*

We will prove these results or sketch proofs using some (multi)-linear algebra.

Example 1.10 (alternating sum of rank sizes) We check the alternating sums in Theorem 1.7 in the three examples above.

1. Necklaces for $n = 6$: the only self-complementary necklaces are those with labels (1) and (2) in Figure 1.2, and in fact the alternating sum of the rank sizes gives $1 - 1 + 3 - 4 + 3 - 1 + 1 = 2$.
2. Ferrers diagrams inside a 2×3 box: we have $1 - 1 + 2 - 2 + 2 - 1 + 1 = 2$. The diagrams (1) and (2) are the only ones in the poset on the left of Figure 1.3 that are self-complementary.

3. Unlabeled simple graphs on 4 vertices: we have $1-1+2-3+2-1+1=1$, and the only self-complementary graph is (1) in the orbit poset on the right of Figure 1.3.

By Theorem 1.8, the generating function in the necklace case can be computed as follows. Let $c=(1,2,3,4,5,6)\in\mathfrak{S}_6$. Then

$$G=\langle c\rangle=\{e,c,c^2,c^3,c^4,c^5\}=\{e\}\cup\{c,c^5\}\cup\{c^2,c^4\}\cup\{c^3\}.$$

Notice that the permutations in each of the latter subsets can be written as a product of the same number of disjoint cycles, 6, 1, 2, and 3, respectively. Thus

$$\begin{aligned}\frac{1}{|G|}\sum_{\sigma\in G}\prod_{\substack{\text{cycles } c\\ \text{of } \sigma}}\left(1+q^{|c|}\right)&=\frac{1}{6}\left((1+q)^6+2(1+q^6)+2(1+q^3)^2+(1+q^2)^3\right)\\&=1+q+3q^2+4q^3+3q^4+q^5+q^6.\end{aligned}$$

1.1.2 Idea of proofs

We **linearize** and

- interpret *cardinalities* as *dimensions*,
- interpret *generating functions* as *graded dimensions* or *Hilbert series*,
- prove *equalities* via *isomorphisms*, and
- prove *inequalities* via *injections* and *surjections*.

Many identities come from *equality of traces* of *conjugate* elements g, hgh^{-1} in a group G acting in a *representation* on V: for homomorphisms $\rho: G\to \mathrm{GL}(V)$, then

$$\mathrm{Trace}_V(\rho(hgh^{-1}))=\mathrm{Trace}_V(\rho(h)\rho(g)\rho(h)^{-1})=\mathrm{Trace}_V(\rho(g)),$$

where we use the identity $\mathrm{Trace}(AB)=\mathrm{Trace}(BA)$, that implies

$$\mathrm{Trace}(PAP^{-1})=\mathrm{Trace}(P^{-1}PA)=\mathrm{Trace}(A).$$

A quick refresher on tensor products

Let $\mathrm{GL}_2(\mathbb{C})$ be the set of 2×2 invertible matrices with complex entries, and $G=\left\{\begin{pmatrix}1&x\\0&1\end{pmatrix}: x\in\mathbb{C}\right\}$. In particular, the identity matrix $e=\begin{pmatrix}1&0\\0&1\end{pmatrix}\in G$. Note that $\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}1&y\\0&1\end{pmatrix}=\begin{pmatrix}1&x+y\\0&1\end{pmatrix}$, and if $g=\begin{pmatrix}1&x\\0&1\end{pmatrix}$ then $g^{-1}=\begin{pmatrix}1&-x\\0&1\end{pmatrix}$. Now $\mathrm{GL}_2(\mathbb{C})$ acts on $V=\mathbb{C}^2$ and also on $V\otimes V$. The elements of $V\otimes V$ are of the form $\sum_{i,j}c_{ij}v_i\otimes v_j$, with $v_i,v_j\in V$ and $c_{ij}\in\mathbb{C}$. If V has a basis $\{b,w\}$ then a basis for

$V \otimes V$ is $\{b \otimes b, b \otimes w, w \otimes b, w \otimes w\}$. The above element g of the group $\mathrm{GL}_2(\mathbb{C})$ acts diagonally on $V \otimes V$ as in the following examples:

$$\begin{aligned} g(b \otimes w) &= g(b) \otimes g(w) = b \otimes (xb + w) = b \otimes xb + b \otimes w = xb \otimes b + b \otimes w \\ g(w \otimes w) &= g(w) \otimes g(w) = (xb + w) \otimes (xb + w) \\ &= x^2 b \otimes b + xb \otimes w + xw \otimes b + w \otimes w. \end{aligned}$$

1.1.3 Proofs of symmetry, alternating sum and generating function

For the beads, start with $V = \mathbb{C}^2$ having $\mathbb{C}$-basis $\{b, w\}$ (black and white). Then elements $T \in \mathrm{GL}(V) = \mathrm{GL}_2(\mathbb{C})$ act on V.

For example, $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ acts via $t(b) = w$ and $t(w) = b$, and $s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ acts via $s(b) = -b$ and $s(w) = w$.

Let $T^n(V) = V \otimes V \otimes \cdots \otimes V = V^{\otimes n}$ be the *n-th tensor power* of the vector space $V = \mathbb{C}^2$ over $\mathbb{C}$. The groups of invertible 2×2 matrices $\mathrm{GL}(V)$ and the symmetric group $\mathfrak{S}_n$ act on $V^{\otimes n}$ as follows:

- $\mathrm{GL}(V)$ acts *diagonally*, i.e.,

$$T(v_1 \otimes \cdots \otimes v_n) = T(v_1) \otimes \cdots \otimes T(v_n),$$

  expanding by multilinearity, and
- $\mathfrak{S}_n$ acts *positionally*, i.e.,

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

These two actions commute, namely

$$\sigma T(v_1 \otimes \cdots \otimes v_n) = T\big(v_{\sigma^{-1}(1)}\big) \otimes \cdots \otimes T\big(v_{\sigma^{-1}(n)}\big) = T\sigma(v_1 \otimes \cdots \otimes v_n).$$

The tensor power $V^{\otimes n}$ has a natural $\mathbb{C}$-basis $\{e_S\}_{S \in 2^{[n]}}$ indexed by subsets $S \in 2^{[n]}$.

Example 1.11 (basis elements indexed by $2^{[n]}$) In the black (b) and white (w) necklaces example in Section 1.1.1, taking $n = 4$, the tensor e_S has b in positions S and w in positions $[4] \setminus S$. We will abbreviate these basis tensors by writing them as words that omit the tensor symbols. For instance,

$$\begin{aligned} e_{\{2\}} &= w \otimes b \otimes w \otimes w \;\longleftrightarrow\; wbww, \\ e_{\{1,4\}} &= b \otimes w \otimes w \otimes b \;\longleftrightarrow\; bwwb. \end{aligned}$$

For a permutation group $G \subset \mathfrak{S}_n$, the *G-fixed subspace* $\big(V^{\otimes n}\big)^G$ has a natural $\mathbb{C}$-basis indexed by G-orbits $\mathcal{O} \in 2^{[n]}/G$ given by

$$\{e_{\mathcal{O}}\}_{\mathcal{O} \in 2^{[n]}/G}, \text{ where } \quad e_{\mathcal{O}} \colon = \sum_{S \in \mathcal{O}} e_S.$$

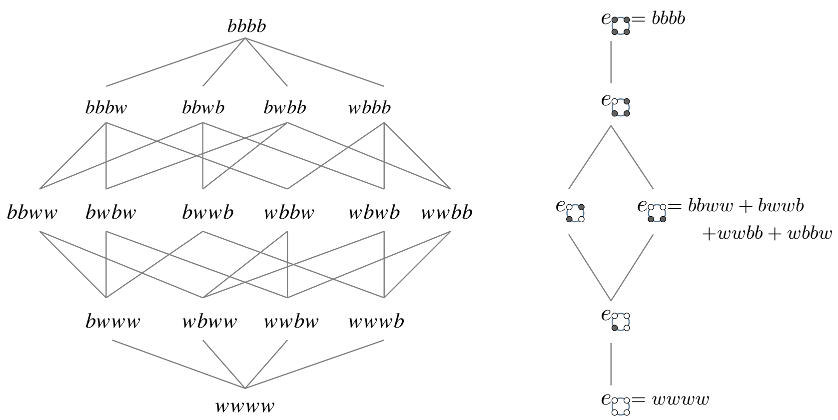


Figure 1.5 Decomposition of $V^{\otimes 4}$ (on the left) and $\left(V^{\otimes 4}\right)^G$ (on the right) into graded $\mathbb{C}$-vector spaces, where $G = \langle(1,2,3,4)\rangle$.

Example 1.12 (basis elements indexed by G-orbits) In the black and white necklace example, taking $G = \langle(1,2,3,4)\rangle \cong \mathbb{Z}/4\mathbb{Z}$, we have

$$e = wbwb + bwbw,$$
$$e = wbbb + bwbb + bbwb + bbbw.$$

Both $V^{\otimes n}$ and $(V^{\otimes n})^G$ are *graded* $\mathbb{C}$*-vector spaces*. Namely, $V^{\otimes n} = \bigoplus_{k=0}^{n} \left(V^{\otimes n}\right)_k$ and $\left(V^{\otimes n}\right)^G = \bigoplus_{k=0}^{n} \left(V^{\otimes n}\right)_k^G$, where $\left(V^{\otimes n}\right)_k$ is the $\mathbb{C}$-span of $\{e_S\}_{S \in \binom{[n]}{k}}$, and $\left(V^{\otimes n}\right)_k^G$ is the $\mathbb{C}$-span of $\{e_{\mathcal{O}}\}_{\mathcal{O} \in \binom{[n]}{k}/G}$. See Figure 1.5 for an example.

Thus the rank sizes $r_0, r_1, \ldots, r_n$ of the orbit poset $2^{[n]}/G$ can now be reinterpreted as dimensions: one has

$$r_k = \dim_{\mathbb{C}}\left(V^{\otimes n}\right)_k^G = \left| \binom{[n]}{k} \Big/ G \right|.$$

Proof of Proposition 1.6. Recall that the matrix $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}(V)$ swaps b and w, and so it permutes the $\mathbb{C}$-basis $\{e_S\}_{S \in 2^{[n]}}$ for $V^{\otimes n}$ by swapping $e_S \overset{t}{\longleftrightarrow} e_{[n]\setminus S}$. This gives a $\mathbb{C}$-linear isomorphism $\left(V^{\otimes n}\right)_k \xrightarrow{t} \left(V^{\otimes n}\right)_{n-k}$. But since $t \in \mathrm{GL}(V)$ commutes with the action of $\mathfrak{S}_n$, and hence with the action of $G \subset \mathfrak{S}_n$, then the map t restricts to a $\mathbb{C}$-linear isomorphism $\left(V^{\otimes n}\right)_k^G \xrightarrow{t} \left(V^{\otimes n}\right)_{n-k}^G$. Therefore $r_k = \dim\left(V^{\otimes n}\right)_k^G = \dim\left(V^{\otimes n}\right)_{n-k}^G = r_{n-k}$, proving the rank sizes are symmetric. $\square$

Proposition 1.13 *The matrix* $s(q) = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}(V)$ *acts on* $\left(V^{\otimes n}\right)^G$ *with* *trace*

$$r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n.$$

In particular, $s = s(-1)$ *acts on* $\left(V^{\otimes n}\right)^G$ *with trace* $r_0 - r_1 + r_2 - \cdots \pm r_n$.

Proof Note that $s(q)$ scales the basis element e_S of $V^{\otimes n}$, in fact, $s(q)(e_S) = q^{|S|} e_S$. For example, if $n = 5$ we have

$$\begin{aligned} s(q)(e_{\{1,3\}}) &= s(q)(b \otimes w \otimes b \otimes w \otimes w) \\ &= qb \otimes w \otimes qb \otimes w \otimes w \\ &= q^2 b \otimes w \otimes b \otimes w \otimes w = q^2 e_{\{1,3\}}. \end{aligned}$$

Hence $s(q)$ scales all of $\left(V^{\otimes n}\right)_k$ by q^k, so $s(q)$ scales $\left(V^{\otimes n}\right)_k^G$ by q^k. Thus its trace on $\left(V^{\otimes n}\right)^G = \bigoplus_{k=0}^n \left(V^{\otimes n}\right)_k^G$ will be $\sum_{k=0}^n q^k \dim_{\mathbb{C}} \left(V^{\otimes n}\right)_k^G = \sum_{k=0}^n q^k r_k$. $\square$

Proof of Theorem 1.7 Note that $s = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are conjugate within $\mathrm{GL}(V)$, since t is diagonalizable with eigenvalues $-1, 1$. Hence in the representation of $\mathrm{GL}(V)$ on $\left(V^{\otimes n}\right)^G$ they must act with the same trace, which is $r_0 - r_1 + r_2 - \cdots \pm r_n$ for s and hence also for t.

Thus it remains to show that t acts on $\left(V^{\otimes n}\right)^G$ with trace equal to the number of self-complementary G-orbits. We saw that t permutes the $\mathbb{C}$-basis $\{e_S\}_{S \in 2^{[n]}}$ for $V^{\otimes n}$ by swapping $e_S \overset{t}{\longleftrightarrow} e_{[n] \setminus S}$. This means that t also permutes the $\mathbb{C}$-basis $\{e_{\mathcal{O}}\}_{\mathcal{O} \in 2^{[n]}/G}$ for $\left(V^{\otimes n}\right)^G$ by fixing $e_{\mathcal{O}}$ if $\mathcal{O}$ is self-complementary and swapping $e_{\mathcal{O}} \overset{t}{\longleftrightarrow} e_{\mathcal{O}'}$ if $S \in \mathcal{O}$ but $[n] \setminus S \in \mathcal{O}' \neq \mathcal{O}$. Hence the trace of t counts these fixed points. $\square$

For example,

$$\begin{aligned} t(e) &= t(bwbw + wbwb) \\ &= wbwb + bwbw \\ &= e, \end{aligned}$$

$$\begin{aligned} t(e) &= t(bwww + wbww + wwbw + wwwb) \\ &= wbbb + bwbb + bbwb + bbbw \\ &= e. \end{aligned}$$

Proof of Theorem 1.8 We use two exercises. From Exercise 1.1.2, we know that for a representation $\rho\colon G \to \mathrm{GL}(U)$ of a finite group, the fixed space U^G has dimension

$$\dim_{\mathbb{C}}\left(U^G\right) = \frac{1}{|G|} \sum_{\sigma \in G} \operatorname{Trace}\left(\rho(\sigma)\right).$$

Thus,

$$\begin{aligned}
r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n &= \sum_{k=0}^{n} \dim_{\mathbb{C}} \big(V^{\otimes n}\big)_k^G \cdot q^k \\
&= \sum_{k=0}^{n} \left(\frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}_{\left(V^{\otimes n}\right)_k}(\sigma) \right) \cdot q^k \\
&= \frac{1}{|G|} \sum_{\sigma \in G} \sum_{k=0}^{n} q^k \mathrm{Trace}_{\left(V^{\otimes n}\right)_k}(\sigma) \\
&= \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} \big(1 + q^{|C|}\big).
\end{aligned}$$

The last equality is proven in Exercise 1.1.3. □

1.1.4 Unimodality

Proof of Theorem 1.9 We want to show for $k < n/2$ that $r_k \leq r_{k+1}$, i.e. $\dim_{\mathbb{C}} \big(V^{\otimes n}\big)_k^G \leq \dim_{\mathbb{C}} \big(V^{\otimes n}\big)_{k+1}^G$. Let's try to find an injective $\mathbb{C}$-linear map

$$\big(V^{\otimes n}\big)_k^G \hookrightarrow \big(V^{\otimes n}\big)_{k+1}^G.$$

We could do this for all permutation groups $G \subseteq \mathfrak{S}_n$ at once if we could find an injective $\mathbb{C}$-linear map

$$\big(V^{\otimes n}\big)_k \xhookrightarrow{U_k} \big(V^{\otimes n}\big)_{k+1}$$

that commutes with the $\mathfrak{S}_n$-action on $V^{\otimes n}$. An obvious candidate is

$$U_k(e_S) = \sum_{\substack{T \in \binom{[n]}{k+1} \\ T \supset S}} e_T.$$

For example for $n = 5$,

$$\begin{aligned}
U_2\big(e_{\{1,3\}}\big) &= e_{\{1,2,3\}} + e_{\{1,3,4\}} + e_{\{1,4,5\}}, \text{ i.e.,} \\
U_2(bwbww) &= bbbww + bwbbw + bwbwb.
\end{aligned}$$

The final step is to show that U_k commutes with the $\mathfrak{S}_n$-action (see Exercise 1.1.4 part (a)), and is injective for $k < n/2$. The latter is Lemma 1.14 below, whose assertions are justified in Exercise 1.1.4 parts (b)-(f). □

We review a few facts from linear algebra. For a square matrix $Q \in \mathbb{R}^{n \times n}$, say that Q is symmetric if $Q^T = Q$. If all eigenvalues ($\in \mathbb{R}$) are ≥ 0 then Q is said to be *positive semidefinite*, which is equivalent to $x^T Q x \geq 0$ for all $x \in \mathbb{R}^n$. If all eigenvalues are > 0 then Q is *positive definite*, which happens if and only if $x^T Q x > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, i.e., if and only if Q is positive semidefinite and nonsingular. Given any (rectangular) matrix A, the matrix $A^T A$ is always positive semidefinite, since $x^T A^T Ax = (Ax)^T Ax = |Ax|^2 \geq 0$.

Lemma 1.14 *For $k < \frac{n}{2}$, the map $U_k\colon \big(V^{\otimes n}\big)_k \to \big(V^{\otimes n}\big)_{k+1}$ defined $\mathbb{C}$-linearly by $U_k(e_S) = \sum_{T\in\binom{[n]}{k+1}} e_T$ is injective.*

Proof (This proof was shown to us by S. Fomin.) It is enough to show that $U_k^T U_k$ is invertible. Note that $U_k^T = D_{k+1}$ where $D_{k+1}(e_T) = \sum_{S\in\binom{[n]}{k},\, S\subset T} e_S$. This formula lets one check that every k-subset S has

$$(D_{k+1}U_k - U_{k-1}D_k)(e_S) = ((n-k)-k)e_S,$$

and therefore

$$D_{k+1}U_k = U_{k-1}D_k + (n-2k)I_{(V^{\otimes n})_k}.$$

From this identity, noting that $U_{k-1}D_k$ is positive semidefinite, and $(n-2k)I_{(V^{\otimes n})_k}$ is positive definite for $k < \frac{n}{2}$, one concludes that $D_{k+1}U_k$ is positive definite for $k < \frac{n}{2}$. Hence $D_{k+1}U_k$ is invertible, so U_k is injective. $\square$

Exercises

Exercise 1.1.1 Give a very short summary of the most important lessons of this lecture. Carry out small examples that illustrate the main definitions and results.

Exercise 1.1.2 Let V be a $\mathbb{C}$-vector space and $\pi\colon V \to V$ a $\mathbb{C}$-linear map which is idempotent, i.e., $\pi^2 = \pi$.

(a) Show that one has a $\mathbb{C}$-vector space decomposition $V = \pi(V) \oplus \big(1_V - \pi\big)V$, where $\pi(V) = \mathrm{image}(\pi)$ and $\big(1_V - \pi\big)(V) = \ker \pi$.

(b) Assume that $\dim_{\mathbb{C}} V$ is finite. Deduce that $\dim_{\mathbb{C}}\big(\mathrm{image}(\pi)\big) = \mathrm{Trace}_V(\pi)$.

(c) Show that for any representation $\rho\colon G \to \mathrm{GL}(V)$ of a finite group G on a finite dimensional $\mathbb{C}$-vector space V, the averaging map $\pi_G\colon V \to V$ defined by $\pi_G(v) = \dfrac{1}{|G|}\sum_{\sigma\in G}\rho(\sigma)(v)$, is idempotent, and its image is the G-fixed subspace V^G.

(d) Deduce that in the setting of (c), one has $\dim_{\mathbb{C}}\big(V^G\big) = \dfrac{1}{|G|}\sum_{\sigma\in G}\mathrm{Trace}_V\big(\rho(\sigma)\big)$.

(e) Use (d) to prove *Burnside's lemma:* when a finite group G permutes a finite set X, the number of G-orbits on X is $\dfrac{1}{|G|}\sum_{\sigma\in G}\big|\{x\in X\colon \sigma(x) = x\}\big|$.

Hint: Consider a vector space V with $\mathbb{C}$-basis $\{e_x\}_{x\in X}$ and $\sigma(e_x) = e_{\sigma(x)}$. What is $\dim_{\mathbb{C}}(V^G)$? Can we compute $\mathrm{Trace}_V(\sigma)$ for $\sigma \in G$?

Exercise 1.1.3 Let $V = \mathbb{C}^2$ with $\mathbb{C}$-basis $\{b, w\}$ and $V^{\otimes n} = \oplus_{k=0}^{n} \big(V^{\otimes n}\big)_k$, where $\big(V^{\otimes n}\big)_k$ has $\mathbb{C}$-basis $\{e_S\}_{S \in \binom{[n]}{k}}$. Let $\mathfrak{S}_n$ act positionally via

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

(a) Prove that a permutation $\sigma \in \mathfrak{S}_n$ has $\sigma(e_S) = e_S$ if and only if the subset S is a union of some of the cycles of σ.

(b) Deduce that $\sum_{k=0}^{n} q^k \operatorname{Trace}_{\left(V^{\otimes n}\right)_k}(\sigma) = \prod_{\substack{\text{cycles } c \\ \text{of } \sigma}} \big(1 + q^{|c|}\big)$.

Exercise 1.1.4 Recall the map $U_k \colon \big(V^{\otimes n}\big)_k \to \big(V^{\otimes n}\big)_{k+1}$ defined by $U_k(e_S) = \sum_{T \in \binom{[n]}{k+1} \colon S \subset T} e_T$.

(a) Prove that U_k commutes with the $\mathfrak{S}_n$-action on $V^{\otimes n}$.

(b) Prove that the map $D_{k+1} \colon \big(V^{\otimes n}\big)_{k+1} \to \big(V^{\otimes n}\big)_k$, defined by $D_{k+1}(e_T) = \sum_{S \in \binom{[n]}{k} \colon S \subset T} e_S$, is actually the transpose/adjoint map $D_{k+1} = U_k^T$ with respect to our usual bases on $\big(V^{\otimes n}\big)_k$.

(c) Explain why (b) implies that both $D_{k+1}U_k$ and $U_{k-1}D_k$ are symmetric and nonnegative definite (all eigenvalues are ≥ 0).

(d) Prove that $\big(D_{k+1}U_k - U_{k-1}D_k\big)(e_S) = (n - 2k)e_S$ for any $S \in \binom{[n]}{k}$, and hence $D_{k+1}U_k = U_{k-1}D_k + (n - 2k)I_{(V^{\otimes n})_k}$.

(e) Explain why (d) implies that $D_{k+1}U_k$ is positive definite for $k < n/2$.

(f) Explain why U_k is injective for $k < n/2$.

Exercise 1.1.5 For $G = \mathfrak{S}_k[\mathfrak{S}_\ell] \subset \mathfrak{S}_{k\ell}$, recall that the G-orbits $2^{[k\ell]}/G$ biject with the Ferrers diagrams inside a $k \times \ell$ box, that is, with the number partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ with $0 \leq \lambda_j \leq \ell$. Let $|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_k$ so that the rank-generating function for $2^{[k\ell]}/G$ is $r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \sum_{\text{such } \lambda} q^{|\lambda|} = \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$, the q-binomial coefficient.

(a) Prove the q-Pascal recurrences

$$\begin{bmatrix} N \\ k \end{bmatrix}_q = q^k \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q \text{ and } \begin{bmatrix} N \\ k \end{bmatrix}_q = \begin{bmatrix} N-1 \\ k \end{bmatrix}_q + q^{N-k} \begin{bmatrix} N-1 \\ k-1 \end{bmatrix}_q,$$

where $N := k + \ell$.

(b) Prove the formula $\begin{bmatrix} N \\ k \end{bmatrix}_q = \dfrac{[N]!_q}{[k]!_q [N-k]!_q}$, where $[N]!_q$ and $[m]_q$ are defined as in Example 1.35.

(c) Prove that when $q = p^d$ is the power of a prime, and hence the cardinality of a finite field $\mathbb{F}_q$, then $\begin{bmatrix} N \\ k \end{bmatrix}_q = \#\big\{k\text{--dimensional } \mathbb{F}_q\text{--linear subspaces of } \mathbb{F}_q^N\big\}$.

1.2 Lecture 2: Representation theory and reflection groups

Definition 1.15 For a group G, a *representation* of G on a $\mathbb{C}$-vector space $V \cong \mathbb{C}^n$ means a *group homomorphism*

$$G \xrightarrow{\rho} \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C}).$$

Example 1.16 (examples in combinatorics) 1. *Permutation representations* come from homomorphisms $G \to \mathfrak{S}_n$ of G into the symmetric group $\mathfrak{S}_n$. Such representations factor as follows:

$$\begin{aligned} G \overset{i}{\hookrightarrow} \mathfrak{S}_n &\xrightarrow{\rho_{\mathrm{perm}}} \mathrm{GL}_n(\mathbb{C}) \\ \sigma &\longmapsto n \times n \text{ permutation matrix of } \sigma, \end{aligned}$$

e.g., in $\mathfrak{S}_5(\mathbb{C})$,

$$\sigma = (2\ 4\ 5)(1\ 3) \xmapsto{\rho_{\mathrm{perm}}} \begin{array}{c} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \\ {\scriptstyle 3} \\ {\scriptstyle 4} \\ {\scriptstyle 5} \end{array} \begin{array}{c} \begin{array}{ccccc} {\scriptstyle 1} & {\scriptstyle 2} & {\scriptstyle 3} & {\scriptstyle 4} & {\scriptstyle 5} \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array} \in \mathrm{GL}_5(\mathbb{C})$$

For example, we saw

$G = \langle (1, 2, \ldots, n) \rangle \hookrightarrow \mathfrak{S}_n$ whose G-orbits on $2^{[n]}$ were necklaces
$\cong \mathbb{Z}/n\mathbb{Z}$

$G = \mathfrak{S}_k[\mathfrak{S}_\ell] \hookrightarrow \mathfrak{S}_{k\ell}$ whose G-orbits on $2^{[k\ell]}$ were Ferrers diagrams in $\left.\underbrace{\begin{array}{|c|c|c|c|}\hline & \cdots & & \\ \hline & \cdots & & \\ \hline\end{array}}_{\ell}\right\} k$

$G = \mathfrak{S}_v \hookrightarrow \mathfrak{S}_{\binom{v}{2}}$ whose G-orbits on $2^{\binom{[v]}{2}}$ were the unlabeled graphs.

Or the *regular representation* ρ_{reg}:

$$G \hookrightarrow \mathfrak{S}_{|G|} \text{ in which } \rho_{\mathrm{reg}}(g)(h) := gh.$$

2. 1-dimensional representations, $G \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ such as the *trivial representation*

$$\begin{aligned} \mathbb{1} = \mathbb{1}_G \colon G &\to \mathbb{C}^* \\ g &\mapsto 1, \end{aligned}$$

or the *determinant representation*

$$\begin{aligned} \det \colon \mathrm{GL}(V) &\xrightarrow{\det} \mathbb{C}^* \\ g &\longmapsto \det(g). \end{aligned}$$

3. Symmetry groups of geometric objects $P \subset \mathbb{R}^n$, where

$$G = \mathrm{Aut}(P) := \{g \in \mathrm{GL}_n(\mathbb{R}) \colon g(P) = P\}$$

If P is the polygon in Figure 1.6, then $G = \mathrm{Aut}(P) = \langle c \rangle \cong \mathbb{Z}/4\mathbb{Z}$. The generator c of G is an element of the orthogonal group $\mathrm{O}_2(\mathbb{R}) \subset \mathrm{GL}_2(\mathbb{R})$ $(\subset \mathrm{GL}_2(\mathbb{C}))$.

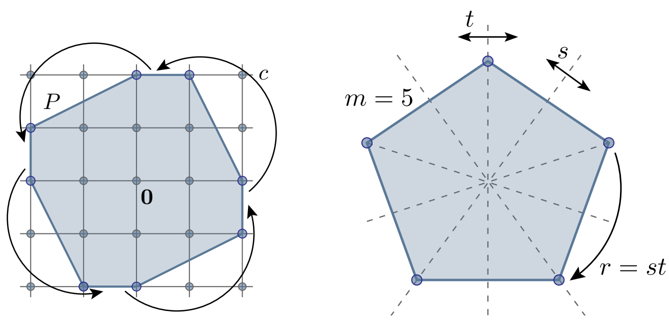


Figure 1.6 On the left, $P \subset \mathbb{R}^2$ is the polygon in Example 1.16.3.. The group of transformations which fix P is generated by a rotation c of $\mathbb{R}^2$ about the origin. On the right, we have a regular pentagon (5-sided polygon), and t, s denote two elements in the group of symmetries of $G = I_2(5) \xhookrightarrow{\rho_{\mathrm{ref}}} \mathrm{O}_2(\mathbb{R})$, namely two Euclidean reflections with respect to the indicated lines in the figure. The composition of these two reflections gives a rotation $r = st \in I_2(5)$ about the center of the polygon.

4. The (real) reflection groups are subgroups $G \subset \mathrm{O}_n(\mathbb{R})$ $(\subset \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C}))$ generated by Euclidean reflections t with a hyperplane H as a set of fixed points; this hyperplane is called the *reflecting hyperplane*. Good examples of reflection groups are $G = \mathrm{Aut}(P)$ for *regular polytopes* P. In this case, G is *transitive* on *maximal flags* of faces, e.g., $G = I_2(m)$ = symmetries of a *regular m-sided polygon*, and $G = \mathfrak{S}_n$ symmetries of a *regular* $(n-1)$-*dimensional simplex*. As examples, see Figure 1.6 (right), Figure 1.7, and Example 1.41 far below.

Definition 1.17 Call two representations $G \xrightarrow{\rho} \mathrm{GL}(V)$ and $G \xrightarrow{\rho'} \mathrm{GL}(V')$ *equivalent* if one has a $\mathbb{C}$-linear isomorphism $V \xrightarrow{\varphi} V'$ with $\varphi^{-1} \circ \rho'(g) \circ \varphi = \rho(g)$ for all $g \in G$.

Question: Can we classify, in any sense, all G-representations *up to equivalence*?

Answer: Yes, when G is *finite* (and we are working over $\mathbb{C}$).

In fact, the indispensable tool here are the *traces* that we have already been using.

Definition 1.18 Given a representation $G \xrightarrow{\rho} \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ its *character* χ_ρ

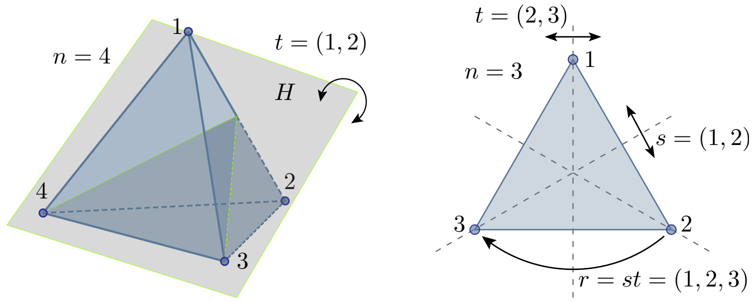


Figure 1.7 Regular 3-dimensional (left) and 2-dimensional (right) simplices. The reflection t, on the left, and the reflections s, t and the rotation st on the right, are elements of the group of symmetries $G = \mathfrak{S}_n \overset{\rho_{\mathrm{ref}}}{\hookrightarrow} \mathrm{O}_{n-1}(\mathbb{R})$ for $n = 4$ and $n = 3$, respectively.

is the (conjugacy) *class function*

$$\begin{aligned} G &\xrightarrow{\chi_\rho} \mathbb{C} \\ g &\longmapsto \chi_\rho(g) := \mathrm{Trace}\big(\rho(g)\big). \end{aligned}$$

Recall that as a class function $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$ for all $h, g \in G$.

1.2.1 Facts from finite group representation theory over $\mathbb{C}$

Theorem 1.19 (Maschke) *One can always decompose* $\rho \cong \bigoplus_{i=1}^{t} \rho_i$, *meaning*

$$\rho(g) = \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_t \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & \cdots & V_t \end{array} \\ \left(\begin{array}{c|c|c|c} \rho_1(g) & 0 & 0 & 0 \\ \hline 0 & \rho_2(g) & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \rho_t(g) \end{array}\right) \end{array}$$

for all $g \in G$, *where* $V = \bigoplus_{i=1}^{t} V_i$ *and each representation* $G \xrightarrow{\rho_i} \mathrm{GL}(V_i)$ *is* simple/irreducible, *i.e., that* V_i *has* no G-stable subspaces, *except* $\{0\}$ *and* V *itself.*

Theorem 1.20 *The list of (inequivalent) irreducible representations* $\{\rho_1, \rho_2, \ldots, \rho_r\}$ *has size* $r = \#G-$*conjugacy classes.*

In fact, the character χ_ρ *determines* ρ *up to equivalence, as* irreducible characters $\{\chi_{\rho_1}, \ldots, \chi_{\rho_r}\}$ *give a* $\mathbb{C}$*-basis for the space of* class functions $f : G \to \mathbb{C}$.

Furthermore, this basis is orthonormal *with respect to this* positive definite Hermitian *inner product on class functions:*

$$\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)} \chi_2(g).$$

Hence to decompose $\rho = \bigoplus_{i=1}^r \rho_i^{\oplus m_i}$ into irreducibles $\rho_1, \ldots, \rho_r$ one can compute the multiplicities m_i from $\chi_\rho = \sum_{i=1}^r m_i \chi_{\rho_i}$, i.e., $\langle \chi_\rho, \chi_{\rho_i} \rangle = m_i$.

Also, $\langle \chi_\rho, \chi_\rho \rangle_G = \sum_{i=1}^r m_i^2$, so χ_ρ is irreducible if and only if $\langle \chi_\rho, \chi_{\rho_i} \rangle_G = 1$.

Example 1.21 (standard examples) 1. 1-dimensional representations $G \xrightarrow{\rho} \mathbb{C}^*$ are the same as their own character $\chi_\rho = \rho$. Hence they are always class functions.

2. For permutation representations $\rho = \rho_{\mathrm{perm}} \circ i \colon G \to \mathrm{GL}_n(\mathbb{C})$, the character $\chi_\rho(\sigma) = \mathrm{Trace}(\sigma) = \#$ of fixed points (1-cycles) of σ as a permutation, and

$$\begin{aligned} \langle \chi_\rho, \chi_{\mathbb{1}} \rangle_G &= \frac{1}{|G|} \sum_{\sigma \in G} \chi_\rho(g) = \frac{1}{|G|} \sum_{\sigma \in G} \#\text{fixed points of } \sigma \\ &= \#\text{ of } G\text{-orbits on } \{1, 2, \ldots, n\}, \end{aligned}$$

where the latter follows by Burnside's lemma.

3. The regular representation $G \overset{\rho_{\mathrm{reg}}}{\hookrightarrow} \mathfrak{S}_{|G|} \to \mathrm{GL}_{|G|}(\mathbb{C})$ having $\rho_{\mathrm{reg}}(g)(h) = gh$ has $\chi_{\mathrm{reg}}(g) = \mathrm{Trace}\big(\rho_{\mathrm{reg}}(g)\big) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{else.} \end{cases}$

Hence

$$\begin{aligned} \big\langle \chi_{\mathrm{reg}}, \chi_{\rho_i} \big\rangle_G &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\mathrm{reg}}(g)} \chi_{\rho_i}(g) = \frac{1}{|G|} \overline{\chi_{\mathrm{reg}}(e)} \chi_{\rho_i}(e) \\ &= \frac{1}{|G|} |G| \dim_{\mathbb{C}}(V_i) = \dim_{\mathbb{C}}(V_i). \end{aligned}$$

Corollary 1.22 *The regular representation ρ_{reg} of G contains every irreducible ρ_i with multiplicity $\dim_{\mathbb{C}}(V_i)$, i.e., $\rho_{\mathrm{reg}} = \bigoplus_{i=1}^r \rho_i^{\oplus \dim_{\mathbb{C}}(V_i)}$.*

Taking dimensions gives $|G| = \sum_{i=1}^r \dim_{\mathbb{C}}(V_i)^2$.

Example 1.23 Let $G = \mathfrak{S}_3 = \{e, (1\ 2), (1\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$. The permutation representation $\rho_{\text{perm}} : G \to \mathrm{GL}_3(\mathbb{C})$ is given by

$$e \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (2\,3) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1\,2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(1\,2\,3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad (1\,3) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad (1\,3\,2) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

There are $r = 3$ conjugacy classes. Namely $\{e\}$, $\{(1\,2), (1\,3), (2\,3)\}$ and $\{(1\,2\,3), (1\,3\,2)\}$. Who are the three irreducible representations?

Since $G = \langle s, t\rangle$, with $s = (1\,2)$, and $t = (2\,3)$, then its 1-dimensional representations χ are determined by the values $\chi(s)$, $\chi(t)$ in $\mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$. But $s^2 = t^2 = e$, so these values $\chi(s)$, $\chi(t)$ lie in $\{\pm 1\}$. Also, since s, t are conjugate in $\mathfrak{S}_3$, these values are equal, either both $+1$ or both -1. This gives two 1-dimensional characters

$$\begin{aligned} \mathbb{1}\colon \mathfrak{S}_3 &\to \mathbb{C}^* & \qquad \mathrm{sgn}\colon \mathfrak{S}_3 &\to \mathbb{C}^* \\ s, t &\mapsto 1 & s, t &\mapsto -1. \end{aligned}$$

Let ρ be the other irreducible character. Then $|G| = \sum_{i=1}^{3} \dim_{\mathbb{C}}(V_i)^2$, i.e., $3! = 1^2 + 1^2 + (\dim \rho)^2$. Hence $\dim \rho = 2$.

We claim that the reflection representation $\rho_{\text{ref}}\colon \mathfrak{S}_3 \to \mathrm{O}_2(\mathbb{R})$ is irreducible. In fact, by computing its character

$$\begin{aligned} \chi_{\text{ref}}(e) &= \mathrm{Trace}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2 \\ \chi_{\text{ref}}((i\ j)) &= \mathrm{Trace}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 \\ \chi_{\text{ref}}((i\ j\ k)) &= \mathrm{Trace}\begin{pmatrix} \xi & 0 \\ 0 & \xi^{-1} \end{pmatrix} = \xi + \xi^{-1} = -1, \end{aligned}$$

where $\xi = e^{2\pi i/3}$, and

$$\langle \chi_{\text{ref}}, \chi_{\text{ref}} \rangle_G = \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} \overline{\chi_{\text{ref}}(\sigma)} \chi_{\text{ref}}(\sigma) = \frac{1}{6}\big(2 \cdot 2 + 3 \cdot 0 \cdot 0 + 2 \cdot (-1)(-1)\big) = 1.$$

The *irreducible character table* for $\mathfrak{S}_3$ is given in Table 1.1. We can also see now that the permutation representation ρ_{perm} must be reducible. In fact from its character values (see Table 1.2), one sees that $\chi_{\text{perm}} = \chi_{\mathbb{1}} + \chi_{\text{ref}}$, and hence $\rho_{\text{perm}} = \mathbb{1} \oplus \rho_{\text{ref}}$.

One can also see that ρ_{perm} is reducible directly from the geometry in $\mathbb{R}^3$ ($\subset \mathbb{C}^3$), as follows. The space $\mathbb{R}^3$ is the orthogonal direct sum of the line $x_1 = x_2 = x_3$ carrying the trivial representation $\mathbb{1}$, and its perpendicular plane $x_1 + x_2 + x_3 =$

	e	$(1\ 2); (1\ 3); (2\ 3)$	$(1\ 2\ 3); (1\ 3\ 2)$
$\mathbb{1}$	1	1	1
sgn	1	-1	1
ρ_{ref}	2	0	-1

Table 1.1 *Character table for* $\mathfrak{S}_3$.

	e	$(1\ 2); (1\ 3); (2\ 3)$	$(1\ 2\ 3); (1\ 3\ 2)$
χ_{perm}	3	1	0

Table 1.2 *Character values for the representation* $\rho_{\mathrm{perm}} \colon \mathfrak{S}_3 \to \mathrm{GL}_3(\mathbb{C})$.

0 carrying ρ_{ref}. In fact, the three standard basis vectors e_1, e_2, e_3 of $\mathbb{R}^3$ project perpendicularly to this plane, giving the three vertices $1, 2, 3$ of the triangle on the right in Figure 1.7.

This example is generalized in Exercise 1.2.4.

Exercises

Exercise 1.2.1 Give a very short summary of the most important lessons of this lecture. Carry out small examples that illustrate the main definitions and results.

Exercise 1.2.2 Let $G = I_2(m)$ the group of linear symmetries of a regular m-sided polygon, with r in G any rotation through angle $\frac{2\pi}{m}$, and s in G any reflection symmetry.

(a) Prove that $G = \{e, r, r^2, \ldots, r^{m-1}\} \sqcup \{s, sr, sr^2, \ldots, sr^{m-1}\}$, the first set corresponding to the rotations and the second to the reflections.

(b) Prove the presentation for G as $G \cong \langle s, r \mid s^2 = r^m = e,\ srs = r^{-1}\rangle$.

(c) Prove the (Coxeter) presentation of G as $G \cong \langle s, r \mid s^2 = t^2 = e = (st)^m\rangle$.

Exercise 1.2.3

(a) Prove that every 1-dimensional irreducible representation for $G = I_2(m)$ sends s, t to values $\{\pm 1\}$, and they exactly are $\{\mathbb{1}, \det\}$ if m is odd, and $\{\mathbb{1}, \det, \rho_s, \rho_t\}$ if m is even, where $\mathbb{1}$ sends s and t to 1, $\det(s) = \det(t) = -1$, $\rho_s(s) = \rho_t(t) = -1$, and $\rho_s(t) = \rho_t(s) = 1$.

(b) Prove that there is a representation $\rho^{(j)} \colon G = I_2(m) \to \mathrm{GL}_2(\mathbb{C})$ for each

$j \in \mathbb{Z}$ uniquely defined by $\rho^{(j)}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\rho^{(j)}(r) = \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$, where $\zeta := e^{2\pi i/m}$.

(c) Prove the following isomorphisms of G-representations:

$$\begin{aligned} \rho^{(j)} &\cong \rho^{(j+m)} \cong \rho^{(m-j)}, \\ \rho^{(0)} &\cong \mathbb{1} \oplus \mathbb{1}, \\ \rho^{(m/2)} &\cong \rho_s \oplus \rho_t \text{ for } m \text{ even}, \\ \rho^{(1)} &\cong \rho_{\mathrm{ref}}. \end{aligned}$$

(d) Prove that $G = I_2(m)$ has as its list of inequivalent irreducible representations the union of $\{\rho^{(j)}\}_{j=1,2,\ldots,\lfloor (m-1)/2 \rfloor}$ together with either $\{\mathbb{1}, \det\}$ if m is odd, or $\{\mathbb{1}, \det, \rho_s, \rho_t\}$ if m is even.

Exercise 1.2.4 Let $G \subset \mathfrak{S}_n$ be a permutation group, and $\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ the associated permutation representation for G acting on $[n]$.

(a) Show that the permutation representation where G permutes the ordered pairs $(i,j) \in [n] \times [n]$ via $g(i,j) = \big(g(i), g(j)\big)$ has character χ_ρ^2.

(b) If ρ is a doubly-transitive permutation representation of G, meaning that it has G acting transitively on the set of pairs $\{(i,j)\colon 1 \le i \ne j \le n\}$, then show $\langle \chi_\rho, \chi_\rho \rangle_G = 2$.

(c) If ρ is doubly-transitive, show $\rho = \mathbb{1} \oplus \rho'$ with ρ' irreducible. Hint: Show $\rho = \mathbb{1} \oplus \rho'$ for some representation ρ', and calculate $\langle \chi_{\rho'}, \chi_{\rho'} \rangle_G$.

(d) Prove $\rho_{\mathrm{perm}}\colon G = \mathfrak{S}_n \to \mathrm{GL}_n(\mathbb{C})$ decomposes as $\rho_{\mathrm{perm}} = \mathbb{1} \oplus \rho_{\mathrm{ref}}$, where ρ_{ref} represents $\mathfrak{S}_n$ as the linear symmetries of a regular $(n-1)$-simplex.

(e) Prove $G = \mathfrak{S}_n$ has ρ_{ref} irreducible.

Exercise 1.2.5 Prove that $G = \mathfrak{S}_4$ has the following list of (inequivalent) irreducibles,

$$\{\mathbb{1},\ \mathrm{sgn},\ \rho_{\mathrm{ref}},\ \mathrm{sgn} \otimes \rho_{\mathrm{ref}},\ \rho_2\},$$

where $\mathrm{sgn} \otimes \rho_{\mathrm{ref}}$ sends σ to $\mathrm{sgn}(\sigma)$ times the permutation matrix of σ, and ρ_2 is the composite $\mathfrak{S}_4 \to \mathfrak{S}_4/V_4 \cong \mathfrak{S}_3 \xrightarrow{\rho_{\mathrm{ref}}} \mathrm{O}_2(\mathbb{R})$. The group V_4 is the Klein-four subgroup $\big\{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\big\}$.

Compute the irreducible character table for $\mathfrak{S}_4$ by giving their characters values on $\big\{e, (ij), (ijk), (ij)(k\ell), (ijk\ell)\big\}$.

1.3 Lecture 3: Molien's theorem and coinvariant algebras

Let us examine the behaviour of *characters* of group representations under various *(multi-) linear constructions.*

1.3.1 Direct sum

Given representations $\rho_1 : G \to \mathrm{GL}(V_1)$ and $\rho_2 : G \to \mathrm{GL}(V_2)$, we have seen that $\rho_1 \oplus \rho_2 : G \to \mathrm{GL}(V_1 \oplus V_2)$ is given by

$$(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2)),$$

which can be written as

$$(\rho_1 \oplus \rho_2)(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right).$$

Then one has the character equality $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$.

1.3.2 Tensor product

Similarly, one can create $\rho_1 \otimes \rho_2 : G \to \mathrm{GL}(V_1 \otimes V_2)$ via

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2).$$

Thus $(\rho_1 \otimes \rho_2)(g)$ is the tensor/Kronecker product of the matrices $\rho_1(g) \otimes \rho_2(g)$.

Recall for matrices $A = (a_{ij})$ and B, the tensor product matrix $A \otimes B = (a_{ij}B)$. If $A : V_1 \to V_1$ and $B : V_2 \to V_2$ where V_1 has a basis $\{v_i\}$ and V_2 has a basis $\{w_j\}$, then $A \otimes B : V_1 \otimes V_2 \to V_1 \otimes V_2$ acts by

$$A \otimes B(v_i \otimes w_j) = Av_i \otimes Bw_j = \cdots + a_{ii}b_{jj}v_i \otimes w_j + \cdots$$

Therefore

$$\mathrm{Trace}(A \otimes B) = \sum_i \sum_j a_{ii}b_{jj} = \left(\sum_i a_{ii}\right)\left(\sum_j b_{jj}\right) = \mathrm{Trace}(A)\mathrm{Trace}(B).$$

Hence $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g)\chi_{\rho_2}(g)$, i.e., $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2}$ as class functions on G.

1.3.3 d-th Tensor power

As in Section 1.1.2 above, we can create the d-th tensor power

$$T^d(V) := V^{\otimes d} = V \otimes \cdots \otimes V (d \text{ factors}).$$

Given a G-representation $\rho \colon G \to \mathrm{GL}(V)$ we can define the *diagonal action* $T^d(\rho) \colon G \to \mathrm{GL}\big(T^d(V)\big) = \mathrm{GL}\big(V^{\otimes d}\big)$ via

$$T^d(\rho)(g)(v_1 \otimes \cdots \otimes v_d) = \rho(g)(v_1) \otimes \cdots \otimes \rho(g)(v_d).$$

Thus $\chi_{T^d(\rho)}(g) = \chi_\rho(g)^d$.

1.3.4 Tensor Algebra

Putting the d-th tensor powers together we get the *tensor algebra*

$$T(V) := \oplus_{d\geq 0} T^d(V) = \bigoplus_{d\geq 0} V^{\otimes d},$$

with a G-representation $T(\rho) : G \to \mathrm{GL}(T(V))$ which now has a *graded character*

$$\chi_{T(\rho)}(g;q) := \sum_{d\geq 0} q^d \cdot \chi_{T^d(\rho)}(g) = \sum_{d\geq 0} q^d \cdot \chi_\rho(g)^d = \frac{1}{1 - q\chi_\rho(g)}.$$

1.3.5 Symmetric powers and symmetric algebra

The *d-th symmetric power of V* is defined by

$$\mathrm{Sym}^d(V) := V^{\otimes d}/\mathrm{span}_{\mathbb{C}}\{v_1\otimes\cdots\otimes v_i\otimes v_{i+1}\otimes\cdots\otimes v_d - v_1\otimes\cdots\otimes v_{i+1}\otimes v_i\otimes\cdots\otimes v_d\}.$$

Denote by $v_1 \cdot v_2 \cdots v_d$ the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_d$ in the quotient. Note that we have commutativity: $v_1 \cdot v_2 \cdot \ldots \cdot v_d = v_{w(1)} \cdot v_{w(2)} \cdot \ldots \cdot v_{w(d)}$ for any $w \in \mathfrak{S}_d$.

Because of the G-action $T^d(\rho) : G \to \mathrm{GL}\big(V^{\otimes d}\big)$ commutes with the $\mathfrak{S}_d$ action on the positions $v_1 \otimes \cdots \otimes v_d$, the subspace modded out above is G-stable, and the G-action makes sense on the quotient. That is, one obtains a G-representation

$$\mathrm{Sym}^d(\rho) : G \to \mathrm{GL}\big(\mathrm{Sym}^d V\big),$$

via $\mathrm{Sym}^d(\rho)(g)(v_1 \cdot v_2 \cdot \ldots \cdot v_d) = \rho(g)(v_1) \cdot \rho(g)(v_2) \cdot \ldots \cdot \rho(g)(v_d)$. Putting them together, on the symmetric algebra $\mathrm{Sym}(V) := \bigoplus_{d\geq 0} \mathrm{Sym}^d(V)$ one also obtains a G-representation $\mathrm{Sym}(\rho) : G \to \mathrm{GL}\big(\mathrm{Sym}(V)\big)$.

Can we compute its graded character $\chi_{\mathrm{Sym}(\rho)}(g;q) := \sum_{d\geq 0} q^d \cdot \chi_{\mathrm{Sym}^d(\rho)}(g)$?

Proposition 1.24 *For any group representation $\rho : G \to \mathrm{GL}(V)$ and $g \in G$ we have*

$$\chi_{\mathrm{Sym}(\rho)}(g;q) := \sum_{d\geq 0} q^d \cdot \chi_{\mathrm{Sym}^d(\rho)}(g) = \frac{1}{\det\big(1_V - q \cdot \rho(g)\big)}.$$

We will prove Proposition 1.24 in Exercise 1.3.2, along with the famous corollary:

Theorem 1.25 (Molien 1897) *Given a finite group representation $\rho : G \to \mathrm{GL}(V)$ (with $V = \mathbb{C}^n$), for any other representation Ψ of G one has*

$$\sum_{d\geq 0} \big\langle \chi_{\mathrm{Sym}^d(\rho)}, \chi_\Psi \big\rangle_G \cdot q^d = \frac{1}{|G|} \sum_{g\in G} \frac{\overline{\chi_\Psi(g)}}{\det\big(1_V - q \cdot \rho(g)\big)}.$$

In particular, taking $\Psi = \mathbb{1}_G$, one obtains the Hilbert series for the G-fixed subalgebra $\mathrm{Sym}(V)^G$*, namely*

$$\mathrm{Hilb}\big(\mathrm{Sym}(V)^G, q\big) := \sum_{d\geq 0} q^d \cdot \dim_{\mathbb{C}} \mathrm{Sym}^d(V)^G = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\det(1_V - q \cdot \rho(g))}.$$

Note that if $V = \mathbb{C}^n$ has a $\mathbb{C}$-basis $x_1, \ldots, x_n$ then $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, \ldots, x_n] =: \mathbb{C}[\boldsymbol{x}]$, the polynomial ring in n variables, and $\mathrm{Sym}(V)^G \cong \mathbb{C}[\boldsymbol{x}]^G$ is the G-invariant subalgebra when $\rho(G) \subset \mathrm{GL}_n(\mathbb{C})$ acts via linear substitution of variables.

Example 1.26 Let $G = \mathfrak{S}_3$ and $\rho_{\mathrm{perm}} : G \to \mathrm{GL}_3(\mathbb{C}) = \mathrm{GL}(V)$, where $V = \mathbb{C}^3$ has basis x_1, x_2, x_3. Then $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, x_2, x_3]$ with $G = \mathfrak{S}_3$ permuting variables. Hence $\mathrm{Sym}(V)^G \cong \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3}$ consists of the symmetric polynomials.
By the *fundamental theorem of symmetric functions* we have $\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, \ldots, e_n]$, where e_d is the d-th elementary symmetric function given by

$$\sum_{1 \leq i_1 < \ldots < i_d \leq n} x_{i_1} x_{i_2} \ldots x_{i_d}.$$

Thus, $\mathrm{Sym}(V)^G \cong \mathbb{C}[e_1, e_2, e_3]$, where $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$ and $e_3 = x_1x_2x_3$. Therefore, we expect

$$\begin{aligned} \mathrm{Hilb}\big(\mathrm{Sym}(V)^G, q\big) &= (1 + q + q^2 + \cdots)\big(1 + q^2 + q^4 + \cdots\big)\big(1 + q^3 + q^6 + \cdots\big) \\ &= \frac{1}{(1-q^1)(1-q^2)(1-q^3)}. \end{aligned}$$

What does Molien's Theorem tell us? Recall the $\mathfrak{S}_3$ character table given in Table 1.1. Molien's Theorem tell us that $\mathrm{Sym}(V) = \mathbb{C}[x_1, x_2, x_3]$ has

$$\begin{aligned} \sum_{d \geq 0} \big\langle \chi_{\mathrm{Sym}^d(\rho)},\ \chi_\psi \big\rangle_{\mathfrak{S}_3} \cdot q^d &= \begin{cases} \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(1)}{(1-q^2)(1-q)} + \frac{2(1)}{1-q^3}\right] & \text{if } \psi = \mathbb{1}, \\ \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(-1)}{(1-q^2)(1-q)} + \frac{2(1)}{1-q^3}\right] & \text{if } \psi = \mathrm{sgn}; \\ \frac{1}{3!}\left[\frac{2}{(1-q)^3} + \frac{3(0)}{(1-q^2)(1-q)} + \frac{2(-1)}{1-q^3}\right] & \text{if } \psi = \rho_{\mathrm{ref}}, \end{cases} \\ &= \begin{cases} \frac{1}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \mathbb{1}; \\ \frac{q^3}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \mathrm{sgn}; \\ \frac{q^1 + q^2}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \rho_{\mathrm{ref}}. \end{cases} \end{aligned}$$

Here we used that for any permutation $\sigma \in \mathfrak{S}_n$ we have (see Exercise 1.3.3):

$$\det\big(1_V - q\rho_{\mathrm{perm}}(\sigma)\big) = \prod_{\substack{\text{cycles } c \\ \text{of } \sigma}} \big(1 - q^{|c|}\big).$$

Notice also that the value obtained here in the case $\psi = \mathbb{1}$ is what we expected since it is $\mathrm{Hilb}(\mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3}, q) = \mathrm{Hilb}(\mathbb{C}[e_1, e_2, e_3], q)$. The numerators $f^\psi(q)$ that appeared in these expressions

$$\sum_{d\geq 0} \langle \chi_{\mathrm{Sym}^d(\rho)},\ \chi_\Psi \rangle_{\mathfrak{S}_3} \cdot q^d = \frac{f^\psi(q)}{(1-q)(1-q^2)(1-q^3)}$$

are called the *fake-degree* polynomials for ψ. They come from viewing $\mathfrak{S}_3$ as a reflection group acting on $V = \mathbb{C}^3$; see Corollary 1.30 below.

Lemma 1.27 *Given any square matrix $A = (a_{ij})$ of variables, viewed as a $\mathbb{C}(a_{ij})$-linear map one has the following identity in the power series ring $\mathbb{C}[\![a_{ij}]\!]$:*

$$\sum_{d\geq 0} \mathrm{Trace}_{\mathrm{Sym}^d(V)} \mathrm{Sym}^d(A) = \frac{1}{\det(1_V - A)}.$$

Remark 1.28 Proposition 1.24 can be deduced from Lemma 1.27 (see Exercise 1.3.2), which in turn is equivalent to MacMahon's Master Theorem (1916). The latter is described and proven in Exercise 1.3.4 and used in Exercise 1.3.5 (d) to prove an interesting identity.

The next result of Shephard and Todd [6] and Chevalley [1] explains the important role played by reflection groups in this story.

Theorem 1.29 (Shephard-Todd 1955/Chevalley 1955) *Given any finite reflection group $G \subset \mathrm{GL}_n(\mathbb{R})$ acting on $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, \ldots, x_n] := \mathbb{C}[\boldsymbol{x}]$ by linear substitutions, where $x_1, \ldots, x_n$ form a basis for V, then*

(a) the G-invariant subalgebra is again a polynomial algebra given by: $\mathbb{C}[\boldsymbol{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ for homogeneous polynomials $f_1, \ldots, f_n$ of degrees $d_1, \ldots, d_n$, and we have

$$\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big) = \frac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})}.$$

(b) As G-representations, the coinvariant algebra $\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})$ where $(\boldsymbol{f}) = (f_1, \ldots, f_n)$ is isomorphic to the regular representation:

$$\mathbb{C}[\boldsymbol{x}]/(f_1, \ldots, f_n) \cong \rho_{\mathrm{reg}}.$$

Note that, for a reflection group G, Theorem 1.29 says the *coinvariant algebra* $\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})$ gives us naturally a *graded version* of the regular representation (!).

Using commutative algebra, namely that $\mathbb{C}[x_1, \ldots, x_n]$ is a Cohen-Macaulay ring and $f_1, \ldots, f_n$ is a system of parameters and hence a regular sequence, one can deduce the following corollary.

Corollary 1.30 *For a reflection group G, as in Theorem 1.29, one has an isomorphism of graded G-representations*

$$\mathbb{C}[\boldsymbol{x}] \cong \mathbb{C}[\boldsymbol{x}]^G \otimes \mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f}), \tag{1.1}$$

and hence for any representation ψ

$$\begin{aligned}\sum_{d\geq 0}\big\langle \chi_{\mathbb{C}[\boldsymbol{x}]_d}, \chi_\psi \big\rangle_G \cdot q^d &= \mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big) \cdot \sum_{d\geq 0}\big\langle \chi_{\left(\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})\right)_d}, \chi_\psi \big\rangle_G \cdot q^d \\ &= \frac{1}{(1-q^{d_1})\cdots(1-q^{d_n})} \cdot f_\psi(q). \qquad (1.2)\end{aligned}$$

In the formula (1.1),

- $\mathbb{C}[\boldsymbol{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ carrying only trivial G-representations $\mathbb{1}$ in all degrees,
- the tensor product is graded, i.e., $(A \otimes B)_d = \oplus_{i+j=d} A_i \otimes B_j$, and
- $\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})$ is the coinvariant algebra, a graded version of the regular representation.

The polynomial $f_\psi(q)$ in (1.2) is (by definition) the *fake-degree* polynomial for ψ.

Example 1.31 (coinvariant algebra for $\mathfrak{S}_3$) What does the coinvariant algebra for $G = \mathfrak{S}_3 \subset \mathrm{GL}_3(\mathbb{C})$ look like?

We have seen that $\mathrm{Sym}(V) = \mathbb{C}[x_1, x_2, x_3]$ and $\mathrm{Sym}(V)^G = \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3} = \mathbb{C}[e_1, e_2, e_3]$. Then the coinvariant algebra is

$$\begin{aligned}\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f}) &= \mathbb{C}[x_1, x_2, x_3]/(e_1, e_2, e_3) \\ &\cong \mathbb{C}[x_1, x_2]/\big(x_1^2 + x_1x_2 + x_2^2, x_1^2x_2 + x_1x_2^2\big).\end{aligned}$$

This quotient has the $\mathbb{C}$-basis given in Table 1.3 in various degrees. Note $G = \mathfrak{S}_3$ has

$$\rho_{\mathrm{ref}} = \mathbb{1} \oplus \rho_{\mathrm{ref}} \oplus \rho_{\mathrm{ref}} \oplus \mathrm{sgn}.$$

Table 1.3 agrees with our calculation that

$$\begin{aligned}f^{\mathbb{1}}(q) &= 1 = q^0, \\ f^{\rho_{\mathrm{ref}}}(q) &= q^1 + q^2, \\ f^{\rho_{\mathrm{sgn}}}(q) &= q^3.\end{aligned}$$

degree	0	1	2	3
$\mathbb{C}$ − basis	1	$x_1,\ x_2$	$x_1^2,\ x_1x_2$	$x_1^2x_2$
$\mathfrak{S}_3$-irreducible decomposition	$\mathbb{1}$	ρ_{ref}	ρ_{ref}	sgn

Table 1.3 $\mathbb{C}$-*basis and irreducible decomposition of the coinvariant algebra for* $\mathfrak{S}_3$.

Exercises

Exercise 1.3.1 Give a very short summary of the most important lessons of this lecture. Carry out small examples that illustrate the main definitions and results.

Exercise 1.3.2 Prove Lemma 1.27. Start by extending the field $\mathbb{C}(a_{ij})$ of rational functions to any algebraically closed field $K \supset \mathbb{C}(a_{ij})$, and extend V to K^n. Then one can triangularize A, i.e., one can choose an ordered K-basis $(x_1, x_2, \dots, x_n)$ for K^n so that the linear map $A\colon K^n \to K^n$ is given by an upper triangular matrix with diagonal entries $\lambda_1, \lambda_2, \dots, \lambda_n$.

(a) Show that the K-basis $\{x_1^{j_1} x_2^{j_2} \dots x_n^{j_n} : j_1 + j_2 + \cdots + j_n = d\}$ for $\mathrm{Sym}^d V$ can be ordered in such a way that $\mathrm{Sym}^d A$ acts triangularly.

(b) Explain way (a) implies that the action of $\mathrm{Sym}^d A$ on $\mathrm{Sym}^d V$ has trace $\sum_{j_1+j_2+\cdots+j_n=d} \lambda_1^{j_1} \lambda_2^{j_2} \cdots \lambda_n^{j_n}$.

(c) Prove Lemma 1.27.

(d) Deduce Proposition 1.24.

(e) Deduce Molien's Theorem from Proposition 1.24.

Exercise 1.3.3

(a) Prove that the permutation representation $\rho_{\mathrm{perm}}\colon \mathfrak{S}_n \hookrightarrow \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$ has the property that for any $\sigma \in \mathfrak{S}_n$,

$$\det\bigl(1_V - q \cdot \rho_{\mathrm{perm}}(\sigma)\bigr) = \prod_{\substack{\text{cycles } c\\ \text{of } \sigma}} \bigl(1 - q^{|c|}\bigr).$$

(b) Prove that for any irreducible character χ_λ of $\mathfrak{S}_n$, one has

$$\sum_{d \geq 0} \langle \chi_{\mathbb{C}[x_1,\dots,x_n]_d}, \chi_\lambda \rangle_{\mathfrak{S}_n} \cdot q^d = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \overline{\chi_\lambda(\sigma)} P_\sigma(1, q, q^2, \dots),$$

where

$$P_\sigma(x_1, x_2, \dots) := \prod_{\substack{\text{cycles } c\\ \text{of } \sigma}} P_{|C|}(x_1, x_2, \dots),$$

and $P_d(x_1, x_2, \dots) = x_1^d + x_2^d + \cdots$ is the d-th power sum symmetric function.

Remark 1.32 For those familiar with $\mathfrak{S}_n$-representations and the relation to symmetric functions, along with *principal specializations* of *Schur functions* $s_\lambda(x_1, x_2, \dots)$ as in [9, §7.18, 7.21], the right side in (b) equals

$$s_\lambda(1, q, q^2, \dots) = \frac{1}{(1-q)(1-q^2)\dots(1-q^n)} f^\lambda(q),$$

with

$$f^\lambda(q) = q^{b(\lambda)} \frac{[n]!_q}{\prod_x [h(x)]_q} = \sum_Q q^{\mathrm{maj}(Q)}, \tag{1.3}$$

where the product is taken over the cells x of the Ferrers diagram of λ, and the sum over the standard Young tableaux Q of shape λ.

See [9, Corollaries 7.21.3, 7.21.5] for the undefined terms here!

Thus we have two interesting expressions for the fake degree polynomials $f^{\psi}(q)$ given by (1.3) in the case $G = \mathfrak{S}_n$.

Exercise 1.3.4 Given a matrix $A = (a_{ij})_{i,j=1,\dots,n} \in \mathbb{C}^{n \times n}$ and nonnegative integers $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$, define $\mathrm{per}_{\boldsymbol{k}}(A)$ as follows: letting $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ be two sets of variables related by $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $\mathrm{per}_{\boldsymbol{k}}(A)$ is defined as the coefficient of $x_1^{k_1} \cdots x_n^{k_n}$ in $y_1^{k_1} \dots y_n^{k_n}$.

(a) Check that for a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one has $\mathrm{per}_{(1,1)}(A) = ad + bc$.

(b) Prove in general that $\mathrm{per}_{(1,1,\dots,1)}(A) = \sum_{\sigma \in \mathfrak{S}_n} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}$, the *permanent* of A.

(c) Deduce from Lemma 1.27 *MacMahon's Master Theorem (1916)*:

$$\sum_{\boldsymbol{k} \in \mathbb{N}^n} \mathrm{per}_{\boldsymbol{k}}(A) t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n} = \frac{1}{\det(I_n - TA)},$$

where T is the diagonal matrix $(t_1, \dots, t_n)$.

Exercise 1.3.5

(a) Fix $d \in \{1, 2, 3, \dots\}$. Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^d = 0$ if n is odd.

(b) When $d = 1$, show one still has $\sum_{k=0}^{n} (-1)^k \binom{n}{k}^1 = 0$ if n is even and $n \geq 2$.

(c) When $d = 2$, show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = (-1)^m \binom{2m}{m} \quad \text{if } n = 2m.$$

Hint: Interpret the left side as

$$\sum_{\substack{(A,B):\ A,B \subset [n] \\ |A|+|B|=n}} (-1)^{|A|}$$

and cancel it down to the terms where $A = B$.

(d) When $d = 3$, *Dixon's identity* says that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = (-1)^m \binom{3m}{m, m, m} \quad \text{if } n = 2m.$$

Deduce this from MacMahon's Master Theorem with $A = \left(\begin{smallmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{smallmatrix}\right)$.

Show that the left side is $\mathrm{per}_{(n,n,n)}(A)$, i.e., the coefficient of $x_1^n x_2^n x_3^n$ in

$(x_2 - x_3)^{2n}(x_3 - x_1)^{2n}(x_1 - x_2)^{2n}$, while the right side is the coefficient of $t_1^n t_2^n t_3^n$ in $1/\det(I_3 - TA) = 1/(1 + t_1t_2 + t_2t_3 + t_1t_3)$.

1.4 Lecture 4: Cyclic sieving phenomena and Springer's theorem

The coinvariant algebra is helpful in combinatorics, since it can give us a grading and a Hilbert series q-count where we had none before. These q-counts often count more things via their evaluations at other values of q, not just $q = 1$.

As motivation, recall this result from our first section.

Theorem 1.33 (de Bruijn 1959) *For a permutation group $G \subseteq \mathfrak{S}_n$ consider its orbits $\mathcal{O} = \{S_1, \dots, S_k\}$ when G acts on the Boolean algebra $2^{[n]}$ and the $\mathbb{Z}/2\mathbb{Z}$-action via complementation sending $c\colon \mathcal{O} \mapsto c(\mathcal{O}) = \big\{[n]\backslash S_1, \dots, [n]\backslash S_t\big\}$. Then the poset of all G-orbits $X := 2^{[n]}/G$ and its rank-generating function $X(q) = r_0 + r_1q + r_2q^2 + \cdots + r_nq^n = \sum_{k=0}^n q^k \cdot \big|\binom{[n]}{k}/G\big|$ have the property that*

$$r_0 - r_1 + r_2 + \cdots \pm r_n = \#\ \textit{self-complementary } G\textit{-orbits},$$

i.e., $\big[X(q)\big]_{q=-1} = \big|\big\{x \in X : c(x) = x\big\}\big|$.

This is an example of what Stembridge (1994) called a "$q = -1$ phenomenon" in [10]: A set X with an action of $\mathbb{Z}/2\mathbb{Z} = \langle c \rangle$ and a polynomial $X(q)$ such that $X(1) = |X|$ and $X(-1) = \big|\big\{x \in X : c(x) = x\big\}\big|$. More generally, one can consider sets X with actions of cyclic groups $\mathbb{Z}/m\mathbb{Z} = C = \langle c \rangle = \{e, c, c^2, \dots, c^{m-1}\}$ for m larger than 2. The following generalization was introduced in [5].

Definition 1.34 (Reiner-Stanton-White 2004) Say that a set X with action of a cyclic group $C = \langle c \rangle \cong \mathbb{Z}/m\mathbb{Z}$ and a polynomial $X(q)$ exhibit a *cyclic sieving phenomenon* (CSP) if for every c^d in C one has

$$\big[X(q)\big]_{q=\zeta^d} = \big|\big\{x \in X : c^d(x) = x\big\}\big|,$$

where $\zeta := e^{2\pi i/m}$.

Example 1.35 Let $X = \binom{[n]}{k}$ the k-element subsets of $[n]$ with the cyclic group $C = \mathbb{Z}/n\mathbb{Z} = \langle c \rangle$ acting on it. Let $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]!_q}{[k]!_q[n-k]!_q}$ the q-binomial coefficient, where $[n]! = [n]_q[n-1]_q \cdots [1]_q$ and $[m]_q = (q^m - 1)/(q - 1)$. Recall from Exercise 1.1.5 that the q-binomial coefficient is the rank generating function for $2^{[k\ell]}/G$ where $G = \mathfrak{S}_k[\mathfrak{S}_\ell]$ and $n = k + \ell$.

Theorem 1.36 (Reiner-Stanton-White 2004) *The set $X = \binom{[n]}{k}$ and the polynomial $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ exhibit a CSP.*

See Exercise 1.4.3 for one of the proofs.

Example 1.37 For $n = 4$ and $k = 2$ we have $X = \binom{[4]}{2}$ and $C = \mathbb{Z}/4\mathbb{Z} = \langle c\rangle$ where $c = (1, 2, 3, 4)$. When c acts on X we obtain the following orbits $\{1, 2\} \xrightarrow{c} \{2, 3\} \xrightarrow{c} \{3, 4\} \xrightarrow{c} \{1, 4\}$ and $\{1, 3\} \xrightarrow{c} \{2, 4\}$. Now

$$\begin{aligned} X(q) = \begin{bmatrix}4\\2\end{bmatrix}_q &= \frac{[4]!_q}{[2]!_q[2]!_q} = \frac{[4]_q[3]_q}{[2]_q[1]_q} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{1+q} \\ &= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4. \end{aligned}$$

Since $m = 4$ then $\zeta = e^{2\pi i/4} = i$.

- Taking $q = \zeta^0$ gives $1 + 1 + 2 + 1 + 1 = 6 = |X| = |X^e|$.
- Taking $q = \zeta^2 = -1$ gives $1 - 1 + 2 - 1 + 1 = 2 = \big|\{\{1, 3\}, \{2, 4\}\}\big| = |X^{c^2}|$.
- Finally, $q = \zeta = i$ or $q = \zeta^3 = -i$ gives $1 + i - 2 - i + 1 = 0 = \big|X^{c^1}\big| = \big|X^{c^3}\big|$.

The previous example comes from a much more general statement about *reflection groups* due to Springer [7], and an enhanced version of the Shephard-Todd/Chevalley isomorphism between the coinvariant algebra and the regular representation.

Theorem 1.38 (Springer 1972) *Given a finite reflection group $G \subset \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$, say that $c \in G$ is a* regular element *if it has an* eigenvector *$v \in V$ (so $c(v) = \zeta \cdot v$) lying on no reflecting hyperplane. Then if we consider the cyclic subgroup $C = \langle c\rangle = \{e, c, c^2, \ldots, c^{m-1}\} \subset G$, one has an isomorphism of $G \times C$-representations*

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \cong \rho_{reg}.$$

On the coinvariant algebra, the group G acts by linear substitutions *and C acts by* scalar substitution *$c(x_i) = \zeta x_i$ for all i. On the regular representation, G acts by* left translation *$g \colon h \mapsto gh$, and C acts by* right translation *$c^d \colon h \mapsto hc^d$.*

Remark 1.39 Equivalently (see Exercise 1.4.4), for any G-representation ρ one has

$$\chi_\rho(c) = [f^\rho(q)]_{q=\zeta},$$

where $f^\rho(q)$ is the *fake degree polynomial* for ρ.

Example 1.40 (regular elements in the symmetric group) One can consider $G = \mathfrak{S}_n$ as a reflection group acting on $V = \mathbb{C}^n$, generated by the *transpositions* (i, j); note that (i, j) acts as a reflection on V with reflecting hyperplane $x_i = x_j$. Then inside $G = \mathfrak{S}_n$, an n-cycle $(1, 2, \ldots, n)$ is *regular* (in Springer's sense) because when c acts on $V = \mathbb{C}^n$, it has an eigenvector

$$\begin{aligned} v &= \begin{pmatrix}1 & \zeta & \zeta^2 & \cdots & \zeta^{n-2} & \zeta^{n-1}\end{pmatrix}^T \\ c(v) &= \begin{pmatrix}\zeta & \zeta^2 & \zeta^3 & \cdots & \zeta^{n-1} & 1\end{pmatrix}^T = \zeta v, \end{aligned}$$

where $\zeta = e^{2\pi i/n}$. This eigenvector v lies on no reflecting hyperplane $x_i = x_j$ since its coordinates are distinct. Similarly, an $(n-1)$-cycle $(1, 2, \ldots, n-1)(n) \in \mathfrak{S}_n$ is also a regular element in Springer's sense, since it has an eigenvector

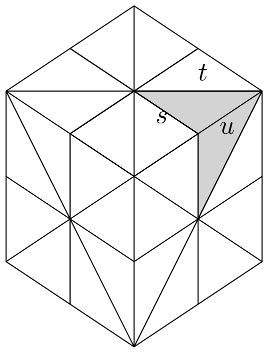
$$\begin{aligned} v &= \begin{pmatrix} 1 & \omega & \omega^2 & \cdots & \omega^{n-3} & \omega^{n-2} & 0 \end{pmatrix}^T \\ c(v) &= \begin{pmatrix} \omega & \omega^2 & \omega^3 & \cdots & \omega^{n-2} & 1 & 0 \end{pmatrix}^T = \omega v \end{aligned}$$

where $\omega = e^{2\pi i/(n-1)}$. Exercise 1.4.5 asks you to check that *powers* of n-cycles and $(n-1)$-cycles are the *only* regular elements in Springer's sense in $\mathfrak{S}_n$.

Example 1.41 (regular polytopes and Coxeter elements) One can show that the group W of symmetries of any *regular polytope* is generated by reflections. Furthermore the reflecting hyperplanes turn out to dissect the boundary faces of the polytope into its *barycentric subdivision*, giving a simplicial complex called the *Coxeter complex.* The group W has a *Coxeter presentation*

$$W = \langle s, t, u \colon s^2 = t^2 = u^2 = e, (st)^{m_{st}} = e, (su)^{m_{su}} = e, (tu)^{m_{tu}} = e \rangle$$

in which $S = \{s, t, u\}$ are the reflections through the hyperplanes spanned by the walls of any particular choice of a *fundamental chamber* C_0 (shaded below). Any element $c = stu$ which is the product of these Coxeter generators in S in any order is called a *Coxeter element.* It turns out that all such Coxeter elements c are conjugate within W, and all are regular elements in Springer's sense.



Example 1.42 (longest elements in real reflection groups) Every finite *real* reflection group W inside $\mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{R})$ contains an element called the *longest element* w_0, with various properties that characterize it uniquely, e.g., it is the unique element carrying the fundamental chamber C_0 to its antipodal chamber $-C_0$. This w_0 is always a regular element in Springer's sense.

For example, in $\mathfrak{S}_n$ the longest element w_0 is the *reversing permutation* $(1, n)(2, n-1)(3, n-2) \cdots$. One can check this w_0 is a power of an n-cycle when n is even, and a power of an $(n-1)$-cycle when n is odd, consistent with Exercise 1.4.5.

Springer's Theorem 1.38 leads to the following general CSP.

Theorem 1.43 (Reiner-Stanton-White 2004) *When a finite reflection group $G \subset \mathrm{GL}_n(\mathbb{C})$ acts* transitively *(with only one orbit) on a set X ($\cong G/H$ for some group H) and $c \in G$ is any regular element, say of order m, then one has a CSP for the action $C = \langle c \rangle \cong \mathbb{Z}/m\mathbb{Z}$ on X, with the polynomial*

$$X(q) := \frac{\mathrm{Hilb}\big(\mathbb{C}[x_1,\ldots,x_n]^H, q\big)}{\mathrm{Hilb}\big(\mathbb{C}[x_1,\ldots,x_n]^G, q\big)} = \prod_{i=1}^{n}\big(1-q^{d_i}\big)\cdot \mathrm{Hilb}\big(\mathbb{C}[x]^H, q\big). \tag{1.4}$$

In other words

$$\big[X(q)\big]_{q=\zeta^d} = \big|\big\{x \in X \colon c^d(x) = x\big\}\big| = \big|\big\{\text{cosets } gH \colon c^d gH = gH\big\}\big|.$$

The second equality in (1.4) follows since $\mathbb{C}[\boldsymbol{x}]^G = \mathbb{C}[f_1,\ldots,f_n]$ where $\deg(f_i) = d_i$.

Why does Theorem 1.43 generalize the example above?

Recall there that $X = \binom{[n]}{k} = G/H$ where $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$, because $G = \mathfrak{S}_n$ acts transitively on the k-subsets and $H = \mathfrak{S}_{\{1,2,\ldots,k\}} \times \mathfrak{S}_{\{k+1,k+2,\ldots,n\}}$ is the stabilizer of a typical k-subset $\{1,2,\ldots,k\}$.

We saw in discussing regular elements in Example 1.40 that inside $\mathfrak{S}_n$ that the n-cycle $c = (1,2,\ldots,n)$ is a regular element, of order n. Hence Theorem 1.43 implies that one should have a CSP for $X = \binom{[n]}{k} = G/H$ with $C = \langle c \rangle$ where $c = (1,2,\ldots,n)$ with the polynomial

$$X(q) = \frac{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^H, q\big)}{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big)}.$$

We know $\mathbb{C}[\boldsymbol{x}]^G = \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1,e_2,\ldots,e_n]$, so

$$\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^n)}. \tag{1.5}$$

On the other hand,

$$\begin{aligned}\mathbb{C}[\boldsymbol{x}]^H =& \mathbb{C}[x_1,\ldots,x_n]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} \\ =& \mathbb{C}\left[e_1(x_1,\ldots,x_k),\ldots,e_k(x_1,\ldots,x_k),\right. \\ & \left. e_1(x_{k+1},\ldots,x_n),\ldots,e_{n-k}(x_{k+1},\ldots,x_n)\right].\end{aligned}$$

Therefore

$$\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^H, q\big) = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)\cdot(1-q)(1-q^2)\cdots(1-q^{n-k})}.$$

Combining this with (1.5) yields

$$X(q) = \frac{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^H, q\big)}{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big)} = \frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{i=1}^{n-k}(1-q^i)} = \begin{bmatrix} n \\ k \end{bmatrix}_q,$$

as desired. The last equality follows via $[m]_q = (1-q^m)/(1-q)$.

The proof idea for deducing the CSP Theorem 1.43 from Springer's Theorem 1.38 is our favorite idea of *comparison of traces*.

Theorem 1.43; sketch of the proof Start with Springer's isomorphism of $G \times C$-representations

$$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \cong \rho_{\mathrm{reg}}.$$

Take H-fixed spaces on both sides, leaving an isomorphism of C-representations

$$\big(\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})\big)^H \cong (\rho_{\mathrm{reg}})^H.$$

Compare the trace of c^d on the two sides:

- The left side is a graded vector space where c^d acts via the scalar $(\zeta^d)^k$ in its k-th graded component. Also, one can show that the Hilbert series is

$$\mathrm{Hilb}\big(\big(\mathbb{C}[\boldsymbol{x}]/(\boldsymbol{f})\big)^H, q\big) = \frac{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^H, q\big)}{\mathrm{Hilb}\big(\mathbb{C}[\boldsymbol{x}]^G, q\big)} = X(q).$$

  Thus, c^d acts with trace $\big[X(q)\big]_{q=\zeta^d}$.
- The right side is the coset space $X = G/H$ with C-action via $c^d(gH) = c^d gH$. Then c^d acts with trace $\big|\{gH : c^d gH = gH\}\big| = \big|\{x \in X : c^d(x) = x\}\big|$.

$\square$

Exercises

Exercise 1.4.1 Give a very short summary of the most important lessons of this lecture. Carry out small examples that illustrate the main definitions and results.

Exercise 1.4.2 We want to understand the coinvariant algebra for the dihedral group $G = I_2(m)$, using the version of its reflection representation

$$\rho^{(1)} : G = I_2(m) \to \mathrm{GL}_2(\mathbb{C})$$

denoted $\rho^{(1)}$ in Exercise 1.2.3 (b), and shown equivalent to ρ_{ref} in Exercise 1.2.3 (c).

(a) Check the inclusion of rings $\mathbb{C}[x, y]^G \supset \mathbb{C}[xy, x^m + y^m]$. It can be shown that the inclusion is actually an equality, but let us assume this. Denote $f_1 = xy$, so that $\deg(f_1) = d_1 = 2$, and denote $f_2 = x^m + y^m$, so that $\deg(f_2) = d_2 = m$.

(b) Explain why the coinvariant algebra

$$\mathbb{C}[x, y]/(f_1, f_2) = \mathbb{C}[x, y]/(xy, x^m + y^m)$$

has the $\mathbb{C}$-basis in various degrees given in Table 1.4.

degree	0	1	2	$\cdots$	$m-1$	m
$\mathbb{C}$ – basis	1	$x,\, y$	$x^2,\, y^2$		$x^{m-1},\, y^{m-1}$	$x^m\ (= -y^m)$

Table 1.4 *Exercise 1.4.2 (b), $\mathbb{C}$-basis for the coinvariant algebra $\mathbb{C}[x,y]/(f_1, f_2)$.*

(c) Prove the following fake degree formulas $f^{\psi}(q)$:

$$\begin{aligned} f^{\mathbb{1}} &= 1, \\ f^{\det}(q) &= q^m \\ f^{\rho_s}(q) &= q^{m/2} = f^{\rho_t}(q) \text{ for } m \text{ even}, \\ f^{\rho^{(j)}}(q) &= q^j + q^{m-j} \text{ for } j = 1, 2, \ldots, \lfloor (m-1)/2 \rfloor. \end{aligned}$$

(d) Check that the answers in (c) are consistent for $m = 3$ with our previous calculations of $f^{\psi}(q)$ for $\mathfrak{S}_3 = I_2(3)$.

Exercise 1.4.3 Let ζ be a primitive d-th root of unity, such as $\zeta = e^{2\pi i/d}$.

(a) Show that for positive integers a, b having $a \equiv b \mod d$, one has

$$\lim_{q \to \zeta} \frac{[a]_q}{[b]_q} = \begin{cases} a/b & \text{if } a \equiv b \equiv 0 \mod d \\ 1 & \text{if } a \equiv b \not\equiv 0 \mod d. \end{cases}$$

(b) We want to understand how a general q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ behaves when one sets $q = \zeta$. Uniquely express

$$\begin{aligned} n &= n'd + n'' \\ k &= k'd + k'' \end{aligned}$$

with $n', n'', k', k'' \in \mathbb{Z}$ and $0 \leq n'', k'' \leq d-1$. In other words, let n', k' be the quotients and n'', k'' be the remainders when dividing n, k by d. Prove

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\zeta} = \binom{n'}{k'} \cdot \begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta}.$$

This implies one only need understand $\begin{bmatrix} n'' \\ k'' \end{bmatrix}_{q=\zeta}$ for $0 \leq k'',\ n'' \leq d-1$.

(c) Use part (b) to prove the CSP result for the action $X = \left(\begin{smallmatrix} [n] \\ k \end{smallmatrix} \right) \circlearrowleft C = \langle (1, 2, \ldots, n) \rangle$ and $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$ via direct evaluation of $\left[X(q) \right]_{q=\zeta^{\ell}}$, and direct enumeration of $\left| \{ x \in X : c^{\ell}(x) = x \} \right|$.

Exercise 1.4.4 Prove that the two statements in Springer's Theorem are equivalent: the isomorphism of $G \times C$-representations versus $\chi_{\rho}(c) = \left[f^{\rho}(q) \right]_{q=\zeta}$.

Exercise 1.4.5 Prove that in $\mathfrak{S}_n$ the only regular elements are the n-cycles, the $(n-1)$-cycles, and their powers.

References

[1] Chevalley, C. 1955. Invariants of finite groups generated by reflections. *Amer. J. Math.*, **77**, 778–782.

[2] de Bruijn, N. 1959. Generalization of Pólya's fundamental theorem in enumerative combinatorial analysis. *Nederl. Akad. Wetensch. Proc. Ser. A 62 = Indag. Math.*, **21**, 59–69.

[3] Pólya, G., and Read, R. 1987. *Combinatorial enumeration of groups, graphs, and chemical compounds.* Springer-Verlag, New York. Pólya's contribution translated from the German by Dorothee Aeppli.

[4] Redfield, J. 1927. The Theory of Group-Reduced Distributions. *Amer. J. Math.*, **49**(3), 433–455.

[5] Reiner, V., Stanton, D., and White, D. 2004. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, **108**(1), 17–50.

[6] Shephard, G., and Todd, J. 1954. Finite unitary reflection groups. *Canad. J. Math.*, **6**, 274–304.

[7] Springer, T. 1974. Regular elements of finite reflection groups. *Invent. Math.*, **25**, 159–198.

[8] Stanley, R. 1982. Some aspects of groups acting on finite posets. *J. Combin. Theory Ser. A*, **32**(2), 132–161.

[9] Stanley, R. 1999. *Enumerative combinatorics. Vol. 2.* Cambridge Studies in Advanced Mathematics, vol. 62. Cambridge University Press, Cambridge.

[10] Stembridge, J. 1994. Some hidden relations involving the ten symmetry classes of plane partitions. *J. Combin. Theory Ser. A*, **68**(2), 372–409.