q-counting and invariant theory

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Lecture

1: Monday — Invitation to q-counts & representation theory
- quotients of Boolean algebras

2: Tuesday — Representation theory review & reflection groups

3: Thursday — Molien's Theorem & coinvariant algebras

4: Thursday — Cyclic Sieving Phenomena (CSP) & Springer's Theorem

see ECCO 2018 lecture notes

5: Friday — More CSP's & the deformation idea

Lecture 1:

Invitation to q-counts
& representation theory
- quotients of Boolean algebras

DEF'N: Boolean algebra

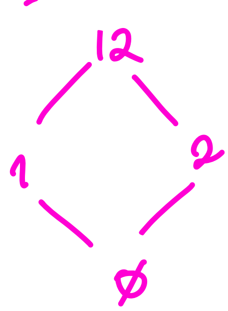
$$2^{[n]} := \{\text{all subsets of } [n] := \{1,2,\ldots,n\}\}$$

thought of as a poset := partially ordered set

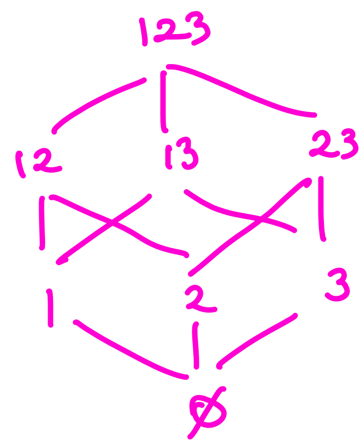
via $A \leq B$ if $A \subseteq B$

EXAMPLES:

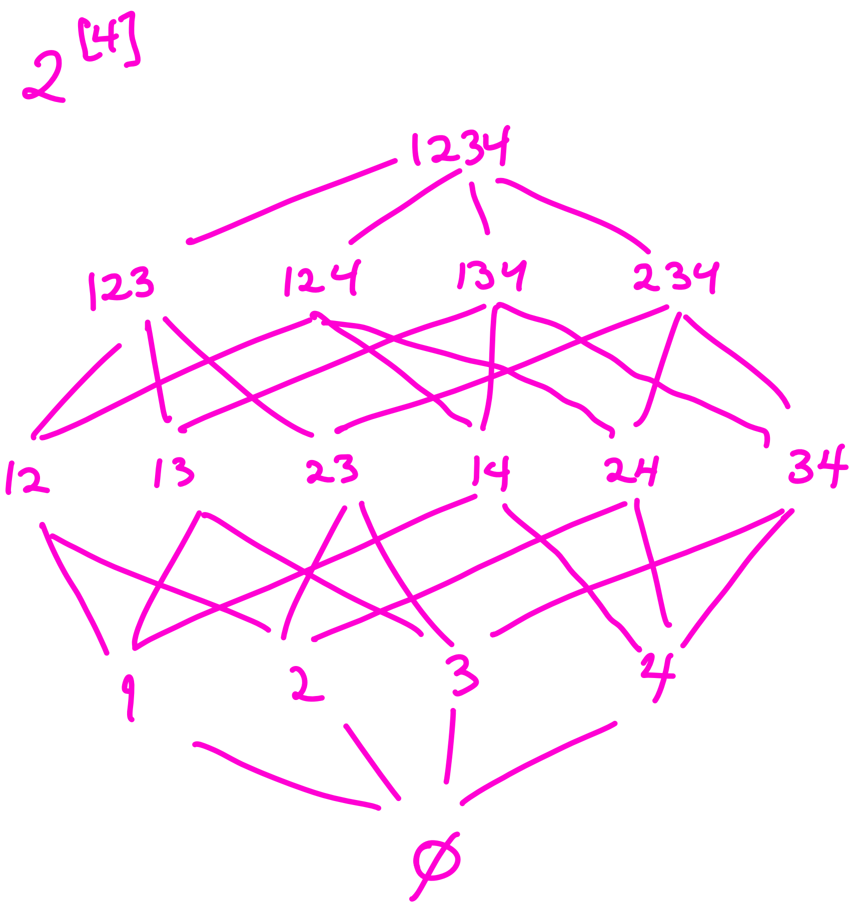
$2^{[2]}$



$2^{[3]}$



$2^{[n]}$ is a ranked poset,
whose rank numbers $r_0, r_1, \ldots, r_n$
$$\begin{array}{ccc} \| & \| & \| \\ \binom{n}{0} & \binom{n}{1} & \binom{n}{n} \end{array}$$
have many nice properties...



rank sizes:

$r_4 = 1 = \binom{4}{4}$

$r_3 = 4 = \binom{4}{3}$

$r_2 = 6 = \binom{4}{2}$

$r_1 = 4 = \binom{4}{1}$

$r_0 = 1 = \binom{4}{0}$

PROPERTIES:

- Symmetry $\binom{n}{k} = \binom{n}{n-k}$

- Alternating sum
$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots \pm \binom{n}{n} = 0$$

- Rank generating function
$$\binom{n}{0} + \binom{n}{1}q^1 + \binom{n}{2}q^2 + \dots = (1+q)^n$$

- Unimodality
$$\binom{n}{0} \leq \binom{n}{1} \leq \dots \leq \binom{n}{\lfloor n/2 \rfloor}$$

For any subgroup G of $\mathfrak{S}_n = \{\text{permutations of } [n]\}$

these properties will generalize to the

quotient poset

$$2^{[n]}/G := G\text{-orbits } \mathcal{O} \text{ of subsets of } [n]$$

ordered via

$$\mathcal{O}_1 \le \mathcal{O}_2 \text{ if } \exists\, S_1 \in \mathcal{O}_1,\ S_2 \in \mathcal{O}_2 \text{ with } S_1 \subseteq S_2$$

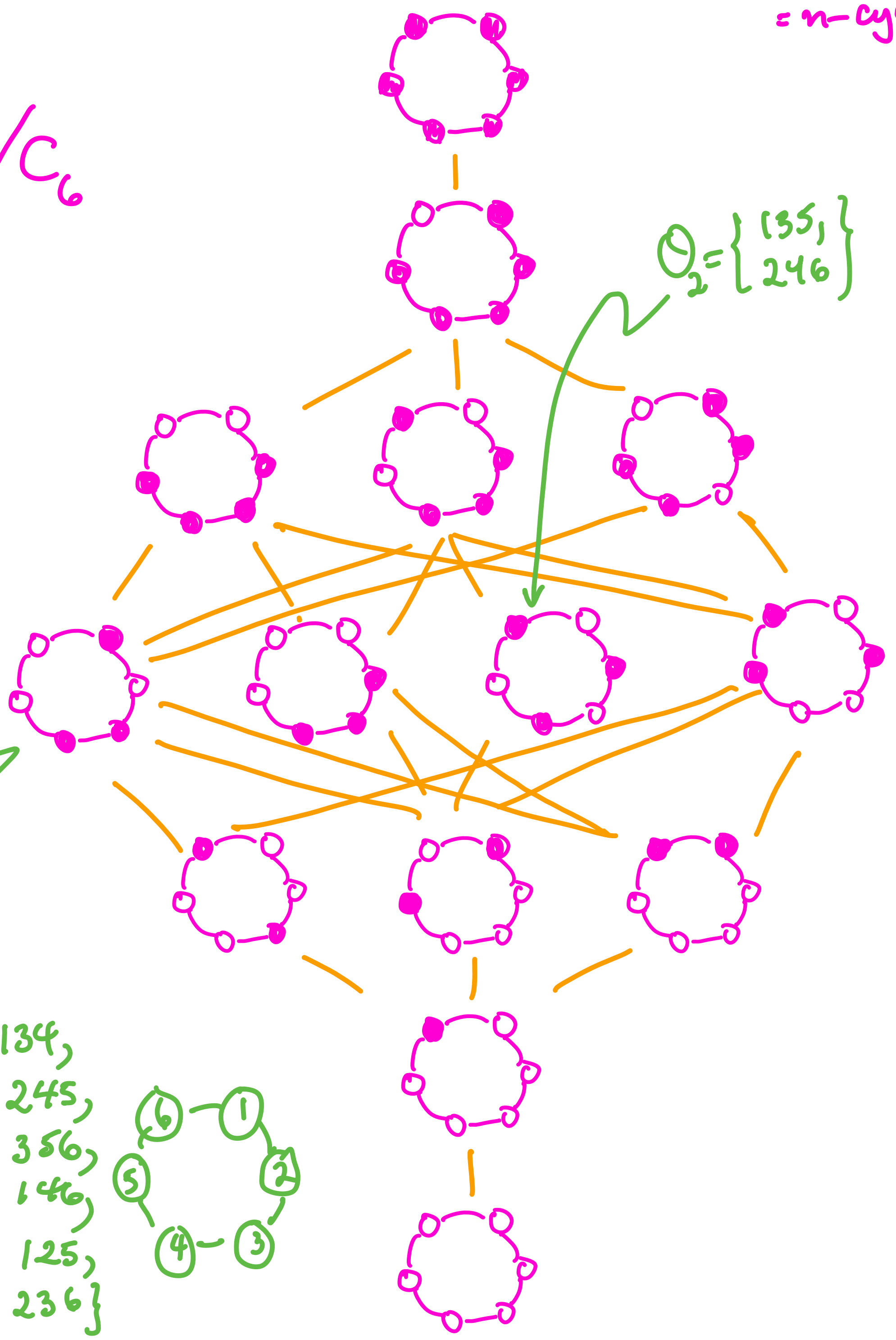
Several interesting combinatorial objects & posets are of the form

$$2^{[n]}/G$$

EXAMPLE Black/white necklaces

$= 2^{[n]}/C_n$ where C_n = cyclic group $\langle (1\ 2\ 3\ \cdots\ n) \rangle$ = n-cycle

$2^{[6]}/C_6$



$r_6 = 1$

$r_5 = 1$

$\mathcal{O}_2 = \{135, 246\}$

$r_4 = 3$

$r_3 = 4$

$r_2 = 3$

$r_1 = 1$

$r_0 = 1$

$\mathcal{O}_1 = \{134, 245, 356, 146, 125, 236\}$

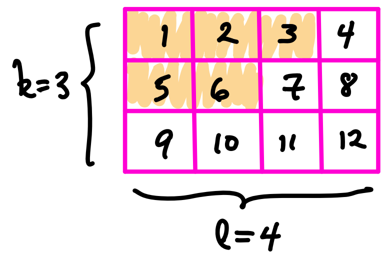
EXAMPLE Ferrers diagrams λ in a $k \times \ell$ rectangle

$$= 2^{[k\times \ell]} / \mathfrak{S}_k[\mathfrak{S}_\ell]$$

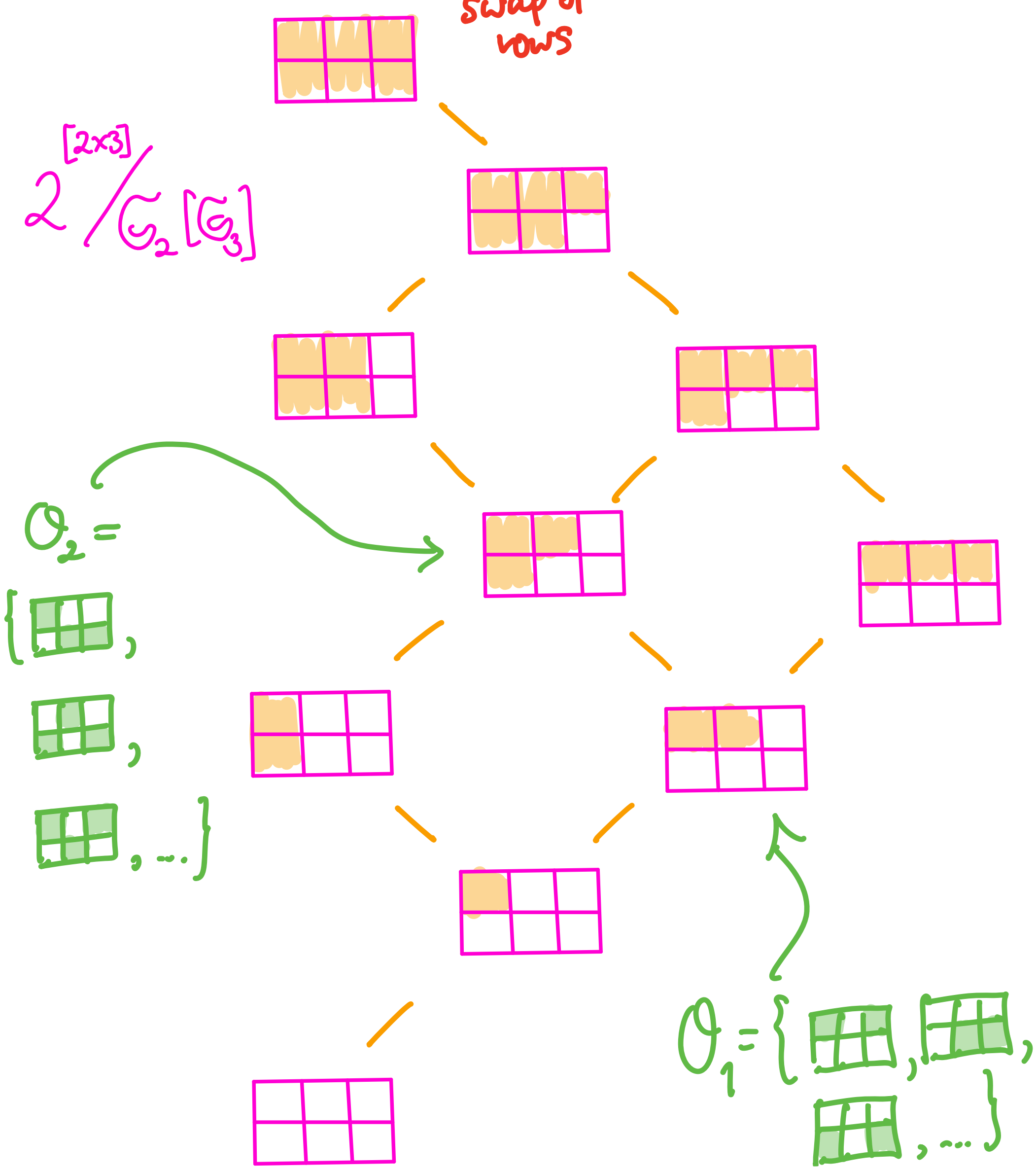
wreath product

wholesale swap of rows

swaps within rows



$2^{[2\times 3]} / \mathfrak{S}_2[\mathfrak{S}_3]$



$r_6 = 1$

$r_5 = 1$

$r_4 = 2$

$r_3 = 2$

$r_2 = 2$

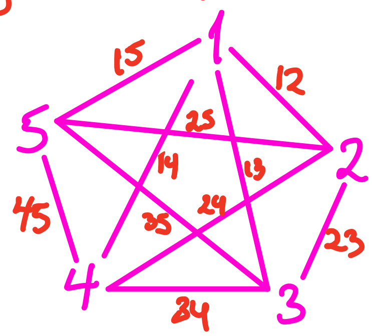
$r_1 = 1$

$r_0 = 1$

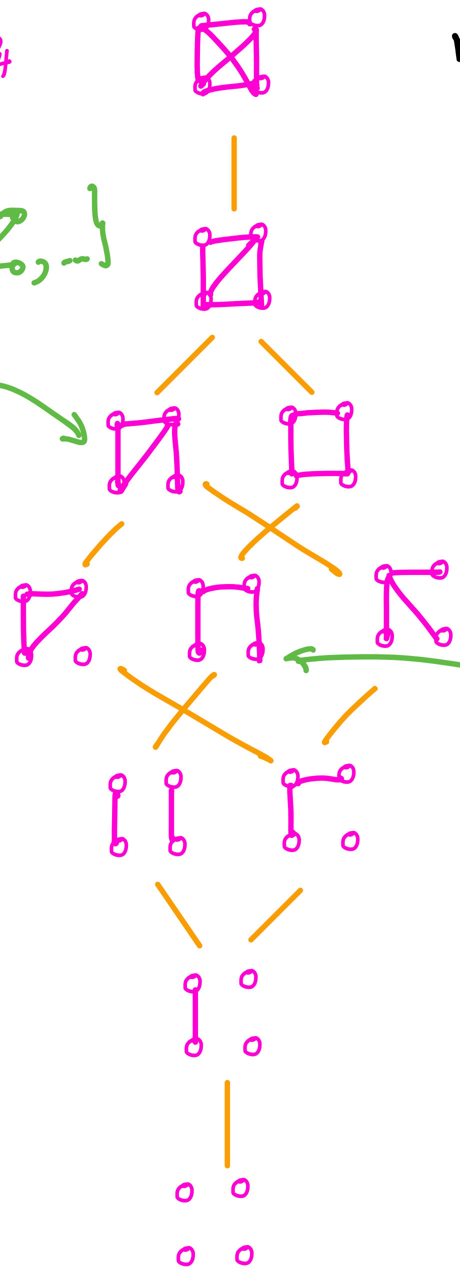
EXAMPLE Unlabeled graphs on n vertices

$$= 2^{\binom{[n]}{2}} / \mathfrak{S}_n$$

permuting the edges of K_n



$2^{\binom{[4]}{2}} / \mathfrak{S}_4$



$r_6 = 1$

$r_5 = 1$

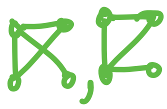
$r_4 = 2$

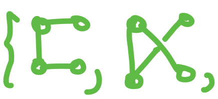
$r_3 = 3$

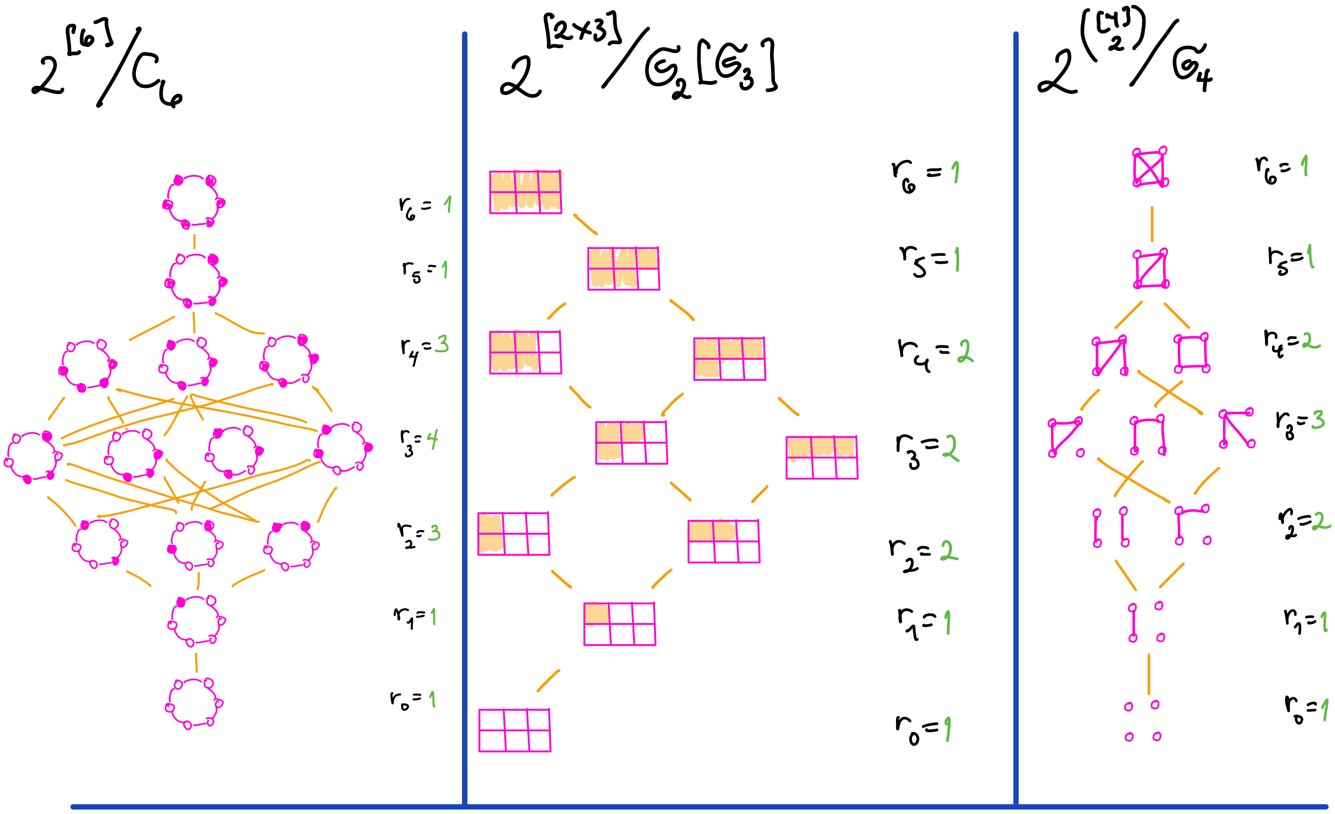
$r_2 = 2$

$r_1 = 1$

$r_0 = 1$

$O_1 = \{$  $, \ldots \}$

$O_2 = \{$  $, \ldots \}$



THEOREM For any subgroup G of $\mathfrak{S}_n$, the rank numbers $r_0, r_1, \ldots, r_n$ for $2^{[n]}/G$ satisfy:

- **Symmetry:** $r_k = r_{n-k}$
- **Alternating sum:** $r_0 - r_1 + r_2 - r_3 + \ldots \pm r_n = \#$ self-complementary orbits ☺
  de Bruijn 1959
- **Rank generating function:**
  Redfield 1927, Pólya 1937
  $$\sum_{k=0}^{n} r_k q^k = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} \left(1 + q^{|C|}\right)$$
- **Unimodality:** $r_0 \leq r_1 \leq \ldots \leq r_{\lfloor n/2 \rfloor}$
  Stanley 1982

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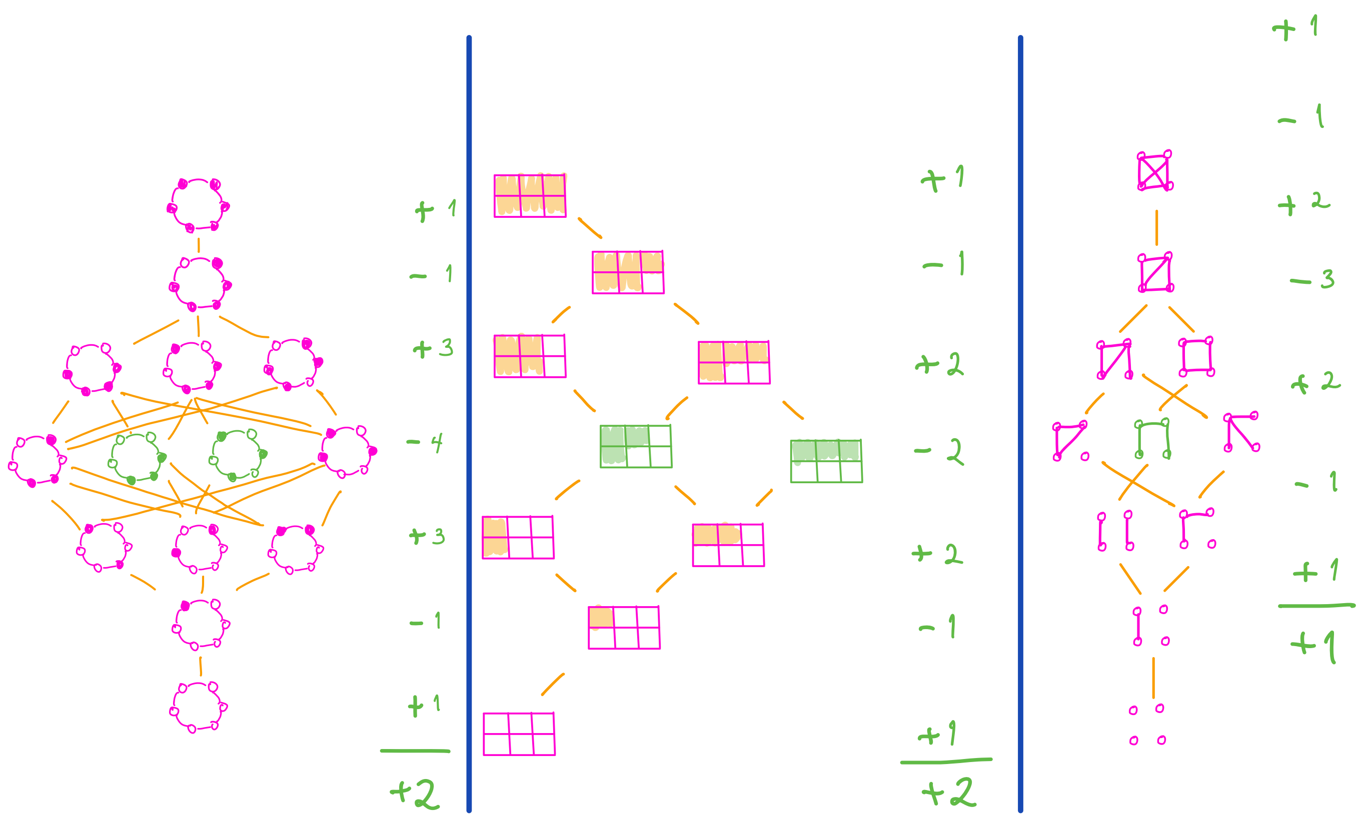
  Stanley 1982

- **Alternating sum:**
  de Bruijn 1959

$$r_0 - r_1 + r_2 - r_3 + \dots \pm r_n = \#\,\text{self-complementary orbits}$$ ☺



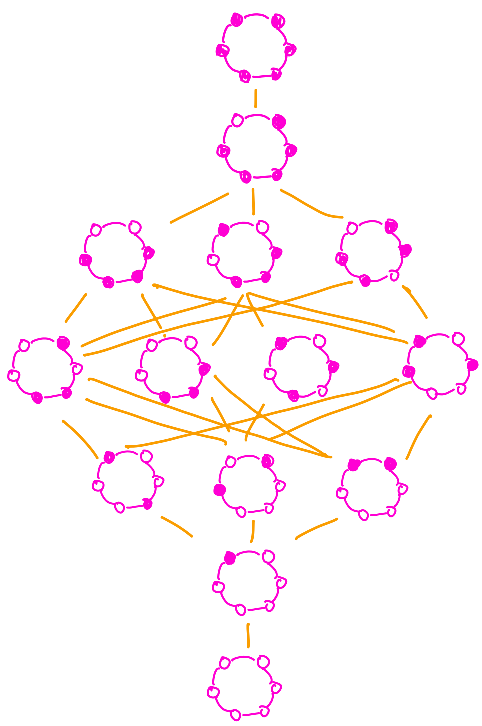
... and it does generalize $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots \pm \binom{n}{n} = 0$.

Rank generating function:

Redfield 1927, Pólya 1937

$$\sum_{k=0}^{n} r_k q^k = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} \left(1+q^{|C|}\right)$$

$2^{[6]}/C_6$



q^6

$+\ q^5$

$+\ 3q^4$

$+\ 4q^3$

$+\ 3q^2$

$+\ q$

1

$C_6 = \{ 1, (123456), (135)(246), (14)(25)(36), (153)(264), (165432) \}$

$$\frac{1}{|C_6|} \sum_{\sigma \in C_6} \prod_{\substack{\text{cycles} \\ c \text{ in } \sigma}} \left(1+q^{|c|}\right) = \frac{1}{6}\Big(\underbrace{(1+q)^6}_{1} + 2\underbrace{(1+q^6)}_{\substack{(123456)\\(165432)}} + 2\underbrace{(1+q^3)^2}_{\substack{(135)(246)\\(153)(264)}} + \underbrace{(1+q^2)^3}_{(14)(25)(36)} \Big)$$

$$= \frac{1}{6}\Big[\, 1 + 6q + 15q^2 + 20q^3 + 15q^4 + 6q^5 + q^6$$

$$+\ 2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +2q^6$$

$$+\ 2 \qquad\qquad\qquad 4q^3 \qquad\qquad\qquad\qquad 2q^6$$

$$+\ 1 \qquad +3q^2 \qquad\qquad\qquad +3q^4 \qquad\qquad +q^6 \Big]$$

$$= \quad 1 + q + 3q^2 + 4q^3 + 3q^4 + q^5 + q^6$$

(Re-usable!) Proof Ideas:

Linearize, and re-interpret

- cardinalities = dimensions

- generating functions or q-counts
  = graded dimensions
  or Hilbert series
  or graded traces

- prove equalities via isomorphisms,
  inequalities via
  linear injections / surjections

- identities can arise from equality of traces for conjugate elements h, ghg^{-1} acting in a representation

group homomorphism

$$G \xrightarrow{\rho} GL(V)$$

general linear group

$$\mathrm{Trace}(\rho(ghg^{-1})) = \mathrm{Trace}(\rho(g)\rho(h)\rho(g)^{-1})$$
$$= \mathrm{Trace}(\rho(h))$$

RECALL: $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$

$\Rightarrow$ $\mathrm{Trace}(PAP^{-1}) = \mathrm{Tr}(P^{-1} \cdot PA) = \mathrm{Tr}(A)$

Linearize...

let $V = \mathbb{C}^2$ with $\mathbb{C}$-basis $\{\overset{\text{black}}{b}, \overset{\text{white}}{w}\}$

Elements $T \in GL(V) \cong GL_2(\mathbb{C})$ 2×2 invertible matrices

act linearly on V.

EXAMPLES

$$t = \begin{array}{c|cc} & b & w \\ \hline b & 0 & 1 \\ w & 1 & 0 \end{array}$$

swaps
$t(b) = w$
$t(w) = b$

$$s = \begin{array}{c|cc} & b & w \\ \hline b & -1 & 0 \\ w & 0 & +1 \end{array}$$

scales
$s(b) = -b$
$s(w) = w$

(and in fact s, t are conjugate in $GL(V)$,
since t has eigenvalues $+1, -1$
eigenvectors $b+w$, $b-w$)

The n^{th} tensor power

$$T^n(V) := V^{\otimes n} := \underbrace{V \otimes V \otimes \ldots \otimes V}_{n \text{ factors}}$$

has actions of ...

- $GL(V)$ diagonally:

$$T(v_1 \otimes \ldots \otimes v_n) := T(v_1) \otimes \ldots \otimes T(v_n)$$

and then expand this multi-linearly

- $\mathfrak{S}_n$ positionally:

$$\sigma(v_1 \otimes \ldots \otimes v_n) := v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(n)}$$

... and the actions commute:

$$\sigma T(v_1 \otimes \ldots \otimes v_n) = T\sigma(v_1 \otimes \ldots \otimes v_n)$$

$$= T(v_{\sigma^{-1}(1)}) \otimes \ldots \otimes T(v_{\sigma^{-1}(n)})$$

Since $V = \mathbb{C}^2$ has $\mathbb{C}$-basis $\{b, w\}$,

$V^{\otimes n}$ has $\mathbb{C}$-basis $\{e_A\}_{A \subset [n]}$

of monomial tensors indexed by words in $\{b,w\}^n$
with $A :=$ positions where b occurs

EXAMPLE $n=4$

$$e_{\{2\}} = \overset{1}{w} \otimes \overset{2}{b} \otimes \overset{3}{w} \otimes \overset{4}{w} \leftrightarrow wbww$$

$$e_{\{1,4\}} = b \otimes w \otimes w \otimes b \leftrightarrow bwwb$$

Permutations $\sigma \in \mathfrak{S}_n$ also permute these basis elements:

$$\sigma(e_A) = e_{\sigma^{-1}(A)}$$

e.g. $(1\,2\,3)(\underbrace{bwwb}_{e_{\{1,4\}}}) = \underbrace{wwbb}_{e_{\{3,4\}}}$

$A = \{1,4\}$ $\qquad$ $\sigma^{-1}(A) = \{3,4\}$

Hence, for any subgroup G of $\mathfrak{S}_n$,

the G-fixed subspace

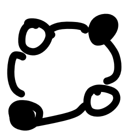
$$(V^{\otimes n})^G := \{ x \in V^{\otimes n} : g(x) = x \ \ \forall x \in G \}$$

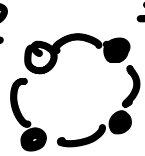
has $\mathbb{C}$-basis $\{ e_{\mathcal{O}} \}_{\mathcal{O} \in 2^{[n]}/G}$

indexed by G-orbits $\mathcal{O}$,

where $e_{\mathcal{O}} := \sum_{A \in \mathcal{O}} e_A$

EXAMPLE $\quad G = C_4$ inside $\mathfrak{S}_4$

e  $= wbwb + bwbw$

e  $= wbbb + bwbb + bbwb + bbbw$

both lie in $(V^{\otimes 4})^G$

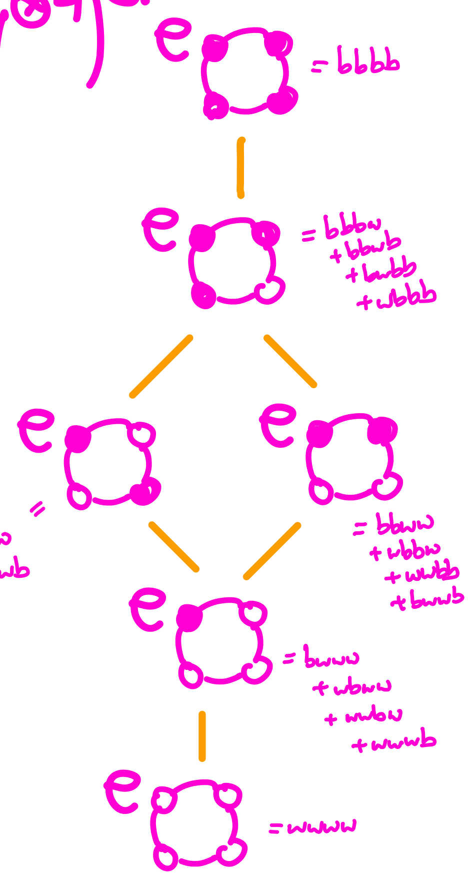
Both $V^{\otimes n}$, $(V^{\otimes n})^G$ are graded $\mathbb{C}$-vector spaces:

$$V^{\otimes n} = \bigoplus_{k=0}^{n} \underbrace{(V^{\otimes n})_k}_{\mathbb{C}\text{-span of } \{e_A\}_{A \in \binom{[n]}{k}}} \qquad (V^{\otimes n})^G = \bigoplus_{k=0}^{n} \underbrace{(V^{\otimes n})^G_k}_{\mathbb{C}\text{-span of } \{e_{\mathcal{O}}\}_{\mathcal{O} \in \binom{[n]}{k}/G}}$$

EXAMPLE $G = C_4$ in $\mathfrak{S}_4$

$V^{\otimes 4}$

$(V^{\otimes 4})^G$



Since the rank sizes $r_0, r_1, \ldots, r_n$ of the orbit poset $2^{[n]}/G$ can be re-interpreted as $r_k = \dim_{\mathbb{C}} (V^{\otimes n})^G_k = \# \binom{[n]}{k}/G$

one can now give a (silly) proof of ...

- **Symmetry:** $r_k = r_{n-k}$

proof: Recall $t = \begin{array}{c|cc} & b & w \\ \hline b & 0 & 1 \\ w & 1 & 0 \end{array} \in GL(V)$ swaps $b \xleftrightarrow{t} w$, and so it permutes the $\mathbb{C}$-basis $\{e_A\}_{A \subseteq [n]}$ for $V^{\otimes n}$

via $e_A \xleftrightarrow{t} e_{[n] \setminus A}$

e.g. $t(\underbrace{bwbww}_{e_{\{1,3\}}}) = \underbrace{wbwbb}_{e_{\{2,4,5\}}}$

Hence it gives a $\mathbb{C}$-linear isomorphism $(V^{\otimes n})_k \xrightarrow[\sim]{t} (V^{\otimes n})_{n-k}$.

But since $t \in GL(V)$ commutes with the action of $G \subseteq G_n$, this isomorphism t restricts to a $\mathbb{C}$-linear isomorphism

$$\underbrace{(V^{\otimes n})^G_k}_{\substack{\text{dimension} \\ r_k}} \xrightarrow[\sim]{t} \underbrace{(V^{\otimes n})^G_{n-k}}_{\substack{\text{dimension} \\ r_{n-k}}}$$

$\blacksquare$

On our way to less silly proofs, let's start by re-interpreting the rank generating function:

PROPOSITION: The matrix $s(q) := \begin{array}{cc} & \begin{array}{cc} b & w \end{array} \\ \begin{array}{c} b \\ w \end{array} & \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix} \end{array}$ in $GL(V)$ acts on $(V^{\otimes n})^G$ with trace $r_0 + r_1 q + r_2 q^2 + \dots + r_n q^n$.

proof: Since $s(q)(b) = q \cdot b$
$s(q)(w) = 1 \cdot w$,
$s(q)$ scales every $\mathbb{C}$-basis element e_A in $V^{\otimes n}$:
$$s(q)(e_A) = q^{|A|} \cdot e_A$$
e.g. $s(q)(bwbww) = qb \otimes w \otimes qb \otimes w \otimes w = q^2\, bwbww$

Hence $s(q)$ scales all of $(V^{\otimes n})_k$ by q^k including the subspace $(V^{\otimes n})^G_k$.

So its trace on $(V^{\otimes n})^G = \bigoplus_{k=0}^{n} (V^{\otimes n})^G_k$

is $\sum_{k=0}^{n} q^k \cdot \underbrace{\dim_{\mathbb{C}} (V^{\otimes n})^G_k}_{r_k} = \sum_{k=0}^{n} r_k q^k$ $\blacksquare$

PROPOSITION: The matrix $S(q) := \begin{array}{c|cc} & b & w \\ \hline b & q & 0 \\ w & 0 & 1 \end{array}$

acts on $(V^{\otimes n})^G$ with trace $r_0 + r_1 q + r_2 q^2 + \ldots + r_n q^n$.

COROLLARY: In particular, $s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = S(-1)$

acts on $(V^{\otimes n})^G$ with trace $r_0 - r_1 + r_2 - \ldots \pm r_n$.

This lets us prove ...

- Alternating sum: $r_0 - r_1 + r_2 - r_3 + \ldots \pm r_n = \#$ self-complementary orbits $\mathcal{O}$
  deBruijn 1959

proof: Since $t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is conjugate to $s = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

in $GL(V)$, it acts with same trace on $(V^{\otimes n})^G$,

so with trace $r_0 - r_1 + r_2 - \ldots \pm r_n$.

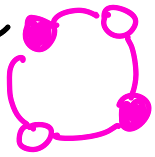
But since we saw it permutes $t(e_A) = e_{[n] \setminus A}$,

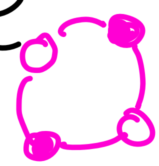
this means it also permutes the $\mathbb{C}$-basis

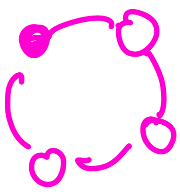
$\{e_{\mathcal{O}}\}_{\mathcal{O} \in 2^{[n]}/G}$ for $(V^{\otimes n})^G$:

$$t(e_{\mathcal{O}}) = \begin{cases} e_{\mathcal{O}} & \text{if } \mathcal{O} \text{ is self-complementary} \\ e_{\mathcal{O}'} & \text{if } \mathcal{O} \text{ is not self-complementary} \\ & \text{and } A \in \mathcal{O} \text{ has } [n] \setminus A \in \mathcal{O}' \end{cases}$$

EXAMPLE

$t\big(e$  $\big) = t(bwbw + wbwb)$

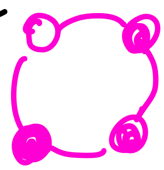
$= wbwb + bwbw = e$ 

$t\big(e$  $\big) = t(bwww + wbww$

$+ wwbw + wwwb)$

$= wbbb + bwbb$

$+ bbwb + bbbw$

$= e$ 

Hence the trace of t on $(V^{\otimes G})^n$ is also this number of t-fixed orbits e_θ, that is, the number which are self-complementary. ■

Let's sketch proofs for the last two properties leaving details for the EXERCISE SESSIONS

- **Rank generating function:**

Redfield 1927, Pólya 1937

$$\sum_{k=0}^{n} r_k q^k = \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ C \text{ of } \sigma}} (1+q^{|C|})$$

Proof sketch:

$$\sum_{k=0}^{n} r_k q^k = \sum_{k=0}^{n} \dim_{\mathbb{C}} \left(V^{\otimes n}\right)_k^G \cdot q^k$$

EXERCISE 1.1.2(d)
For a finite group rep. $G \xrightarrow{\rho} GL(\mathcal{U})$, $\dim_{\mathbb{C}} \mathcal{U}^G = \frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}(\rho(\sigma))$

$$= \sum_{k=0}^{n} \left(\frac{1}{|G|} \sum_{\sigma \in G} \mathrm{Trace}_{(V^{\otimes n})_k}(\sigma) \right) q^k$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \underbrace{\left(\sum_{k=0}^{n} q^k \cdot \mathrm{Trace}_{(V^{\otimes n})_k}(\sigma) \right)}_{\displaystyle \| \atop \displaystyle \prod_{\substack{\text{cycles} \\ c \text{ of } \sigma}} (1+q^{|C|})}$$

EXERCISE 1.1.3(c)
These two are equal

$$= \frac{1}{|G|} \sum_{\sigma \in G} \prod_{\substack{\text{cycles} \\ c \text{ of } \sigma}} (1+q^{|C|})$$

$\blacksquare$

- Unimodality: $r_0 \leq r_1 \leq \ldots \leq r_{\lfloor n/2 \rfloor}$
  Stanley 1982

Proof sketch: We want to show that

$$\underbrace{r_k}_{\dim_{\mathbb{C}} (V^{\otimes n})^G_k} \leq \underbrace{r_{k+1}}_{\dim_{\mathbb{C}} (V^{\otimes n})^G_{k+1}} \qquad \text{for } k < \frac{n}{2}$$

so look for a G-linear injection

$$(*) \qquad (V^{\otimes n})^G_k \hookrightarrow (V^{\otimes n})^G_{k+1} \quad \text{when } k < \frac{n}{2}.$$

It would be even better to have a $\mathbb{G}$-linear injection

$$(V^{\otimes n})_k \overset{U_k}{\hookrightarrow} (V^{\otimes n})_{k+1} \qquad \text{for } k < \frac{n}{2}$$

that commutes with the action of $\mathfrak{S}_n$ on $(V^{\otimes n})_k$, and hence with every subgroup G in $\mathfrak{S}_n$.

Then it would restrict to an injection as in $(*)$.

There is an obvious candidate for U_k, namely $(V^{\otimes n})_k \xrightarrow{U_k} (V^{\otimes n})_{k+1}$

given by $e_A \longmapsto \sum_{\substack{B \subseteq [n]: \\ |B|=k+1, \\ B \supseteq A}} e_B$

EXAMPLE $n=5$

$$\mathcal{U}_2(e_{\{1,3\}}) = e_{\{1,2,3\}} + e_{\{1,3,4\}} + e_{\{1,3,5\}}$$

i.e. $\mathcal{U}_2(bwbww) = bbbww + bwbbw + bwbwb$

EXERCISE 1.1.4(a) asks you to check U_k that commutes with the action of $\mathfrak{S}_n$.

EXERCISE 1.1.4 (b) - (f) takes you through a proof that U_k is injective for $k < n/2$, by showing the map $D_k(e_A) := \sum_{\substack{B \subseteq [n]: \\ |B| = k-1, \\ B \subset A}} e_B$

satisfies the commutation relation

$$D_{k+1} U_k - U_{k-1} D_k = (n-2k) \cdot Id_{\binom{[n]}{k}}$$

which leads to a proof that

$D_{k+1} U_k$ is positive definite for $k < n/2$,

$\Longrightarrow$ $D_{k+1} U_k$ is non-singular,

$\Longrightarrow$ U_k is injective. $\blacksquare$

REMARK 1: This commutation relation

$$D_{k+1} U_k - U_{k-1} D_k = (n-2k) \cdot \mathrm{Id}_{(V^{\otimes n})_k}$$

and injectivity of U_k are closely related to representations of $\mathfrak{sl}_2(\mathbb{C})$ on $V = \mathbb{C}^2$ and on $V^{\otimes n}$

and theory of crystal bases as in Prof. Schilling's lectures.

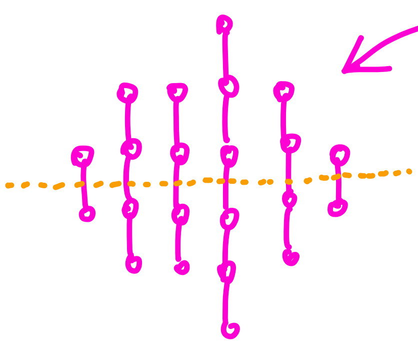
REMARK 2: One can extend the above theory to show (Stanley 1984) that the posets $2^{[n]}/G$ are all

Peck := $\begin{cases} \text{rank-symmetric} & \text{(we saw)} \\ \text{rank-unimodal} & \text{(we saw)} \\ \text{strongly Sperner} \end{cases}$

strongly Sperner := for all k, the max size $|A_1|+|A_2|+\dots+|A_k|$ of a union of k antichains in $2^{[n]}/G$ is the max of $r_{i_1}+\dots+r_{i_k}$

Hard!

(CONJECTURE) All posets $2^{[n]}/G$ have the stronger property of a symmetric chain decomposition



:= decomposition into disjoint saturated chains, each symmetric about the middle rank

THEOREM (Hersh & Schilling 2011)

$2^{[n]}/C_n$ cyclic group

does have an explicit symmetric chain decomposition

(inspired by the theory of crystal bases!)