q-counting and invariant theory

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Summer School in
Algebraic Combinatorics
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Lecture

1: Monday
Invitation to q-counts
& representation theory
- quotients of Boolean algebras

2: Tuesday
Representation theory review
& reflection groups

3: Thursday
Molien's Theorem
& coinvariant algebras

4: Thursday
Cyclic Sieving Phenomena (CSP)
& Springer's Theorem

see ECCO 2018 lecture notes

5: Friday
More CSP's
& the deformation idea

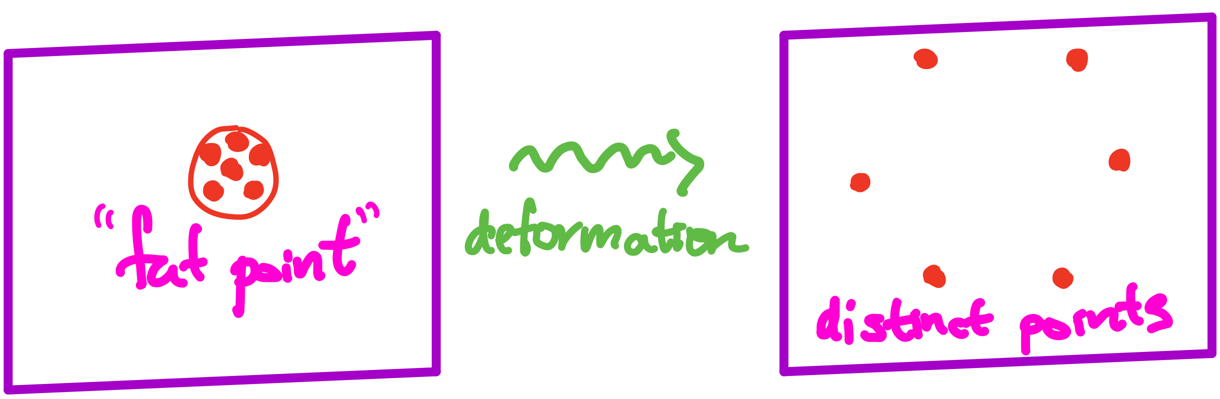
TODAY

GOAL: Discuss three counts

$$\binom{n}{k}, \quad n^{n-2}, \quad \frac{1}{n+1}\binom{2n}{n}$$

having ...

- q-counts via Hilbert series
- reflection group generalizations
- cyclic actions with CSP's
- a common proof idea:



1. Three counts

We know $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ counts

$X = \binom{[n]}{k}$ = k-element subsets of $[n]$

$\circlearrowright$

$C = \langle \overset{c=}{\underbrace{(1\,2\cdots n)}_{n\text{-cycle}}} \rangle \cong \mathbb{Z}/n\mathbb{Z}$

EXAMPLE $n=4$ $k=2$

$X = \binom{[4]}{2}$

$\circlearrowright$

$C = \langle \overset{c=}{(1\,2\,3\,4)} \rangle$

$1 \to 2 \to 3 \to 4 \to 1$

$= \{e, c, c^2, c^3\}$

$\{1,2\} \xrightarrow{c} \{2,3\} \xrightarrow{c} \{3,4\} \xrightarrow{c} \{1,4\} \xrightarrow{c} \{1,2\}$

$\{1,3\} \xrightarrow{c} \{2,4\} \xrightarrow{c} \{1,3\}$

X

THEOREM

Hurwitz 1891

n^{n-2} counts

$$X = \left\{ \begin{array}{c} \text{factorizations } c = (1,2,\ldots,n) = t_1 t_2 \cdots t_{n-1} \\ \text{of the } n\text{-cycle into} \\ n-1 \text{ transpositions } t_k = (i,j) \end{array} \right\}$$

X carries a natural action of a cyclic group $C = \langle \psi \rangle \cong \mathbb{Z}/n(n-1)\mathbb{Z}$:

$$t_1 t_2 \cdots t_{n-2} t_{n-1} \xmapsto{\psi} c t_{n-1} c^{-1} \cdot t_1 t_2 \cdots t_{n-2}$$

$$\xmapsto{\psi^{n-1}} c t_1 c^{-1} \cdot c t_2 c^{-1} \cdots c t_{n-1} c^{-1}$$

EXAMPLE

$n=4$

$n^{n-2} = 4^2 = 16$ factorizations in X

$C = \langle \psi \rangle \cong \mathbb{Z}/3\cdot 4\mathbb{Z}$
$= \mathbb{Z}/12\mathbb{Z}$

has 2 orbits on X

$c = (1,2,3,4)$
$= (12)(23)(34)$

$$(12)(23)(34) \xrightarrow{\psi} (14)(12)(23) \xrightarrow{\psi} (34)(14)(12) \xrightarrow{\psi} (23)(34)(14) \xrightarrow{\psi} (12)(23)(34)$$

$$(13)(34)(12) \xrightarrow{\psi} (23)(13)(34) \xrightarrow{\psi} (14)(23)(13) \xrightarrow{\psi} (24)(14)(23) \xrightarrow{\psi} (34)(24)(14) \xrightarrow{\psi} (12)(34)(24) \xrightarrow{\psi} (13)(12)(34)$$

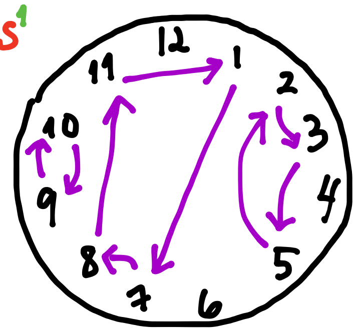
$$(13)(12)(34) \xrightarrow{\psi} (14)(13)(12) \xrightarrow{\psi} (23)(14)(13) \xrightarrow{\psi} (24)(23)(14) \xrightarrow{\psi} (12)(24)(23) \xrightarrow{\psi} (34)(12)(24) \xrightarrow{\psi} (13)(34)(12)$$

THEOREM (Kreweras 1972, Biane 1997)

The Catalan number $\frac{1}{n+1}\binom{2n}{n}$ counts

$$X = \left\{ \begin{array}{l} \text{permutations } w \text{ that can be factored} \\ \qquad w = t_1 t_2 \cdots t_k \text{ as a prefix of} \\ \text{a factorization } c = t_1 t_2 \cdots t_k t_{k+1} \cdots t_{n-1} \\ \qquad \text{of } c=(1,2,\ldots,n) \text{ into } n\text{-}1 \text{ transpositions.} \end{array} \right\}$$

(equivalently, non-crossing set partitions[1] of $\{1,2,\ldots,n\}$)



X has a natural cyclic group
$C = \langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z}$ acting via $w \overset{\varphi}{\longmapsto} cwc^{-1}$

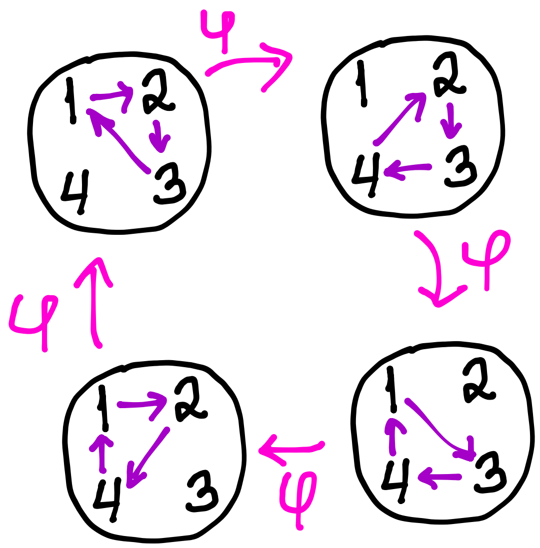
(= rotation of noncrossing partitions)

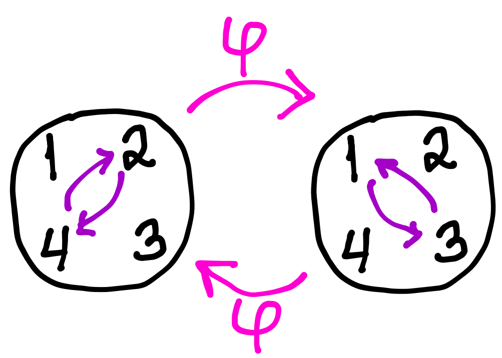
[1] See Stanley, "Catalan Numbers" #159

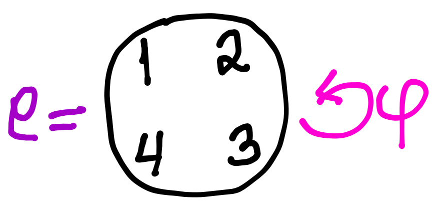
$n=4$

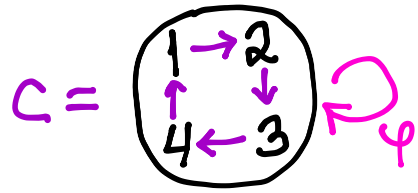
$$\frac{1}{n+1}\binom{2n}{n}=\frac{1}{5}\binom{8}{4}=14$$

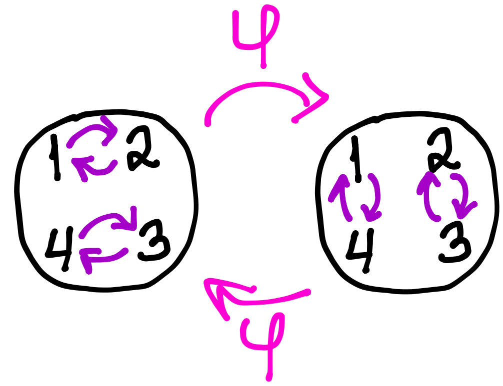
$C=\langle\varphi\rangle\cong\mathbb{Z}/4\mathbb{Z}$ has 6 orbits

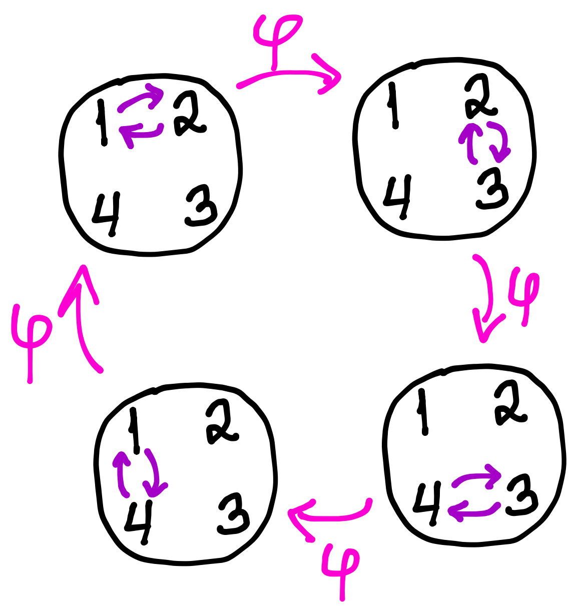


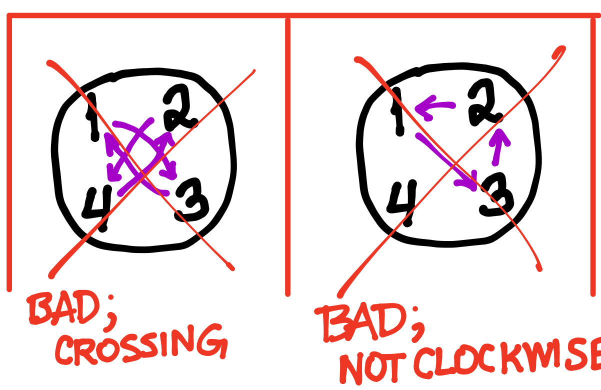












Their q-counts

$$\binom{n}{k} \overset{q=1}{\leftsquigarrow} \begin{bmatrix} n \\ k \end{bmatrix}_q$$ q-binomial (Euler, Gauss)

$$n^{n-2} \overset{q=1}{\leftsquigarrow} [n]_{q^2} [n]_{q^3} \cdots [n]_{q^{n-1}}$$

$$\frac{1}{n+1}\binom{2n}{n} \overset{q=1}{\leftsquigarrow} \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$ MacMahon's 1915 q-Catalan

where as before,

$$[n]_q := 1+q+q^2+\cdots+q^{n-1}$$

$$[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q \, [n-k]!_q}$$

$$6 = \binom{4}{2} \overset{q=1}{\leftsquigarrow} \begin{bmatrix}4\\2\end{bmatrix}_q = \frac{[4]_q[3]_q\cancel{[2]_q}\cancel{[1]_q}}{[2]_q[1]_q\cdot\cancel{[2]_q}\cancel{[1]_q}}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)\,(1)}$$

$$= (1+q^2)(1+q+q^2)$$

$$16 = 4^2 \overset{q=1}{\leftsquigarrow} [4]_{q^2}[4]_{q^3}$$

$$= (1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$$

$$14 = \frac{1}{5}\binom{8}{4} \overset{q=1}{\leftsquigarrow} \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q$$

$$= \frac{1}{\cancel{[5]_q}}\,\frac{[8]_q[7]_q[6]_q\cancel{[5]_q}}{[4]_q[3]_q[2]_q\cancel{[1]_q}}$$

$$= (1-q+q^2)(1+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)$$

Each of these q-counts $X(q)$

gives a CSP for the appropriate

set X and

$\circlearrowleft$

cyclic group $C \cong \mathbb{Z}/m\mathbb{Z}$,

meaning for all $c^d \in C$ one has

$$\#\{x \in X : c^d(x) = x\} = \left[X(q)\right]_{q=\zeta^d}$$

where $\zeta = e^{2\pi i/m}$ = primitive m^{th} root of 1 in $\mathbb{C}^\times$

We saw...

THEOREM

RSW 2004

- X = k-element subsets of $\{1,2,\ldots,n\}$ $\circlearrowright$
- $C = \langle \overset{c}{\overset{\shortparallel}{(1,2,\ldots,n)}} \rangle \cong \mathbb{Z}/n\mathbb{Z}$
- $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$

exhibit a CSP.

$n=4$

$k=2$

$\zeta = e^{\frac{2\pi i}{4}} = i$

$$
\begin{tikzcd}
 & \{1,2\} \arrow[dr, "c"] & & \{1,3\} \arrow[d, bend left, "c"] \\
\{1,4\} \arrow[ur, "c"] & & \{2,3\} \arrow[dl, "c"] & \{2,4\} \arrow[u, bend left, "c"] \\
 & \{3,4\} \arrow[ul, "c"] & &
\end{tikzcd}
$$

$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = (1+q^2)(1+q+q^2)$$

$q = \zeta^0 = 1$ $\rightsquigarrow$ 6

$q = \zeta^2 = -1$ $\rightsquigarrow$ 2

$q = \zeta^1 = i$ $\rightsquigarrow$ 0

THEOREM

Douvropoulos 2017

Conj. by N. Williams 2013

- X = factorizations $c = t_1 t_2 \cdots t_{n-1}$ of n-cycle c into $n-1$ transpositions

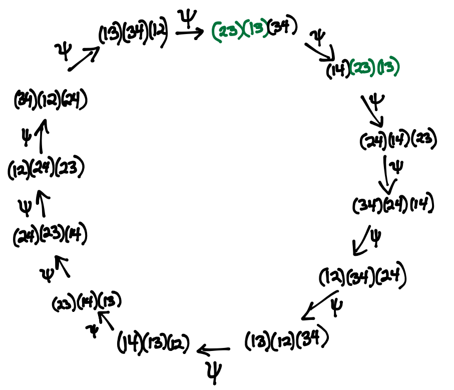
$\circlearrowright$

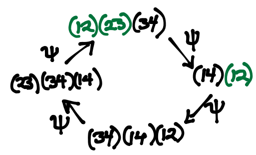
- $C = \langle \psi \rangle \cong \mathbb{Z}/(n-1)n\mathbb{Z}$
- $X(q) = [n]_{q^2}[n]_{q^3} \cdots [n]_{q^{n-1}}$

exhibit a CSP.

$n=4$

$\zeta = e^{\frac{2\pi i}{12}}$





$X(q) = [4]_{q^2}[4]_{q^3} = (1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$

$q = \zeta^0 = 1 \rightsquigarrow 16$

$q = \zeta^4 = e^{\frac{2\pi i}{3}} \rightsquigarrow 4$

$q = \zeta^1 = e^{\frac{2\pi i}{12}} \rightsquigarrow 0$

$q = \zeta^2 = e^{\frac{2\pi i}{6}} \rightsquigarrow 0$

$q = \zeta^3 = e^{\frac{2\pi i}{4}} = i \rightsquigarrow 0$

$q = \zeta^6 = e^{\frac{2\pi i}{2}} = -1 \rightsquigarrow 0$

THEOREM

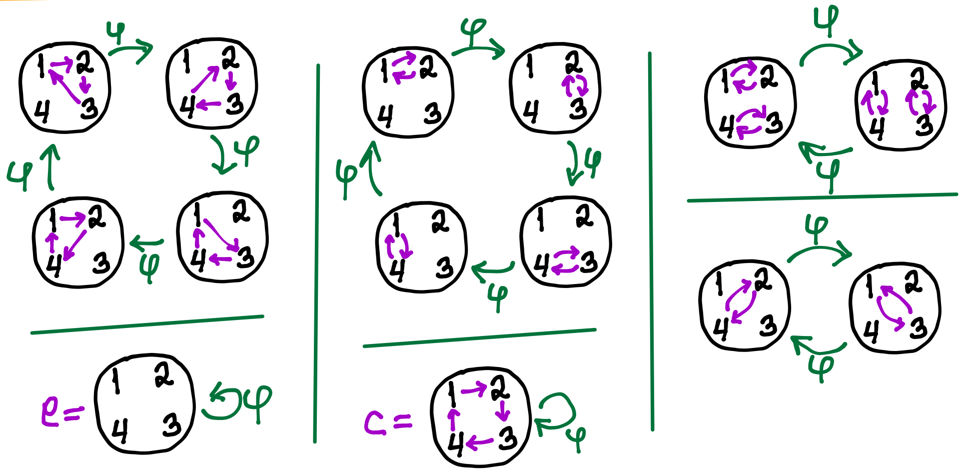
RSW 2004

- $X =$ permutations w factored $w=t_1t_2\cdots t_k$ as prefixes of factorizations $c=t_1t_2\cdots t_k\cdots t_{n-1}$ (noncrossing partitions)
- $C=\langle \varphi \rangle \cong \mathbb{Z}/n\mathbb{Z}$
- $X(q)=\dfrac{1}{[n+1]_q}\begin{bmatrix}2n\\ n\end{bmatrix}_q$

exhibit a CSP.

$n=4$

$\xi = e^{\frac{2\pi i}{4}} = i$



$$X(q)=\frac{1}{[5]_q}\begin{bmatrix}8\\ 4\end{bmatrix}_q=(1-q+q^2)(1+q^4)(1+q+q^2+q^3+q^4+q^5+q^6)$$

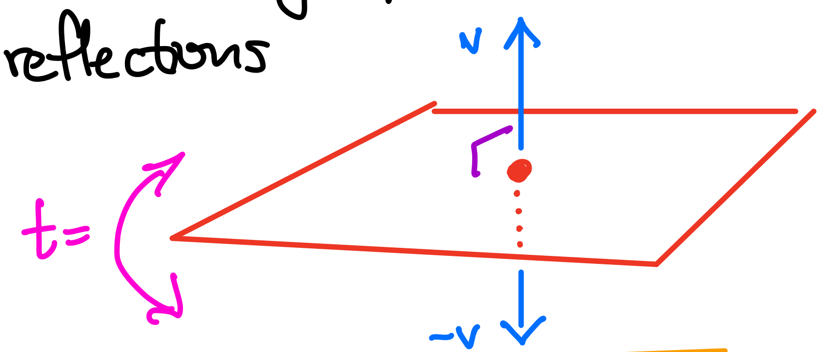
$q=\xi^0=1$ ⇝ 14

$q=\xi^2=-1$ ⇝ 6

$q=\xi^1=i$ ⇝ 2

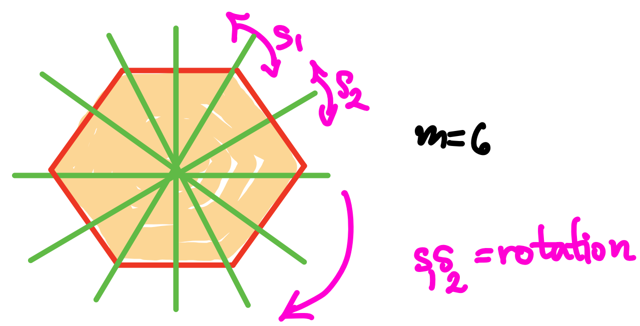
How to generalize to real reflection groups G?

Recall this means a finite group $G \subset GL_n(\mathbb{R})$ generated by reflections

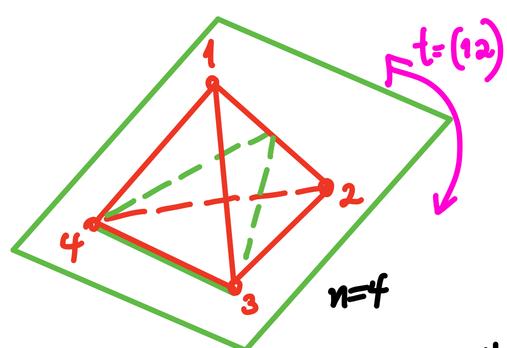


EXAMPLES: symmetry groups of regular polytopes

G = dihedral group = symmetries of regular m-gon



$G = \mathfrak{S}_n$ = symmetries of regular $(n-1)$-simplex



reflections = transpositions (i,j)

Much numerology and q-counting comes from the (fundamental) degrees $d_1, d_2, \ldots, d_n$ of basic homogeneous G-invariant polynomials that already appeared in...

THEOREM

Shephard-Todd 1955
Chevalley 1955

A reflection group G acting on

$$\mathrm{Sym}(V) \cong \mathbb{C}[\underbrace{x_1, \ldots, x_n}_{\text{basis for } V}] = \mathbb{C}[\underline{x}]$$

has its G-invariant subalgebra again a polynomial algebra $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1, f_2, \ldots, f_n]$

for some homogeneous $f_1, f_2, \ldots, f_n$

of degrees $d_1, d_2, \ldots, d_n$

What was so special about the n-cycles $c=(1\,2\cdots n)$ in $G=\mathfrak{S}_n$?

We've seen that they are regular elements in the sense of Springer: they have an eigenvector v lying on none of the reflecting hyperplanes for $G=\mathfrak{S}_n$

But they are even more special...

THEOREM Coxeter 1948 Real reflection groups G contain a special conjugacy class of regular elements,

called Coxeter elements

with multiplicative order $h := \max\{d_1,\ldots,d_n\}$ called the Coxeter number of G

represented by $c=s_1 s_2 \cdots s_n$,

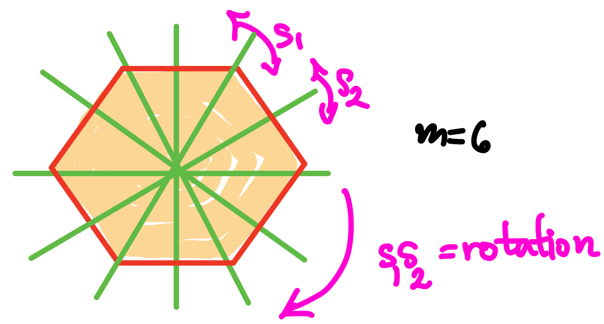
where $G \cong \langle s_1, s_2, \ldots, s_n \mid s_i^2 = e = (s_i s_j)^{m_{ij}} \rangle$

with $m_{ij} \in \{2,3,\ldots\}$

called the Coxeter presentation for G

EXAMPLES

$G =$ dihedral group $=$ symmetries of regular m-gon



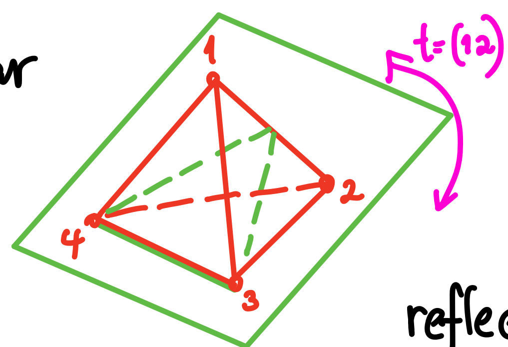
$$\cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = (s_1 s_2)^m = 1 \rangle$$

Coxeter presentation

– fundamental degrees $(d_1, d_2) = (2, m)$ $\Rightarrow$ $h = m$ Coxeter number

– and Coxeter element $c = s_1 s_2 =$ rotation

$G = \mathfrak{S}_n =$ symmetries of regular $(n-1)$-simplex



$$\cong \langle \overset{(12)}{s_1}, \overset{(23)}{s_2}, \dots, \overset{(n-1,n)}{s_{n-1}} \;\Big|\; s_i^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = 1 \rangle$$

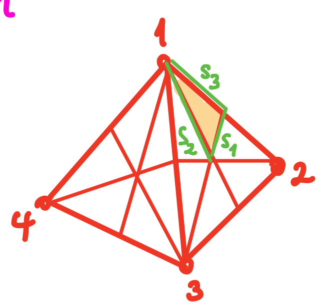
if $|i-j| \geq 2$

Coxeter presentation

reflections = transpositions (i,j)

– fundamental degrees $(d_1, d_2, \dots, d_n) = (1, 2, \dots, n)$

$\Rightarrow$ $h = n$ Coxeter number



– and Coxeter element

$$c = s_1 s_2 \cdots s_{n-1} = (12)(23)\cdots(n-1,n)$$

$$= (12\cdots n) = n\text{-cycle}$$

The q-counts for reflection groups G:

$$\binom{n}{k} \xleftarrow{q=1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q\,[n-k]!_q} \qquad \overset{G=\mathfrak{S}_n}{\underset{H=\mathfrak{S}_k\times\mathfrak{S}_{n-k}}{\longleftarrow}} \qquad \prod_{i=1}^{n} \frac{[d_i^G]_q}{[d_i^H]_q}$$

$$n^{n-2} \xleftarrow{q=1} [n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}} \qquad \xleftarrow{G=\mathfrak{S}_n} \qquad \prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$$

$$\frac{1}{n+1}\binom{2n}{n} \xleftarrow{q=1} \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \qquad \xleftarrow{G=\mathfrak{S}_n} \qquad \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$$

q-Catalan number for G

Q: Why are these q-counts polynomials, in $\mathbb{Z}[q]$?

A: All are Hilbert series $\mathrm{Hilb}(A,q) := \sum_{d=0}^{\infty} \dim_{\mathbb{C}}(A_d)\cdot q^d$

for various graded rings $A = \bigoplus_{d=0}^{\infty} A_d$

EXAMPLE

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \qquad \begin{matrix} G = \mathfrak{S}_n \\ \leftsquigarrow \\ H = \mathfrak{S}_k \times \mathfrak{S}_{n-k} \end{matrix} \qquad \prod_{i=1}^{n} \frac{[d_i^G]_q}{[d_i^H]_q}$$

$$\prod_{i=1}^{n} \frac{[d_i^G]_q}{[d_i^H]_q} = \mathrm{Hilb}\left(\mathbb{C}[f_1^H,\ldots,f_n^H] \big/ (f_1^G,\ldots,f_n^G), q \right)$$

where $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1^G,\ldots,f_n^G]$

$$\cap$$

$\mathbb{C}[\underline{x}]^H = \mathbb{C}[f_1^H,\ldots,f_n^H]$

for a reflection subgroup $H \subset G$

$$[n]_{q^2}[n]_{q^3}\cdots[n]_{q^{n-1}} \xleftarrow{G=\mathfrak{S}_n} \prod_{i=1}^{n}\frac{[ih]_q}{[d_i]_q}$$

$$\prod_{i=1}^{n}\frac{[ih]_q}{[d_i]_q} = \mathrm{Hilb}\left(\frac{\mathbb{C}[f_1,\ldots,f_{n-1}]}{(\alpha_2(\underline{f}),\ldots,\alpha_n(\underline{f}))}, q\right)$$

where the G-discriminant in $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1,\ldots,f_n]$

is expressed $\Delta_G^2 = f_n^n + \alpha_2(\underline{f})f_n^{n-2} + \alpha_3(\underline{f})f_n^{n-3} + \ldots + \alpha_n(\underline{f})$

with $\Delta_G := \prod \ell_H(x_1,\ldots,x_n)$ $\left(\text{if } \overset{G=\mathfrak{S}_n}{=} \prod_{1\le i<j\le n}(x_i - x_j)\right)$

reflection hyperplanes H for G

(a bit technical; uses work of Bessis and others!)

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q \quad G=\mathfrak{S}_n \quad \leftsquigarrow \quad \prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q}$$

$$\prod_{i=1}^{n}\frac{[h+d_i]_q}{[d_i]_q} = \mathrm{Hilb}\left(\left(\mathbb{C}[\underline{x}] \big/ (\theta_1,\ldots,\theta_n) \right)^G , q \right)$$

where $\theta_1,\ldots,\theta_n$ in $\mathbb{C}[\underline{x}]$

- each have same degree $h+1$
- form a system of parameters for $\mathbb{C}[\underline{x}]$
- have the map $x_i \longmapsto \theta_i$ G-equivariant

Existence of such magical $\theta_1,\ldots,\theta_n$ provided by rep'n theory of rational Cherednik algebras (Gordon, Berest-Etingof-Ginzburg 2002)

Deformation proof idea

(for some CSP's with $X \circlearrowleft C$ and $X(q)$)

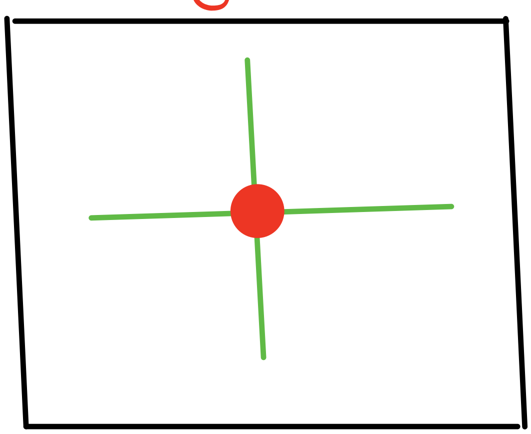
Let $X(q) = \mathrm{Hilb}(A, q)$ for graded ring

$$A = \mathbb{C}[x_1, \ldots, x_n] / \underbrace{(h_1, \ldots, h_n)}$$

homogeneous ideal I

= coordinate ring for the fat point $h_1(\underline{x}) = \ldots = h_n(\underline{x}) = 0$ at the origin in $\mathbb{C}^n$

$\mathbb{C}^2$



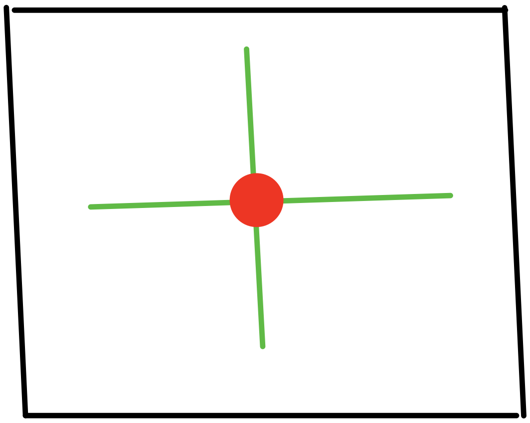
We now ask for a lot ...

Deform

$I=(h_1,\dots,h_n)$ $\rightsquigarrow$ $J=(h_1',\dots,h_n')$

homogeneous | inhomogeneous

$h_1(\underline{x})=\dots=h_n(\underline{x})=0$



fat point of multiplicity $X(1)=\#X$, with coordinate ring $A=\mathbb{C}[x]/I$

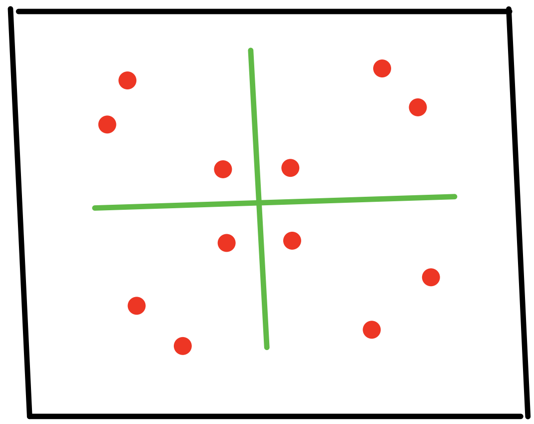
$\circlearrowleft$ $g(x_i)=\zeta x_i$

$C=\mathbb{Z}/m\mathbb{Z}$

$\parallel$

$\langle g\rangle$

$h_1'(\underline{x})=\dots=h_n'(\underline{x})=0$



$\#X$ reduced points with coordinate ring $\mathbb{C}[x]/J$

$\circlearrowleft$ $g(x_i)=\zeta x_i$

$C=\mathbb{Z}/m\mathbb{Z}$ permuting as in $C \circlearrowright X$

This would prove the CSP:

$$\begin{array}{ccc} \#\{x\in X : g^d(x)=x\} & \overset{?}{=} & [X(q)]_{q=\zeta^d} \\ \| & & \| \\ & & \sum_i \dim(A_i)\cdot(\zeta^d)^i \\ \| & & \| \\ \text{Trace of } g^d & & \text{Trace of } g^d \\ \text{acting on } \mathbb{R}[\underline{x}]/J & = & \text{acting on } \underbrace{\mathbb{R}[\underline{x}]/I}_{A} \end{array}$$

assuming $\mathbb{R}[\underline{x}]/I$ and $\mathbb{R}[\underline{x}]/J$ agree up to a G-stable filtration

How does this go in our examples?

THEOREM RSW 2004 Given a Coxeter element c in G, and reflection subgroup $H < G$,

- $X = G/H$ $\circlearrowleft$
- $C = \langle c \rangle \cong \mathbb{Z}/h\mathbb{Z}$ via $c^d(gH) = c^d gH$
- $X(q) = \prod_{i=1}^{n} \frac{[d_i^G]_q}{[d_i^H]_q}$

exhibits a CSP.

Proof sketch: Deform

$$\mathbb{C}[f_1^H, \ldots, f_n^H] / \underbrace{(f_1^G, \ldots, f_n^G)}_{I}$$

$$\rightsquigarrow \quad \mathbb{C}[f_1^H, \ldots, f_n^H] / \underbrace{(f_1^G - f_1^G(v), \ldots, f_n^G - f_n^G(v))}_{J}$$

where $v \in V = \mathbb{C}^n$ is an eigenvector for c avoiding all the reflecting hyperplanes for G. $\blacksquare$

THEOREM

Douvropoulos 2017

(Conj. by N. Williams 2013)

- X = factorizations $c = t_1 t_2 \cdots t_n$ of a Coxeter element c into n reflections

  $\circlearrowleft$

- $C = \langle \psi \rangle \cong \mathbb{Z}/nh\mathbb{Z}$

- $X(q) = \displaystyle\prod_{i=1}^{n} \frac{[ih]_q}{[d_i]_q}$

exhibit a CSP.

Proof sketch. Deform

$$\mathbb{C}[f_1, f_2, \ldots, f_{n-1}] \Big/ \overbrace{(\alpha_2(\underline{f}), \ldots, \alpha_n(\underline{f}))}^{I}$$

$$\rightsquigarrow \mathbb{C}[f_1, f_2, \ldots, f_{n-1}] \Big/ \overbrace{(\alpha_2(\underline{f}) - c_2, \ldots, \alpha_n(\underline{f}) - c_n)}^{J}$$

for particular choices of $c_2, \ldots, c_n$, making heavy use of Bessis's 2007 results on Lyashko-Looijenga morphism. $\blacksquare$

"Parking Space"
CONJECTURE
Armstrong-R.-Rhoades 2012

One can explain a known CSP for

- X = $g \in G$ factored $g = t_1 t_2 \cdots t_k$ as prefixes of factorizations $c = t_1 t_2 \cdots t_k \cdots t_n$ of a Coxeter element c

  $\circlearrowleft$

  W-noncrossing partitions

- $C = \langle \overset{w \mapsto cwc^{-1}}{\varphi} \rangle \cong \mathbb{Z}/h\mathbb{Z}$

- $X(q) = \prod_{i=1}^{n} \frac{[h+d_i]_q}{[d_i]_q}$

via this deformation:

$$\Big(\mathbb{C}[\underline{x}] \big/ \overbrace{(\Theta_1, \dots, \Theta_n)}^{I} \Big)^G$$

$$\rightsquigarrow \Big(\mathbb{C}[\underline{x}] \big/ \overbrace{(\Theta_1 - x_1, \dots, \Theta_n - x_n)}^{J} \Big)^G$$

REMARKS

1. There are many generalizations of

Hurwitz's formula

$$n^{n-2} = \#\{\text{factorizations } \underset{n\text{-cycle}}{c} = t_1 t_2 \cdots t_{n-1}\}$$

$\uparrow t_i$ transpositions

with recent generalizations in the reflection group direction

e.g. by Chapuy & Stump
Michel
Chapuy & Douvropoulos
Lewis & Morales
Lewis, Morales, Douvropoulos
⋮

2. Another beautiful, less understood thread:

symmetric group $\mathfrak{S}_n$ $\underset{\text{"}q=1\text{"}}{\leftsquigarrow}$ finite general linear group $GL_n(\mathbb{F}_q)$

$\mathbb{C}[\underline{x}]^{\mathfrak{S}_n}$
$= \mathbb{C}[e_1, \ldots, e_n]$ $\leftsquigarrow$ $\mathbb{F}_q[x_1, \ldots, x_n]^{GL_n(\mathbb{F}_q)}$
$= \mathbb{F}_q[D_0, D_1, \ldots, D_{n-1}]$
Dickson's invariants

$c = (1\ 2\ \cdots\ n)$
n-cycle $\leftsquigarrow$ c_q = Singer cycle
:= cyclic generator of
$\langle c_q \rangle = \mathbb{F}_{q^n}^{\times} \hookrightarrow GL_{\mathbb{F}_q}(\mathbb{F}_{q^n}) \cong GL_n(\mathbb{F}_q)$

There are lots of results & conjectures!

Thanks for your attention

and

thank you,

SSACK organizers!

Extra References (beyond those in the ECCO chapter):

On cyclic sieving:

Bruce Sagan "The cyclic sieving phenomenon: a survey" In 'Surveys in Combinatorics 2011' London Math. Soc. Lec. Notes Series Vol 392, 183-233

R.-Stanton-White "What is ... cyclic sieving?" Notices of the AMS 61 (2): 169-171

On the analogy between $\mathfrak{S}_n$ and $GL_n(\mathbb{F}_q)$:

Lewis-R.-Stanton "Reflection factorizations of Singer cycles", J. Algeb. Combin. (2014) 40, 663-691.

R.-Stanton-Webb "Springer's regular elements over arbitrary fields", Math. Proc. Camb. Phil. Soc. 141 (2006), 209-229