Sandpiles and Representation Theory

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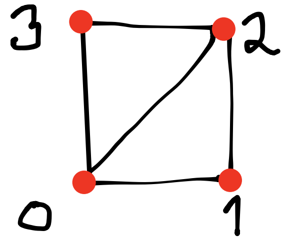
OUTLINE

Laplacian & sandpile group for a...

- ... graph
- ... group representation
- ... module over a Hopf algebra

Graphs

$\Gamma = (\underset{\{0,1,2,\ldots,\ell\}}{\underset{\shortparallel}{V}}, E)$ an undirected (multi-) graph



$L_\Gamma := D_\Gamma - A_\Gamma$

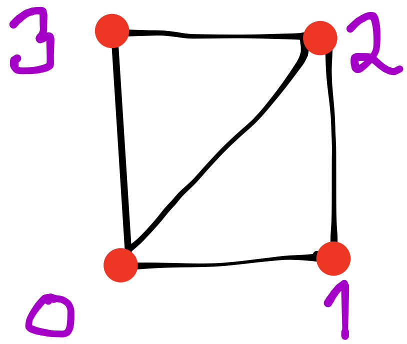
graph Laplacian

diagonal matrix of vertex degrees

adjacency matrix

$$(L_\Gamma)_{i,j} = \begin{cases} \deg_\Gamma(i) & \text{if } i=j \\ -\#\{\text{edges } i \text{ to } j\} & \text{if } i\neq j \end{cases}$$

EXAMPLE

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

The graph Laplacian L_Γ

- is positive semidefinite

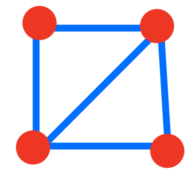
$$\left(L_\Gamma = \partial \partial^T \text{ where } \begin{tikzcd} \mathbb{R}^E \arrow[r, "\partial"] \arrow[d, equal] & \mathbb{R}^V \arrow[d, equal] \\ C_1(\Gamma, \mathbb{R}) & C_0(\Gamma, \mathbb{R}) \end{tikzcd} \right)$$

- has $\mathbb{R}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \subseteq \ker(L_\Gamma)$

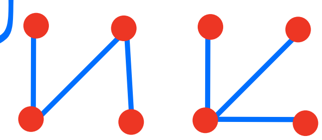
- equality $\boxed{\text{here}}$ $\Leftrightarrow$ Γ connected

- From spectrum (=eigenvalues) of L_Γ

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\ell$$



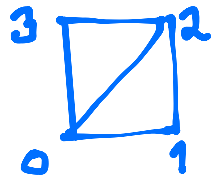
one can count the spanning trees in Γ:

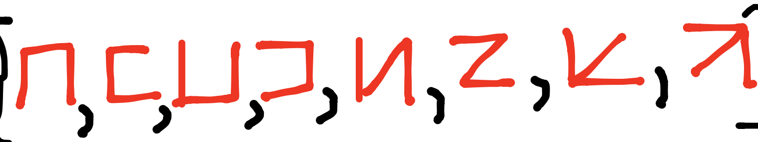


$$\underbrace{\tau(\Gamma)}_{\substack{\#\text{spanning}\\ \text{trees in } \Gamma}} = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell + 1}$$

- Alternatively,

$$\tau(\Gamma) \underset{\substack{\uparrow\\ \text{Kirchhoff's}\\ \text{Matrix-Tree}\\ \text{Theorem}\\ (1845)}}{=} \det\Bigg(\underbrace{L_\Gamma - \begin{Bmatrix} 0^{\text{th}}\ \text{row}, \\ 0^{\text{th}}\ \text{column} \end{Bmatrix}}_{\substack{\text{reduced Laplacian}\\ \overline{L}_\Gamma}}\Bigg)$$

EXAMPLE $\Gamma =$  has

$\tau(\Gamma) = \#\{$  $\} = 8$

$$L_\Gamma = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

has eigenvalues

$$\underset{\lambda_0}{0} \le \underset{\lambda_1}{2} \le \underset{\lambda_2}{4} \le \underset{\lambda_3}{4}$$

So $\tau(\Gamma) = \dfrac{\lambda_1\lambda_2\lambda_3}{4} = \dfrac{2\cdot 4\cdot 4}{4} = 8$ ✓

Or, $\tau(\Gamma) = \det \overline{L}_\Gamma = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$= 2(3\cdot 2 - 1) + (-1\cdot 2 - 0)$

$= 10 - 2 = 8$ ✓

REMARK:

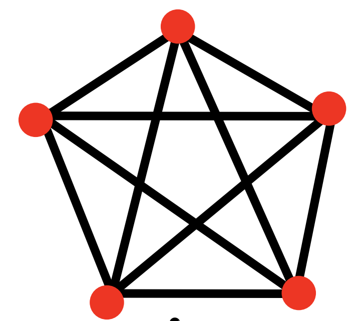
Eigenvalues of L_Γ are known for several families of graphs, letting one compute $\tau(\Gamma)$:

usually graphs with large symmetry or with inductive structure

- complete graphs, complete multipartite graphs
- cubes, Cartesian products
- distance-regular graphs
- threshold graphs, co-graphs

EXAMPLE

Complete graphs K_n



K_5

have L_{K_n} eigenvalues

$$\lambda_0 < \lambda_1 = \lambda_2 = \ldots = \lambda_{n-1}$$

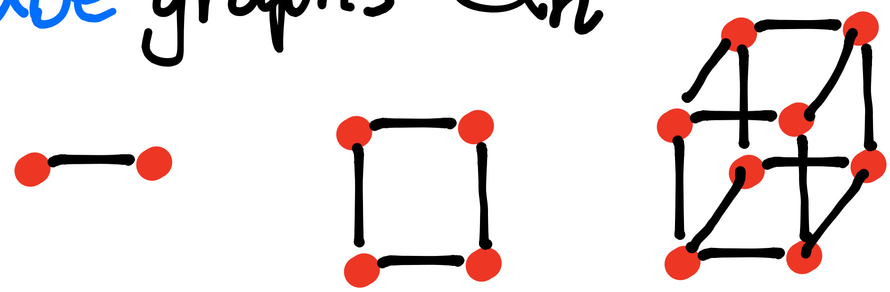
$$(0, n, n, \ldots, n)$$

COROLLARY

$$\tau(K_n) = \frac{n^{n-1}}{n} = n^{n-2}$$

Cayley 1889
Borchardt 1860

EXAMPLE n-dimensional **cube** graphs Q_n



Q_1 $\quad$ Q_2 $\quad$ Q_3

have L_{Q_n} eigenvalues

λ	0	2	4	$\cdots$	$2n-2$	$2n$
mult.	1	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

COROLLARY

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

REMARK

Eigenvectors of L_{Γ} are also important in applications to

- optimal graph-drawing
- clustering of data

(see articles and surveys by Dan Spielman)

What about the Laplacian L_Γ considered as a map $R^V \xrightarrow{L_\Gamma} R^V$ for other rings R, e.g. what is $\operatorname{rank}(L_\Gamma)$ when reduced mod p?

To answer this, one can work with $R = \mathbb{Z}$ and compute

$$\operatorname{coker}\left(\mathbb{Z}^V \xrightarrow{L_\Gamma} \mathbb{Z}^V\right)$$

$$:= \mathbb{Z}^V / \operatorname{im}(L_\Gamma)$$

$$\cong \mathbb{Z} \oplus K(\Gamma)$$

critical group or sandpile group

Alternatively, one can show that

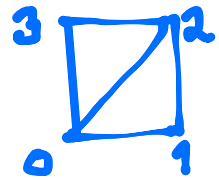
$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^{\ell} \xrightarrow{L_\Gamma} \mathbb{Z}^{\ell}\right)$$

with

$K(\Gamma)$ finite $\iff$ Γ connected

Kirchhoff's Thm. then implies

$\# K(\Gamma) = \tau(\Gamma) = \#$ spanning trees in Γ

EXAMPLE $\Gamma =$ 

has $L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 3 & -1 \\ 3 & -1 & 0 & -1 & 2 \end{array}$ with

$$\operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{L_\Gamma} \mathbb{Z}^4\right) \cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/8\mathbb{Z}}_{K(\Gamma)}$$

because one can compute L_Γ has

Smith normal form

$$P L_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $P, Q \in GL_4(\mathbb{Z})$

Alternatively, using the reduced Laplacian $\overline{L}_\Gamma$

$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/8\mathbb{Z}$$

via an equivalent Smith normal form calculation.

So, for example,

$$\operatorname{rank}_{\mathbb{F}_2}(L_\Gamma) = 2 \quad (\text{not } 0 \text{ or } 1)$$

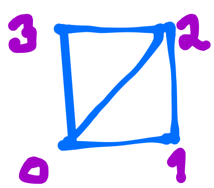
Why sandpile group?

The reduced Laplacian $\tilde{L}_\Gamma$ is an avalanche-finite matrix:

- entries in $\mathbb{Z}$
- off-diagonal entries ≤ 0
- invertible, with inverse entries ≥ 0

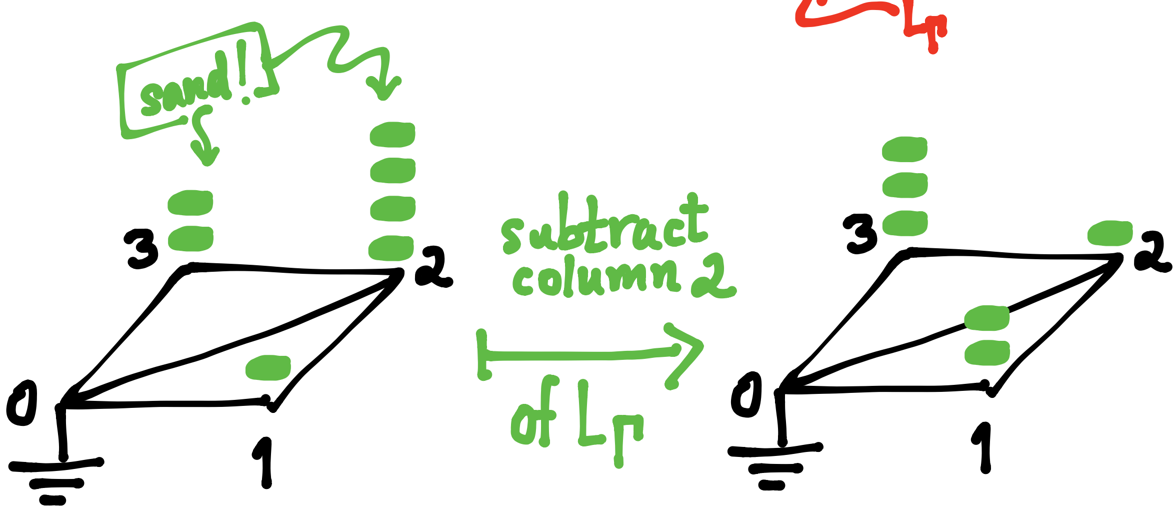
(Also known as nonsingular M-matrices)

This implies every vector $x \in \mathbb{N}^{\ell}$ can be brought via a finite sequence of steps that subtract columns of $\tilde{L}_{\Gamma}$, keeping it in $\mathbb{N}^{\ell}$, until no such subtraction is possible; x is stable.

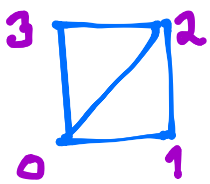
$\Gamma =$ 

$$L_\Gamma = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{c|ccc} 3 & -1 & -1 & -1 \\ \hline -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{array}\right] \end{array}$$

$\tilde{L}_\Gamma$

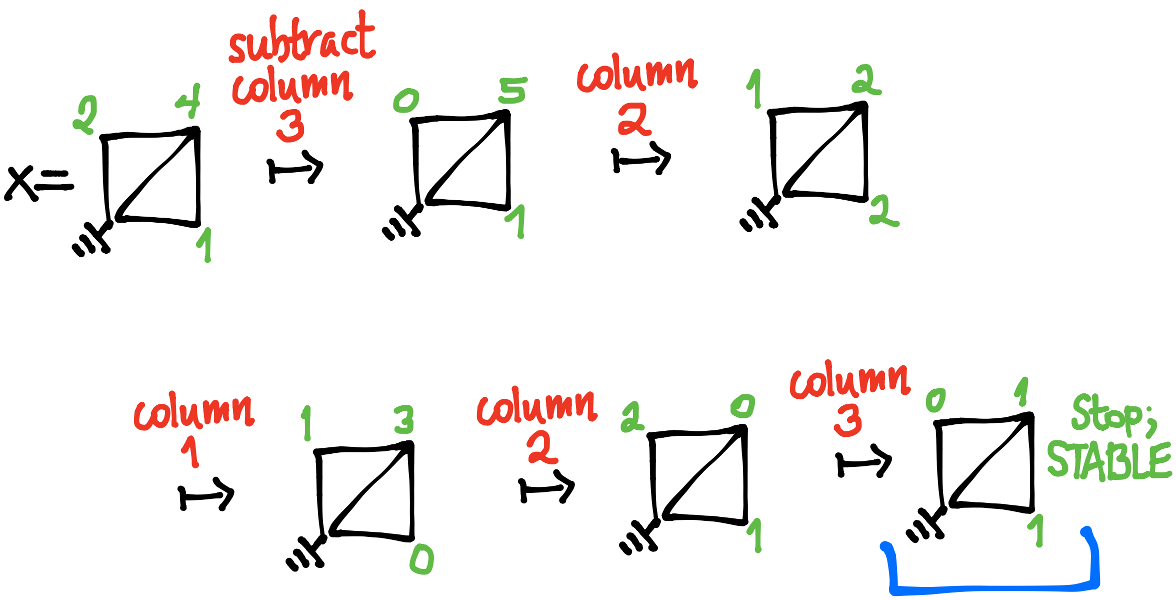


EXAMPLE

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

$\tilde{L}_\Gamma$

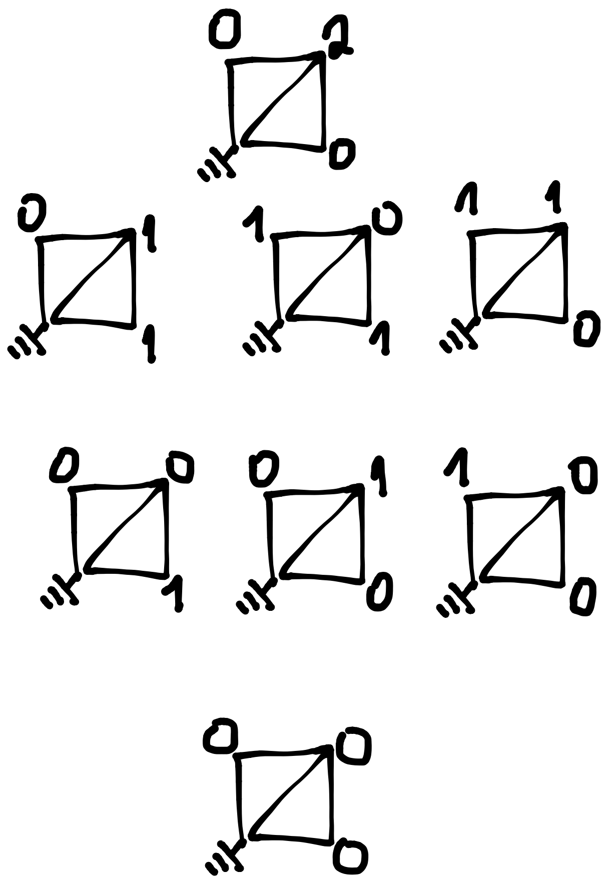
$x =$ 

The stabilization is unique, independent of choices of firings.

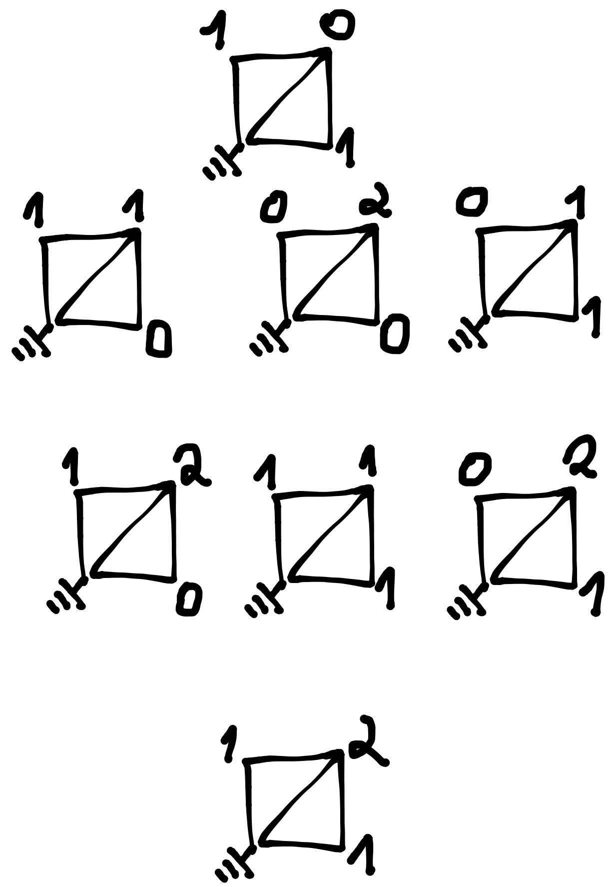
Leads to two interesting classes of coset representatives in $\mathbb{N}^\ell$ for $K(\Gamma) = \mathbb{Z}^\ell / \mathrm{im}\, L_\Gamma$

- critical configurations (= stable + recurrent)
- superstable configurations

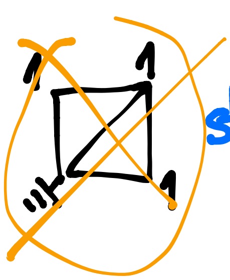
1987 Bak-Tang-Wiesenfeld
1990 Dhar
1991 Lorenzini
1993 Gabrielov*
2007 Baker-Norine
2009 Shokrieh*
2012 Levine*-Pegden-Smart
2013 Holroyd-Levine*-Mészáros*-Peres-Propp-Wilson
2013 Chan*(Swee Hong)
2014 Kiss*-Tóthmérész*
2016 Strichartz*+REUs



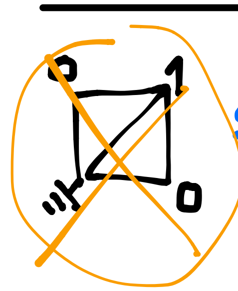
8 superstable configurations



8 critical configurations



stable, but not superstable



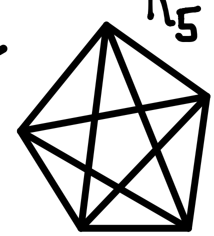
stable, but not recurrent

The exact structure of the sandpile group $K(\Gamma) = \mathbb{Z}^{\ell}/\mathrm{im}\,\overline{L}_{\Gamma}$ is known for very few graphs Γ, even when eigenvalues and eigenvectors and $\tau(\Gamma) = \# K(\Gamma)$ are easy.

(easy)

EXAMPLE Complete graphs K_n

K_5

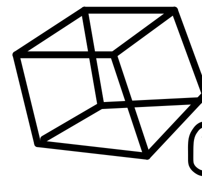


have $\tau(K_n) = n^{n-2}$

and $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$

(frustrating!)

EXAMPLE n-dimensional cubes Q_n



Q_3

have L_{Q_n} eigenspaces easy

and

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

The p-primary/p-Sylow structure of $K(Q_n)$ is known for p odd

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p \bigoplus_{k=1}^{n} \left(\mathbb{Z}/k\mathbb{Z}\right)^{\binom{n}{k}}$$

but for $p=2$

$\mathrm{Syl}_2 K(Q_n)$ is an unknown mess!

Now for (ordinary)

Finite group representations

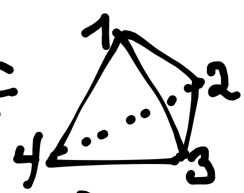
- G a finite group
- irreducible/simple complex G-representations / $\mathbb{C}G$-modules

  $\mathbb{1}_G = S_0, S_1, S_2, \ldots, S_\ell$

  trivial G-rep

- characters $\chi_0, \chi_1, \ldots, \chi_\ell$

EXAMPLE

$G = \mathcal{A}_4 =$ rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

$\omega = e^{2\pi i/3}$

DEFINITION:

Given a representation

$$G \xrightarrow{\rho} GL_n(\mathbb{C})$$

define its McKay matrix $M_\rho = (m_{ij})$ via

$$\left(\chi_{S_i \otimes \rho} =\right) \chi_i \, \chi_\rho = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

or

$$S_i \otimes \rho = \bigoplus_{j=0}^{\ell} S_j^{\oplus m_{ij}}$$

$$(\chi_{s_i \otimes \rho^=})\chi_i \chi_\rho = \sum_{j=0}^{\ell} m_{ij}\chi_j \quad \text{defines } M_\rho$$

Then…

- $L_\rho := nI_{\ell+1} - M_\rho$

- $\overline{L}_\rho := L_\rho - \begin{Bmatrix} \chi_0 \text{ row} \\ \chi_0 \text{ column} \end{Bmatrix}$

- $K(\rho) := \operatorname{coker}\left(\mathbb{Z}^\ell \xrightarrow{\overline{L}_\rho} \mathbb{Z}^\ell\right)$

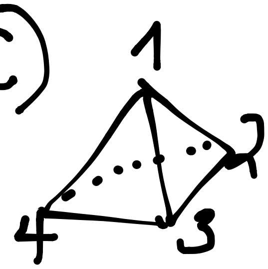
  sandpile group

  or

  $\mathbb{Z} \oplus K(\rho) = \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_\rho} \mathbb{Z}^{\ell+1}\right)$

EXAMPLE

$$G = \mathcal{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subseteq GL_3(\mathbb{C})$$

via rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\chi_\rho = \chi_3$	3	0	0	-1

$$\chi_0 \chi_\rho = \chi_1 \chi_\rho = \chi_2 \chi_\rho = \chi_\rho = 1\chi_3$$

$$\chi_3 \chi_\rho = 1\chi_0 + 1\chi_1 + 1\chi_2 + 2\chi_3$$

$$\rightsquigarrow \quad M_\rho = \begin{array}{c|cccc} & \chi_0 & \chi_1 & \chi_2 & \chi_3 \\ \hline \chi_0 & & & & \\ \chi_1 & & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} & & \\ \chi_2 & & & & \\ \chi_3 & & & & \end{array}$$

$$M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

McKay matrix

$$L_\rho = 3I_4 - M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

$$\overline{L}_\rho = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{normal form}]{\text{Smith}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$K(\rho) = \operatorname{coker} \overline{L}_\rho \cong \mathbb{Z}/3\mathbb{Z}$$

Sandpile group

Why is $\operatorname{coker} L_\rho = \mathbb{Z} \oplus \underbrace{\operatorname{coker} \overline{L}_\rho}_{K(\rho)}$?

L_ρ has

$$\bar{s} = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix} = \begin{bmatrix} 1 \\ \dim(S_1) \\ \vdots \\ \dim(S_\ell) \end{bmatrix}$$

as right- and left-nullvector.

EXAMPLE

$$L_\rho \bar{s} = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM

The inclusion $\mathbb{R}\bar{s} \subseteq \ker L_\rho$ is an equality

$\iff G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ is faithful

(analogous of Γ connected)

More generally, the columns of the character table

$$\bar{s}(g) := \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix},$$

and re-ordered columns

$$\bar{s}^*(g) := \begin{bmatrix} \chi_0^*(g) \\ \chi_1^*(g) \\ \vdots \\ \chi_\ell^*(g) \end{bmatrix}$$

give right- and left-eigenbases for M_ρ, L_ρ:

$$\sum_{j=0}^{\ell} m_{ij} \chi_j(g) = \chi_i(g) \, \chi_\rho(g)$$

$$\Longrightarrow \quad M_\rho \, \bar{s}(g) = \chi_\rho(g) \, \bar{s}(g)$$

$$L_\rho \, \bar{s}(g) = \underbrace{\left(n - \chi_\rho(g)\right)}_{\text{eigenvalues of } L_\rho} \bar{s}(g)$$

THEOREMS & EXAMPLES

THEOREM (Benkart–Klivans–R.)

For faithful abelian group reps $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$

$$K(\rho) = \underbrace{K(\Gamma)}_{\text{(di-)graph sandpile group}}$$

where Γ = Cayley digraph for the dual group $\check{G} = \mathrm{Hom}(G, \mathbb{C}^\times) = \{\chi_0, \chi_1, \ldots, \chi_\ell\}$

with respect to generators $\{\chi_{i_1}, \ldots, \chi_{i_n}\}$,

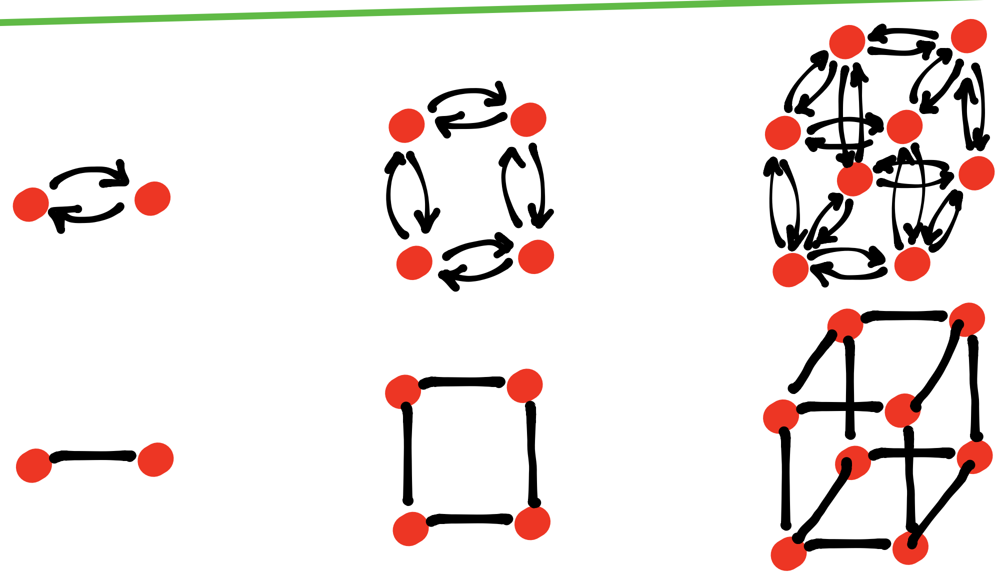
if $\chi_\rho = \chi_{i_1} + \ldots + \chi_{i_n}$.

EXAMPLE

$$G = (\mathbb{Z}/2\mathbb{Z})^n \xhookrightarrow{\rho} GL_n(\mathbb{C})$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \longmapsto \begin{bmatrix} (-1)^{\epsilon_1} & & 0 \\ & (-1)^{\epsilon_2} & \\ 0 & & \ddots \\ & & & (-1)^{\epsilon_n} \end{bmatrix}$$

has $K(\rho) = K(\overset{n\text{-cube}}{Q_n})$



The analogue of $\#K(\Gamma) = \tau(\Gamma) = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell+1}$ is

THEOREM (Gaetz) For any faithful representation ρ of G,

$$\#K(\rho) = \frac{1}{\#G} \cdot \prod_{\substack{G\text{-conj. classes} \\ [g] \neq \{e\}}} (n - \chi_\rho(g))$$

EXAMPLE $G = \mathfrak{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$

had $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

	e	(123)	(132)	$(12)(34)$
χ_ρ	3	0	0	-1

$$\#K(\rho) = \frac{1}{12}(3-0)(3-0)(3-(-1)) = 3$$

THEOREM (Gaetz) If $n = \#G$,

regular representation of G

$$G \xhookrightarrow{\ reg_G\ } GL_n(\mathbb{C}) \quad \text{has}$$

$$K(reg_G) \cong (\mathbb{Z}/n\mathbb{Z})^{\#\left(\substack{G\text{-conjugacy}\\ \text{classes}}\right)-2}$$

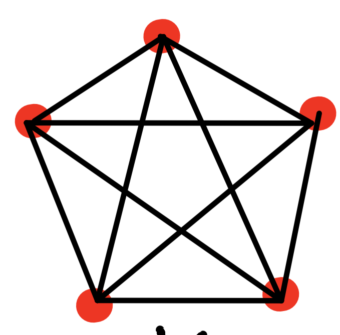
$\Big\downarrow$ G abelian

$$K(reg_G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

$\Big\downarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

complete graph



K_5

THEOREM (Benkart-Klivans-R)

For **faithful** G-reps ρ, $\tilde{L}_\rho$ is an **avalanche-finite** matrix, so one can compute in $K(\rho) = \mathrm{coker}(\tilde{L}_\rho)$ via toppling with **superstable** or **critical** coset representatives in $\mathbb{N}^\ell$

EXAMPLE (continued)

$$\tilde{L}_\rho = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline x_1 & 3 & 0 & -1 \\ x_2 & 0 & 3 & -1 \\ x_3 & -1 & -1 & 1 \end{array} \quad \text{with } K(\rho) = \mathbb{Z}/3\mathbb{Z}$$

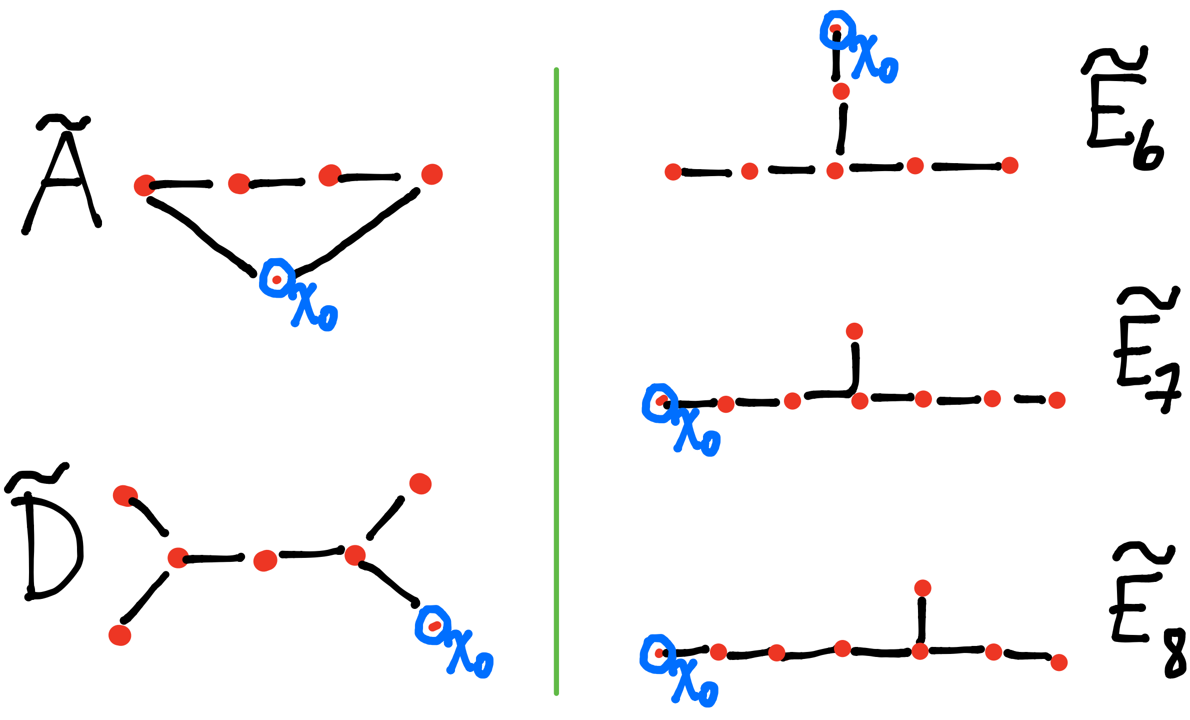
superstables (x_1 x_2 x_3)	criticals (x_1 x_2 x_3)
[0 0 0]	[2 2 0]
[1 0 0]	[1 2 0]
[0 1 0]	[2 1 0]

McKay's original theorem (1980)

When $G \xhookrightarrow{\rho} SL_2(\mathbb{C})$, then

$\overline{L}_\rho$, L_ρ are the Cartan, extended Cartan

matrices for a simply-laced root system Φ



THEOREM (Benkart-Klivans-R)

In McKay's $G \xrightarrow{\rho} SL_2(\mathbb{C})$ setting

$$K(\rho) \cong \mathrm{Hom}(G, \mathbb{C}^\times)$$
$$= \text{1-dim'l characters } \chi_i \text{ of } G$$

$$\left(\overset{\text{Pontryagin dual}}{\cong} \underbrace{G^{ab}}_{\text{abelianization of } G} = G/[G,G] \right)$$

$$\left(\cong \underbrace{P(\Phi)}_{\text{weight lattice}} \Big/ \underbrace{Q(\Phi)}_{\text{root lattice}} \cong \underbrace{\pi_1 \begin{pmatrix} \text{adjoint form} \\ \text{of compact} \\ \text{Lie group} \\ \text{associated to } \Phi \end{pmatrix}}_{\text{fundamental group of } \Phi} \right)$$

THEOREM (Benkart-Klivans-R)

More generally, when $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$

one has a surjection

$$K(\rho) \twoheadrightarrow \mathrm{Hom}(G, \mathbb{C}^\times)$$

THEOREM (Gaetz)

When $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$,

$$\left\{ \begin{array}{c} \text{1-dimensional} \\ \text{characters } \chi_i \end{array} \right\} \hookrightarrow \left\{ \begin{array}{c} \text{superstable} \\ \text{configurations} \\ \text{for } \bar{L}_\rho \end{array} \right\}$$

$$\begin{array}{c} \| \\ \mathrm{Hom}(G, \mathbb{C}^\times) \end{array}$$

Do we really need complex G-representations that is, $\mathbb{C}G$-modules ?

Why not representations
$$G \xrightarrow{\rho} GL_n(\mathbb{F}),$$
that is, $\mathbb{F}G$-modules ?

Much of it works replacing $A = \mathbb{F}G$ with...

Hopf algebras

Let A be a

finite dimensional Hopf algebra

over an algebraically closed field $\mathbb{F}$

so it has product $A \otimes A \xrightarrow{\mu} A$

and A-modules V,

but also...

- coproduct $A \xrightarrow{\Delta} A \otimes A$
- counit $A \xrightarrow{\epsilon} \mathbb{F}$
- antipode $A \xrightarrow{\alpha} A$

defines

$V \otimes W$

Trivial A-mod S_0 on $\mathbb{F}$

Left and right duals $^*V$, V^*

EXAMPLE

$A = \mathbb{F}G =$ group algebra for a finite group G, with

- coproduct $g \xmapsto{\Delta} g \otimes g$
- counit $g \xmapsto{\epsilon} 1$
- antipode $g \xmapsto{\alpha} g^{-1}$

$\mathbb{Z}^{\ell+1} \cong$ virtual characters of G

$\mathbb{Z}^{\ell+1} \cong G_0(A) =$ Grothendieck group of A-modules

$\left(\begin{array}{l} 0 \to U \to V \to W \to 0 \\ \quad \Rightarrow [V] = [U] + [W] \end{array} \right)$

with $\mathbb{Z}$-basis $[S_0], [S_1], \ldots, [S_\ell]$

where $S_0, S_1, \ldots, S_\ell$ are the simple A-mods

and $[V] = \sum_{i=0}^{\ell} [V : S_i]\, [S_i]$.

composition multiplicity of S_i in V

$G_0(A)$ has multiplication from

$[V][W] := [V \otimes W]$.

DEFINITION: For an A-module V, let

- $M_V \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$ express the map

  McKay matrix

$$\begin{array}{ccc} G_0(A) & \xrightarrow{(-)\cdot[V]} & G_0(A) \\ \cong & & \cong \\ \mathbb{Z}^{\ell+1} & & \mathbb{Z}^{\ell+1} \end{array}$$

that is, $(M_V)_{i,j} := [S_j \otimes V : S_i]$

- $L_V = nI_{\ell+1} - M_V$ where $n := \dim V$

- $\mathbb{Z} \oplus K(V) := \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_V} \mathbb{Z}^{\ell+1}\right)$

  sandpile group

When is $K(V)$ finite, generalizing $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ being faithful ?

THEOREM (Grinberg-Huang-R)

$K(V)$ is finite

$\iff$ V is tensor-rich

every A-simple S_i occurs in at least one $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k\text{-fold}}$

$\iff$ $\overline{L}_V := L_V - \left\{ \begin{array}{c} \text{row, column} \\ \text{indexed by} \\ S_0 = \epsilon \end{array} \right\}$ avalanche-finite

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REMARK: For $\mathbb{C}G$-modules V

V tensor-rich $\underset{\text{Burnside}}{\iff}$ V faithful

REMARK: In general,

$$\operatorname{coker}(L_V) = \mathbb{Z} \oplus \underbrace{K(V)}_{\not\cong \, \operatorname{coker}(\bar{L}_V)}$$

unless A is semisimple as an algebra

But in the semisimple case, one can again compute in

$$K(V) = \operatorname{coker}(\bar{L}_V)$$

via sandpiles in $\mathbb{N}^\ell$ and $\bar{L}_V$.

Recall

$$A \cong \bigoplus_{i=0}^{\ell} P_i^{\oplus \dim S_i}$$

left-regular A-module

where $P_0, P_1, \ldots, P_\ell$ are the indecomposable projective A-modules (so P_i = projective cover of S_i)

PROPOSITION

Let $\bar{s} := [s_0, s_1, \ldots, s_\ell]^t$ where $s_i := \dim S_i$
$\bar{p} := [p_0, p_1, \ldots, p_\ell]^t$ $p_i := \dim P_i$

Then $\bar{p}, \bar{s}$ are left, right nullvectors for L_V.

PROPOSITION For $A = \mathbb{F}G$ one knows all the eigenspaces:

the Brauer character table columns

$$\bar{s}(g) := [\chi_{S_0}(g), \ldots, \chi_{S_\ell}(g)]^t \text{ for } p\text{-regular } g \in G$$

and

(permuted) indecomposable projective Brauer character table columns

$$\bar{p}(g) := [\chi_{*P_0}(g), \ldots, \chi_{*P_\ell}(g)]^t$$

give left and right eigenbases for L_V.

For tensor-rich A-modules V, what is $\#K(V)$?

A lemma of Lorenzini implies this:

PROPOSITION If L_V has eigenvalues $0=\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\ell$ then

$$\#K(V) = \frac{\gamma(A)}{\dim A} \lambda_1 \lambda_2 \cdots \lambda_\ell$$

where $\gamma(A) := \gcd\{\dim P_i\}_{i=0,1,\ldots,\ell}$

QUESTION:

Does $\gamma(A)$ have more meaning in terms of structure of A?

PROPOSITION: For $A = \mathbb{F}G$, with $\operatorname{char}\mathbb{F} = p$

$\gamma(A)$ = the size of any p-Sylow subgroup

$= p^a$ where $\#G = p^a q$

with $\gcd(p,q) = 1$

COROLLARY: For $A = \mathbb{F}G$,

and an A-module V of dimension n,

$$\#K(V) = \frac{p^a}{\#G} \prod_{\substack{p\text{-regular} \\ G\text{-conj. classes} \\ [g] \neq \{e\}}} \left(n - \underbrace{\chi_V}_{\text{Brauer character}}(g) \right)$$

The left regular A-module A itself is always tensor-rich.

THEOREM (Grinberg-Huang-R)

For any finite dim'l Hopf algebra A,

$$K(A) \cong \mathbb{Z}/\gamma\mathbb{Z} \oplus \left(\mathbb{Z}/d\mathbb{Z}\right)^{\ell-1}$$

where $\gamma := \gamma(A)$

$d := \dim A$

$\ell := \#\{\text{non-trivial simple } A\text{-modules } S_1, \ldots, S_\ell\}$

Various questions on finite-dimensional Hopf algebras ensue...

Thanks for
your
attention!