q-counting and invariant theory

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Lecture

1: Monday — Invitation to q-counts
& representation theory
- quotients of Boolean algebras

2: Tuesday — Representation theory review
& reflection groups

3: Thursday — Molien's Theorem
& coinvariant algebras

TODAY

4: Thursday — Cyclic Sieving Phenomena (CSP)
& Springer's Theorem

see ECCO 2018 lecture notes

5: Friday — More CSP's
& the deformation idea

TODAY'S GOAL:

Discover how to add in a "q" into the regular representation $\mathbb{C}G$ for a finite reflection group G, by mucking around with graded traces on polynomial rings

$$\mathbb{C}[x_1, \ldots, x_n] = \mathrm{Sym}(V)$$

symmetric algebra

where $V = \mathbb{C}^n$ has $\mathbb{C}$-basis $x_1, \ldots, x_n$

Let's examine behavior of characters under multilinear constructions applied to group representations ...

1. Direct sum

We've noted (and used) that if

$\rho = \rho_1 \oplus \rho_2$ on $V = V_1 \oplus V_2$ so

$$\rho(g) = \begin{array}{c|c} & \begin{array}{cc} V_1 & V_2 \end{array} \\ \begin{array}{c} V_1 \\ V_2 \end{array} & \left[\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right] \end{array}$$

then $\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$

(since $\mathrm{Trace}\left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right] = \mathrm{Trace}(A_1) + \mathrm{Trace}(A_2)$)

2. Tensor product

Given G-representations $G \xrightarrow{\rho} GL(V)$

$G \xrightarrow{\rho'} GL(V')$

one can also define

$$G \xrightarrow{\rho\otimes\rho'} GL(V\otimes V')$$

where $(\rho\otimes\rho')(g)\,(v\otimes v') := \rho(g)(v) \otimes \rho'(g)(v')$

Not hard to check the matrix for $(\rho\otimes\rho')(g)$ is the tensor/Kronecker product of matrices $\rho(g)\otimes\rho'(g)$, after picking $\mathbb{C}$-bases $v_1, \ldots, v_n$ for V

$v_1', \ldots, v_m'$ for V'

and $\{v_i \otimes v_j'\}_{\substack{i=1,\ldots,n\\ j=1,\ldots,m}}$ for $V\otimes V'$

Recall tensor/Kronecker product of matrices $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{bmatrix}$ and B

is
$$A \otimes B := \left[\begin{array}{c|c|c} a_{11}B & a_{12}B & \cdots \\ \hline a_{21}B & a_{22}B & \\ \hline \vdots & & \ddots \end{array}\right]$$

with $\mathrm{Trace}(A \otimes B) = a_{11}\mathrm{Trace}(B) + a_{22}\mathrm{Trace}(B) + \ldots$
$$= \mathrm{Trace}(A) \cdot \mathrm{Trace}(B).$$

Hence
$$\begin{aligned} \chi_{\rho \otimes \rho'}(g) &= \mathrm{Trace}\big((\rho \otimes \rho')(g)\big) \\ &= \mathrm{Trace}\big(\rho(g) \otimes \rho'(g)\big) \\ &= \mathrm{Trace}(\rho(g)) \cdot \mathrm{Trace}(\rho'(g)) \\ &= \chi_\rho(g) \cdot \chi_{\rho'}(g) \end{aligned}$$

3. d^{th} tensor power

Given a G-representation $G \xrightarrow{\rho} GL(V)$

we saw $T^d(V) := V^{\otimes d} = \underbrace{V \otimes \dots \otimes V}_{d \text{ factors}}$

also has one where G acts diagonally:

$$G \xrightarrow{T^d(\rho)} GL(V^{\otimes d}) \text{ with}$$

$$T^d(\rho)(g)(v_1 \otimes v_2 \otimes \dots \otimes v_d)$$
$$:= \rho(g)(v_1) \otimes \dots \otimes \rho(g)(v_d)$$

Can check similarly that

$$\chi_{T^d(\rho)}(g) = \chi_\rho(g) \cdot \chi_\rho(g) \cdots \chi_\rho(g)$$
$$= \chi_\rho(g)^d$$

4. Tensor algebra

Consequently, the tensor algebra

$$T(V) := \bigoplus_{d=0}^{\infty} T^d(V)$$

$$= \underbrace{\mathbb{C}}_{T^0(V)} \oplus \underbrace{V}_{T^1(V)} \oplus \underbrace{V \otimes V}_{T^2(V)} \oplus \underbrace{V \otimes V \otimes V}_{T^3(V)} \oplus \cdots$$

has a G-representation $G \xrightarrow{T(\rho)} GL(T(V))$

with graded character/trace

$$\chi_{T(\rho)}(g\,;q) \overset{\text{DEF'N}}{:=} \sum_{d=0}^{\infty} q^d \cdot \chi_{T^d(\rho)}(g)$$

$$= \sum_{d=0}^{\infty} q^d \cdot \chi_\rho(g)^d$$

$$= \frac{1}{1 - q\,\chi_\rho(g)}$$

5. Symmetric algebra

Picking a $\mathbb{C}$-basis $x_1, \dots, x_n$ for V, one can view the (commutative) polynomial ring

$$\mathbb{C}[x_1, \dots, x_n] = \bigoplus_{d=0}^{\infty} \underbrace{\mathbb{C}[\underline{x}]_d}_{\text{homogeneous degree } d \text{ polynomials}}$$

as the symmetric algebra

$$\mathrm{Sym}(V) = \bigoplus_{d=0} \underbrace{\mathrm{Sym}^d(V)}_{d^{\text{th}} \text{ symmetric power}}$$

$$\left[\underset{\text{more formally}}{:=} T(V) \Big/ \mathrm{span}_{\mathbb{C}} \left\{ \begin{array}{l} v_1 \otimes \dots \otimes v_d \\ \quad - v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(d)} \\ \forall d \text{ and } \sigma \in \mathfrak{S}_d \end{array} \right\} \right]$$

call the image of $v_1 \otimes \dots \otimes v_n$ in this quotient $v_1 \cdot v_2 \cdot \ldots \cdot v_n$

Given a G-representation $G \xrightarrow{\rho} GL(V)$, one can define one on $\mathbb{C}[\underline{x}] \cong \mathrm{Sym}(V)$ and each $\mathbb{C}[\underline{x}]_d \cong \mathrm{Sym}^d(V)$

via diagonal action as usual:

$$\mathrm{Sym}^d(\rho)(g)(v_1 \cdot v_2 \cdot \ldots \cdot v_d)$$
$$:= \rho(g)(v_1) \cdot \rho(g)(v_2) \cdot \ldots \cdot \rho(g)(v_d)$$

$\underline{\underline{Q}}$: On $\mathrm{Sym}(V)$, can we compute the graded character/trace

$$\chi_{\mathrm{Sym}(\rho)}(g\,;q) := \sum_{d=0}^{\infty} q^d \cdot \chi_{\mathrm{Sym}^d(\rho)}(g) \,?$$

A: Yes, and it's interesting.

PROPOSITION: Exercise 1.3.2(d)

$$\chi_{\mathrm{Sym}(\rho)}(g;q) = \frac{1}{\det\left(I_V - q\cdot\rho(g)\right)}$$

COROLLARY (Molien 1897)

Given a finite group representation

$$G \xrightarrow{\rho} GL(V) \qquad (\text{with } V=\mathbb{C}^n)$$

for any other G-representation ψ, one has

$$\sum_{d=0}^{\infty} \left\langle \chi_{\mathrm{Sym}^d(\rho)}, \chi_{\psi} \right\rangle_G \cdot q^d = \frac{1}{|G|}\sum_{g\in G} \frac{\overline{\chi_{\psi}(g)}}{\det\left(I_V - q\cdot\rho(g)\right)}$$

In particular, taking $\psi = 1\!\!1_G$ = trivial rep gives

COROLLARY

$$\mathrm{Hilb}(\mathrm{Sym}(V)^G, q) := \sum_{d=0}^{\infty} q^d \cdot \dim_{\mathbb{C}} \mathrm{Sym}^d(V)^G$$

Hilbert series for the G-fixed subalgebra $\mathrm{Sym}(V)^G$ $= \mathbb{C}[x_1,\ldots,x_n]^G$

$$= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I_V - q \cdot \rho(g))}$$

EXAMPLE

Let $G = \mathfrak{S}_3 \xrightarrow{\rho_{perm}} GL_3(\mathbb{C}) = GL(V)$

where $V = \mathbb{C}^3$ has $\mathbb{C}$-basis x_1, x_2, x_3

Then $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, x_2, x_3]$ with

$\mathfrak{S}_3$ permuting the variables,

so $\mathrm{Sym}(V)^G \cong \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3}$

= symmetric polynomials in x_1, x_2, x_3

degrees: 1, 2, 3

$$= \mathbb{C}[e_1, e_2, e_3]$$

$e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$

elementary symmetric polynomials

FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS

$\mathbb{C}[\underline{x}]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

Hence we expect here that

$$\mathrm{Hilb}(\mathrm{Sym}(V)^G, q) = \mathrm{Hilb}(\mathbb{C}[x_1,x_2,x_3]^{S_3}, q)$$

$$= \mathrm{Hilb}(\mathbb{C}[e_1,e_2,e_3], q)$$

$$= (1+q+q^2+q^3+\dots)(1+q^2+q^4+q^6+q^8+\dots)(1+q^3+q^6+\dots)$$

picking a term from each parenthesis corresponds to picking a monomial

$$e_1^{\alpha_1} e_2^{\alpha_2} e_3^{\alpha_3} = e_1^1 e_2^4 e_3^2$$

in this example

$$= \frac{1}{1-q^1} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3}$$

$$= \frac{1}{(1-q)(1-q^2)(1-q^3)}$$

Q: What does Molien's Theorem actually tell us here?

Recall $\mathfrak{S}_3$ has irreducible character table

	e	$(12),(13),(23)$	$(123),(132)$
$\mathbb{1}$	1	1	1
sgn	1	-1	1
ρ_{ref}	2	0	-1

and hence Molien gives us

$$\sum_{d=0}^{\infty} \langle \chi_{\mathrm{Sym}^d(V)}, \chi_\psi \rangle_{\mathfrak{S}_3} \cdot q^d = \begin{cases} \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(1)}{(1-q^2)(1-q)} + \frac{2(1)}{(1-q^3)} \right] & \text{if } \psi = \mathbb{1} \\ \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(-1)}{(1-q^2)(1-q)} + \frac{2(1)}{(1-q^3)} \right] & \text{if } \psi = \mathrm{sgn} \\ \frac{1}{3!}\left[\frac{2}{(1-q)^3} + \frac{3(0)}{(1-q^2)(1-q)} + \frac{2(-1)}{(1-q^3)} \right] & \text{if } \psi = \rho_{\mathrm{ref}} \end{cases}$$

$$= \begin{cases} \frac{1}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \mathbb{1}, \text{ as expected} \\ \frac{q^3}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \mathrm{sgn} \\ \frac{q+q^2}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \rho_{\mathrm{ref}} \end{cases}$$

$\boxed{\text{Very suggestive!}}$

What are those mysterious numerators $f_\psi(q)$ appearing in

$$\sum_{d=0}^{\infty} \left\langle \chi_{\mathbb{C}[x_1,x_2,x_3]_d}, \chi_\psi \right\rangle_{\mathfrak{S}_3} \cdot q^d = \frac{f_\psi(q)}{(1-q)(1-q^2)(1-q^3)} \; ?$$

numerator $f_\psi(q)$	ψ	e	(12), (13), (23)	(123), (132)
1	$\mathbb{1}$	1	1	1
q^3	sgn	1	-1	1
$q+q^2$	ref	2	0	-1

A: They are the fake-degree polynomials that come from the reflection action of $\mathfrak{S}_3$ and the Shephard-Todd/Chevalley Thm.

THEOREM
Shephard-Todd 1955
Chevalley 1955

Given a finite reflection group
$G \subset GL_n(\mathbb{R}) \; (\subset GL_n(\mathbb{C}) = GL(V))$
acting on $Sym(V) \cong \mathbb{C}[\underbrace{x_1, \ldots, x_n}_{\text{basis for } V}] = \mathbb{C}[\underline{x}]$

(a) the G-invariant subalgebra $\mathbb{C}[\underline{x}]^G$ is again a polynomial algebra $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1, \ldots, f_n]$ for some homogeneous $f_1, f_2, \ldots, f_n$
of some degrees $d_1, d_2, \ldots, d_n$

so that $\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q) = \dfrac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})}$

(b) and as G-representations,

$$\underbrace{\mathbb{C}[\underline{x}]/(f_1, f_2, \ldots, f_n)}_{\text{called the coinvariant algebra}} \cong \underbrace{\rho_{reg}}_{\text{regular representation of } G}$$

MORAL: When G is a reflection group, the coinvariant algebra $\mathbb{C}[\underline{x}]/(f_1, \ldots, f_n)$ gives us a naturally graded version of the regular representation of G

– it tells us how to add in a "q" !?

Using a bit of commutative algebra

- $\mathbb{C}[\underline{x}]$ is a Cohen-Macaulay ring
- $f_1, \ldots, f_n$ are a system of parameters,
- hence also a regular sequence

one can deduce the following :

COROLLARY: In the above setting of a reflection G acting on V, for any G-representation ψ

$$\sum_{d=0}^{\infty} \langle \chi_{\mathbb{C}[\underline{x}]_d}, \chi_\psi \rangle_G \cdot q^d$$

$$= \mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q) \cdot \underbrace{\sum_{d=0}^{\infty} \langle \chi_{(\mathbb{C}[\underline{x}]/(\underline{f}))_d}, \chi_\psi \rangle_G \cdot q^d}_{\overset{\text{DEF'N}}{=:}\ f_\psi(q)}$$

$$= \frac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})} \cdot f_\psi(q)$$

called the fake-degree polynomial for ψ

$$= \frac{f_\psi(q)}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})}$$

EXAMPLE For $G = \mathfrak{S}_3 \hookrightarrow GL_3(\mathbb{C})$ what does the coinvariant algebra look like?

$$\mathrm{Sym}(V) = \mathbb{C}[x_1, x_2, x_3]$$

$$\mathrm{Sym}(V)^G = \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3} = \mathbb{C}[\overset{f_1}{e_1}, \overset{f_2}{e_2}, \overset{f_3}{e_3}]$$

$$e_1 = x_1+x_2+x_3, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad e_3 = x_1x_2x_3$$

So the coinvariant algebra is

$$\mathbb{C}[\underline{x}]/(\underline{f}) = \mathbb{C}[x_1,x_2,x_3]/(\overset{f_1}{x_1+x_2+x_3}, \overset{f_2}{x_1x_2+x_1x_3+x_2x_3}, \overset{f_3}{x_1x_2x_3})$$

use $x_1+x_2+x_3=0$ to substitute $x_3 = -x_1-x_2$

$$\cong \mathbb{C}[x_1,x_2]/(x_1x_2 - x_1^2 - x_1x_2 - x_2^2 - x_1x_2, -x_1^2x_2 - x_1x_2^2)$$

$$= \mathbb{C}[x_1,x_2]/(x_1^2 + x_1x_2 + x_2^2, x_1^2x_2 + x_1x_2^2)$$

degree	0	1		2		3
$= \mathrm{span}_{\mathbb{C}}\{$	$1,$	$x_1,$	$x_2,$	$x_1^2,$	$x_2^2,$	$x_1^2x_2\}$
$\mathfrak{S}_3$-rep:	$\mathbb{1}$	ρ_{ref}		ρ_{ref}		sgn
	$f_{\mathbb{1}}(q) = 1 = q^0$	$f_{\rho_{\mathrm{ref}}}(q) = q^1 + q^2$				$f_{\mathrm{sgn}}(q) = q^3$

REMARK:

A recent paper of Sagan & Swanson (arXiv: 2205.14078) conjectures the Hilbert series for the coinvariant algebra when $\mathfrak{S}_n$ acts on

$$A := \mathbb{C}[\underbrace{x_1, \dots, x_n}_{\text{commuting variables}}]\langle \underbrace{\theta_1, \dots, \theta_n}_{\substack{\text{anticommuting variables} \\ \theta_i\theta_j = -\theta_j\theta_i \\ \theta_i^2 = 0}} \rangle \simeq \overset{\text{symmetric algebra}}{\mathrm{Sym}(V)} \otimes \overset{\text{exterior algebra}}{\Lambda(V)}$$

that is,

$$\mathrm{Hilb}\left(A/(A_+^{\mathfrak{S}_n}), \underset{\substack{\uparrow \\ \text{tracks deg in } x_i}}{q}, \underset{\substack{\uparrow \\ \text{tracks deg in } \theta_i}}{t} \right).$$

The conjecture is elegantly formulated in terms of q-Stirling numbers.