Sandpiles and Representation Theory

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(joint with Benkart & Klivans,
Gaetz,
Grinberg & Huang)

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OUTLINE

Laplacian & sandpile group for a...

- ... graph
- ... group representation
- ... module over a Hopf algebra

Graphs

$\Gamma = (V, E)$ an undirected multigraph

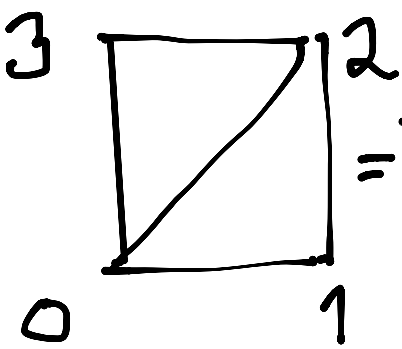
$V = \{0, 1, 2, \dots, \ell\}$

$$L_\Gamma := \underbrace{D_\Gamma}_{\text{diagonal matrix of vertex degrees}} - \underbrace{A_\Gamma}_{\text{adjacency matrix}}$$

L_Γ: graph Laplacian

$$(L_\Gamma)_{ij} = \begin{cases} \deg_\Gamma(i) & \text{if } i = j \\ -\#\{\text{edges } i \text{ to } j\} & \text{if } i \neq j \end{cases}$$

EXAMPLE



$= \Gamma$

$$L_\Gamma = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

The graph Laplacian L_Γ

- is positive semidefinite
  ($L_\Gamma = \partial\partial^T$ where $\mathbb{R}^E \xrightarrow{\partial} \mathbb{R}^V$)

- has $\mathbb{R}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \subseteq \ker(L_\Gamma)$

- equality $\boxed{\text{here}}$ $\Leftrightarrow$ Γ connected

- From spectrum (=eigenvalues) of L_Γ

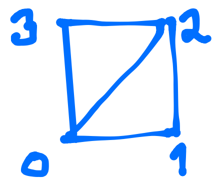
$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\ell$$

one can count the spanning trees in Γ:

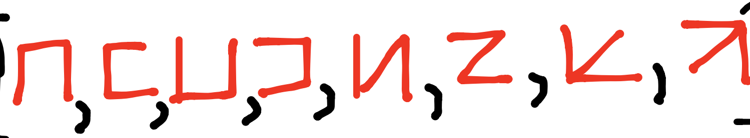
$$\tau(\Gamma) = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell + 1}$$

- Alternatively,

$$\tau(\Gamma) \underset{\substack{\uparrow \\ \text{Kirchhoff's} \\ \text{Matrix-Tree} \\ \text{Theorem} \\ (1845)}}{=} \det\Bigg(\underbrace{L_\Gamma - \left\{ \begin{matrix} 0^{\text{th}} \text{ row}, \\ 0^{\text{th}} \text{ column} \end{matrix} \right\}}_{\substack{\text{reduced Laplacian} \\ \overline{L_\Gamma}}} \Bigg)$$

EXAMPLE $\Gamma =$ 

has

$\tau(\Gamma) = \#\{$  $\} = 8$

$$L_\Gamma = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

has eigenvalues

$$\underset{\lambda_0}{0} \leq \underset{\lambda_1}{2} \leq \underset{\lambda_2}{4} \leq \underset{\lambda_3}{4}$$

So $\tau(\Gamma) = \dfrac{\lambda_1\lambda_2\lambda_3}{4} = \dfrac{2\cdot 4\cdot 4}{4} = 8$ ✓

Or, $\tau(\Gamma) = \det \overline{L}_\Gamma = \det \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$$= 2(3\cdot 2 - 1) + (-1\cdot 2 - 0)$$

$$= 10 - 2 = 8 \ ✓$$

REMARK:

Eigenvalues of L_Γ are known for many families of graphs, letting one compute $\tau(\Gamma)$:

graphs with large symmetry

- complete graphs, complete multipartite graphs
- distance-regular graphs

graphs with inductive structure

- threshold graphs, co-graphs
- cubes, Cartesian products

REMARK

Eigenvectors of L_Π are also important in applications to

- optimal graph-drawing
- clustering of data

(see articles and surveys by Dan Spielman)

What about the Laplacian L_Γ considered as a map $R^V \xrightarrow{L_\Gamma} R^V$ for other rings R, e.g. what is $\operatorname{rank}(L_\Gamma)$ when reduced mod p?

To answer this, one can work with $R=\mathbb{Z}$ and compute

$$\operatorname{coker}\left(\mathbb{Z}^V \xrightarrow{L_\Gamma} \mathbb{Z}^V\right)$$

$$:= \mathbb{Z}^V / \operatorname{im}(L_\Gamma)$$

$$\cong \mathbb{Z} \oplus K(\Gamma)$$

critical group
or sandpile group

Alternatively, one can show that

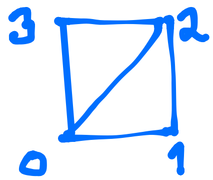
$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^{\ell} \xrightarrow{L_\Gamma} \mathbb{Z}^{\ell}\right)$$

with

$K(\Gamma)$ finite $\iff$ Γ connected

in which case, Kirchhoff's Thm. implies

$$\# K(\Gamma) = \tau(\Gamma) = \#\text{spanning trees in } \Gamma$$

EXAMPLE $\Gamma =$  has

$$L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & \multicolumn{4}{c}{\multirow{4}{*}{}} \end{array}\quad L_\Gamma = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

with $\operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{L_\Gamma} \mathbb{Z}^4\right)$

$$\cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/8\mathbb{Z}}_{K(\Gamma)}$$

because L_Γ has Smith normal form

$$P L_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad P, Q \in GL_4(\mathbb{Z})$$

Alternatively, using reduced Laplacian $\overline{L}_\Gamma$

$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/8\mathbb{Z}$$

via an equivalent Smith calculation.

So, e.g., $\operatorname{rank}_{\mathbb{F}_2}(L_\Gamma) = 2$ (not 0 or 1)

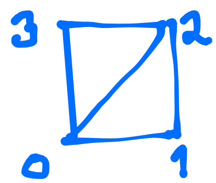
Why sandpile group?

The reduced Laplacian $\bar{L}_\Gamma$ is an avalanche-finite matrix:

- entries in $\mathbb{Z}$
- off-diagonal entries ≤ 0
- invertible, with inverse entries ≥ 0

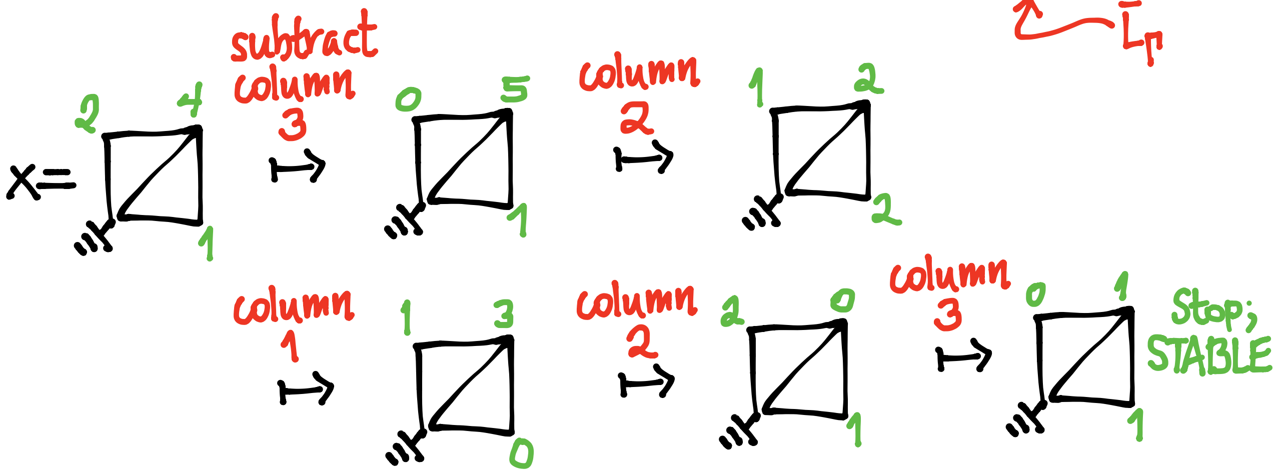
This implies every vector $x \in \mathbb{N}^\ell$ can be brought via a finite sequence of steps that subtract columns of $\bar{L}_\Gamma$, keeping it in $\mathbb{N}^\ell$, until no such subtraction is possible; x is stable.

EXAMPLE

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 3 & -1 \\ 3 & -1 & 0 & -1 & 2 \end{array}$$

$\overline{L}_\Gamma$

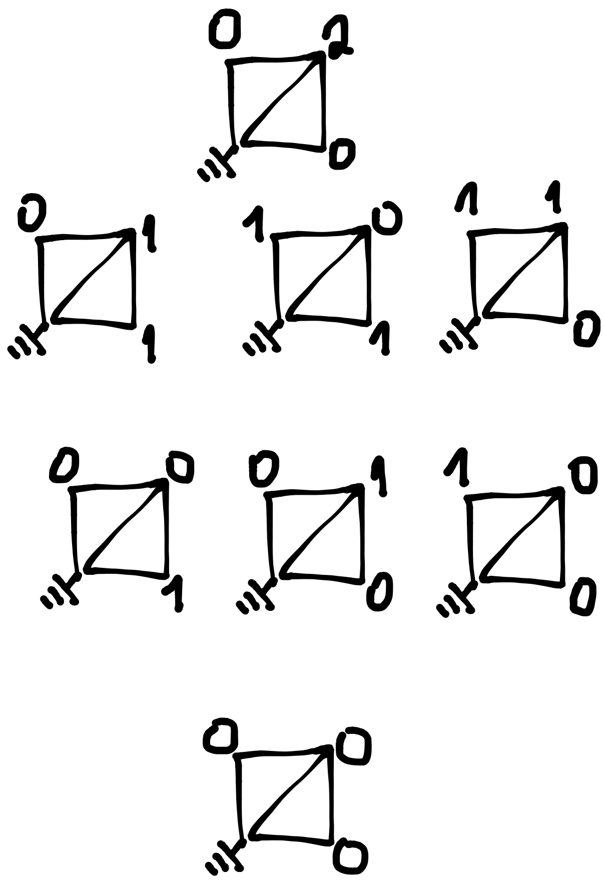


Leads to two interesting classes of coset representatives for

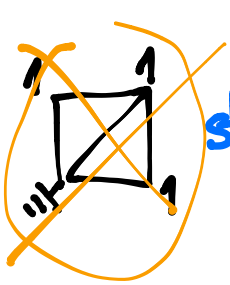
$$K(\Gamma) = \mathbb{Z}^\ell / \operatorname{im} \overline{L}_\Gamma$$

lying in $\mathbb{N}^\ell$

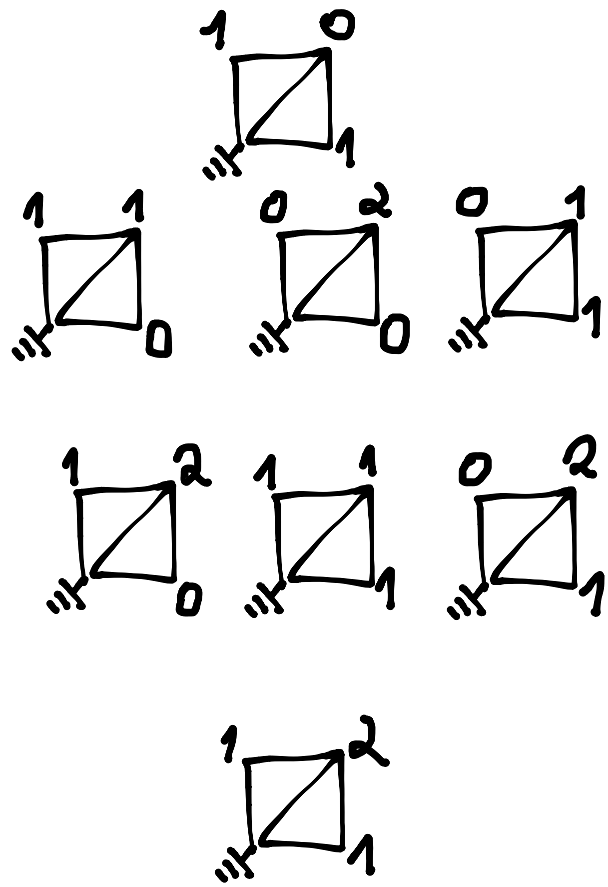
- critical (= stable, recurrent) configurations
- superstable configurations



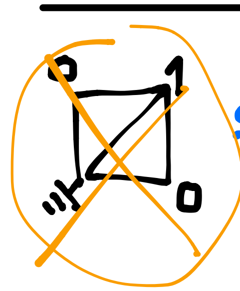
superstable configurations



stable, but not superstable



critical configurations



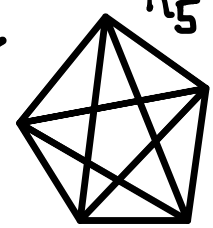
stable, but not recurrent

The exact structure of the sandpile group $K(\Gamma)=\mathbb{Z}^{\ell}/\mathrm{im}\,\overline{L}_{\Gamma}$ is not known for very many graphs Γ, including some where $\tau(\Gamma)$ and spectrum of L_{Γ} is known completely.

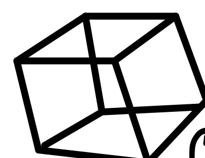
EXAMPLE Complete graphs K_n have $K(K_n)\cong(\mathbb{Z}/n\mathbb{Z})^{n-2}$

$$\Rightarrow \quad \tau(K_n)=n^{n-2} \text{ (Cayley)}$$

K_5



EXAMPLE n-dimensional cubes Q_n have known L_{Q_n} spectrum and p-primary/p-Sylow structure of $K(Q_n)$ known for p odd, but an unknown mess for $p=2$!



Q_3

Now for (ordinary)

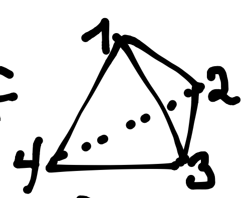
Finite group representations

- G a finite group
- irreducible/simple complex G-representations/$\mathbb{C}G$-modules

  $\mathbb{1}_G = S_0, S_1, S_2, \ldots, S_\ell$

  trivial G-rep
- characters $\chi_0, \chi_1, \ldots, \chi_\ell$

EXAMPLE

$G = \mathcal{A}_4$ = rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

$\omega = e^{2\pi i/3}$

DEFINITION: Given a representation $G \xrightarrow{\rho} GL_n(\mathbb{C})$, define...

- McKay matrix $M_\rho = (m_{ij})$

$$\left(\chi_{S_i \otimes \rho} =\right) \chi_i \chi_\rho = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

- $L_\rho := n I_{\ell+1} - M_\rho$

- $\overline{L}_\rho := L_\rho - \begin{Bmatrix} \chi_0 \text{ row} \\ \chi_0 \text{ column} \end{Bmatrix}$

- $K(\rho) := \operatorname{coker}\left(\mathbb{Z}^\ell \xrightarrow{\overline{L}_\rho} \mathbb{Z}^\ell\right)$

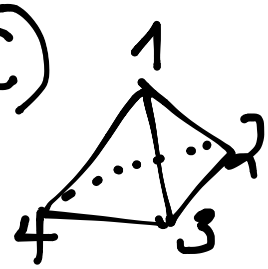
sandpile group

or

$$\mathbb{Z} \oplus K(\rho) = \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_\rho} \mathbb{Z}^{\ell+1}\right)$$

EXAMPLE

$$G = \mathcal{A}_4 \xhookrightarrow{\rho} SO_3(\mathbb{R}) \subseteq GL_3(\mathbb{C})$$

via rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\chi_\rho = \chi_3$	3	0	0	-1

.

$$\chi_0\chi_\rho = \chi_1\chi_\rho = \chi_2\chi_\rho = \chi_\rho = 1\chi_3$$

$$\chi_3\chi_\rho = 1\chi_0 + 1\chi_1 + 1\chi_2 + 2\chi_3$$

$$\rightsquigarrow \quad M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

McKay matrix

$$L_\rho = 3I_4 - M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

$$\overline{L}_\rho = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{normal form}]{\text{Smith}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$K(\rho) = \operatorname{coker} \overline{L}_\rho \cong \mathbb{Z}/3\mathbb{Z}$$

Sandpile group

Why is $\operatorname{coker} L_\rho = \mathbb{Z} \oplus \underbrace{\operatorname{coker} \overline{L}_\rho}_{K(\rho)}$?

Because L_ρ has

$$\bar{s} = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix} = \begin{bmatrix} 1 \\ \dim(S_1) \\ \vdots \\ \dim(S_\ell) \end{bmatrix}$$

as both right- and left-nullvector.

EXAMPLE

$$L_\rho \bar{s} = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM

The inclusion $\mathbb{R}\bar{s} \subseteq \ker L_\rho$ is an equality

$\iff G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ is faithful

More generally, the columns of the character table

$$\bar{s}(g) := \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix},$$

and re-ordered columns

$$\bar{s}^*(g) := \begin{bmatrix} \chi_0^*(g) \\ \chi_1^*(g) \\ \vdots \\ \chi_\ell^*(g) \end{bmatrix}$$

give right- and left-eigenbases for M_ρ, L_ρ:

$$\sum_{j=0}^{\ell} m_{ij} \chi_j(g) = \chi_i(g)\, \chi_\rho(g)$$

$$\Rightarrow \quad M_\rho\, \bar{s}(g) = \chi_\rho(g)\, \bar{s}(g)$$

$$L_\rho\, \bar{s}(g) = \underbrace{\left(n - \chi_\rho(g)\right)}_{\text{eigenvalues of } L_\rho} \bar{s}(g)$$

THEOREMS & EXAMPLES

THEOREM (Benkart-Klivans-R.)

For faithful abelian group reps $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$

$$K(\rho) = \underbrace{K(\Gamma)}_{\substack{\text{(di-)graph} \\ \text{sandpile group}}}$$

where Γ = Cayley digraph for the dual group $\check{G} = \mathrm{Hom}(G, \mathbb{C}^\times) = \{\chi_0, \chi_1, \ldots, \chi_\ell\}$

with respect to generators $\{\chi_{i_1}, \ldots, \chi_{i_n}\}$,

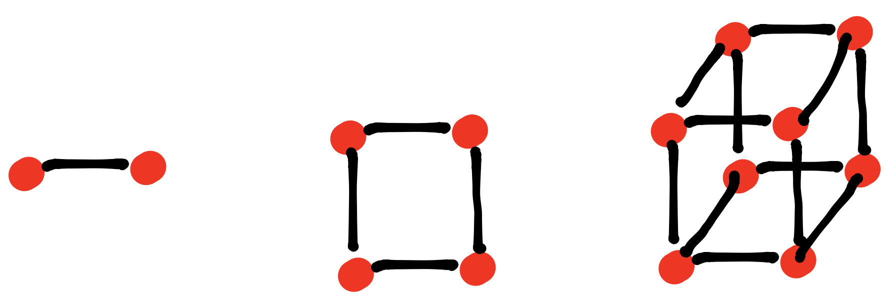
if $\chi_\rho = \chi_{i_1} + \cdots + \chi_{i_n}$.

EXAMPLE

$$G = (\mathbb{Z}/2\mathbb{Z})^n \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \longmapsto \begin{bmatrix} (-1)^{\epsilon_1} & & & 0 \\ & (-1)^{\epsilon_2} & & \\ & & \ddots & \\ 0 & & & (-1)^{\epsilon_n} \end{bmatrix}$$

has $K(\rho) = K(\overset{n\text{-cube}}{Q_n})$



THEOREM (Gaetz) For any faithful representation ρ of G,

$$\#K(\rho) = \frac{1}{\#G} \cdot \prod_{\substack{G\text{-conj. classes} \\ [g] \neq \{e\}}} \left(n - \chi_\rho(g)\right)$$

EXAMPLE $G = \mathfrak{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$

had $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

	e	(123)	(132)	$(12)(34)$
χ_ρ	3	0	0	-1

$$\#K(\rho) = \frac{1}{12}(3-0)(3-0)(3-(-1)) = 3$$

EXAMPLE (Goetz) If $n = \#G$,
regular representation of G

$$G \xhookrightarrow{\mathrm{reg}_G} GL_n(\mathbb{C}) \quad \text{has}$$

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{\#\binom{G\text{-conjugacy}}{\text{classes}} - 2}$$

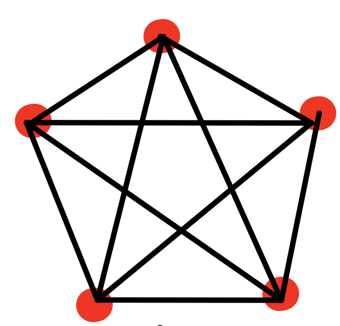
$\rightsquigarrow$ G abelian

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

$\rightsquigarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

complete graph



K_5

THEOREM (Benkart-Klivans-R)

For faithful G-reps ρ,
$\overline{L}_\rho$ is an avalanche-finite matrix,
so one can compute in
$K(\rho) = \mathrm{coker}(\overline{L}_\rho)$ via toppling
with superstable or critical
coset representatives in $\mathbb{N}^\ell$

EXAMPLE (continued)

$$\overline{L}_\rho = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \quad \text{with } K(\rho) = \mathbb{Z}/3\mathbb{Z}$$

x_1	x_2	x_3
0	0	0
1	0	0
0	1	0

superstables

x_1	x_2	x_3
2	2	0
1	2	0
2	1	0

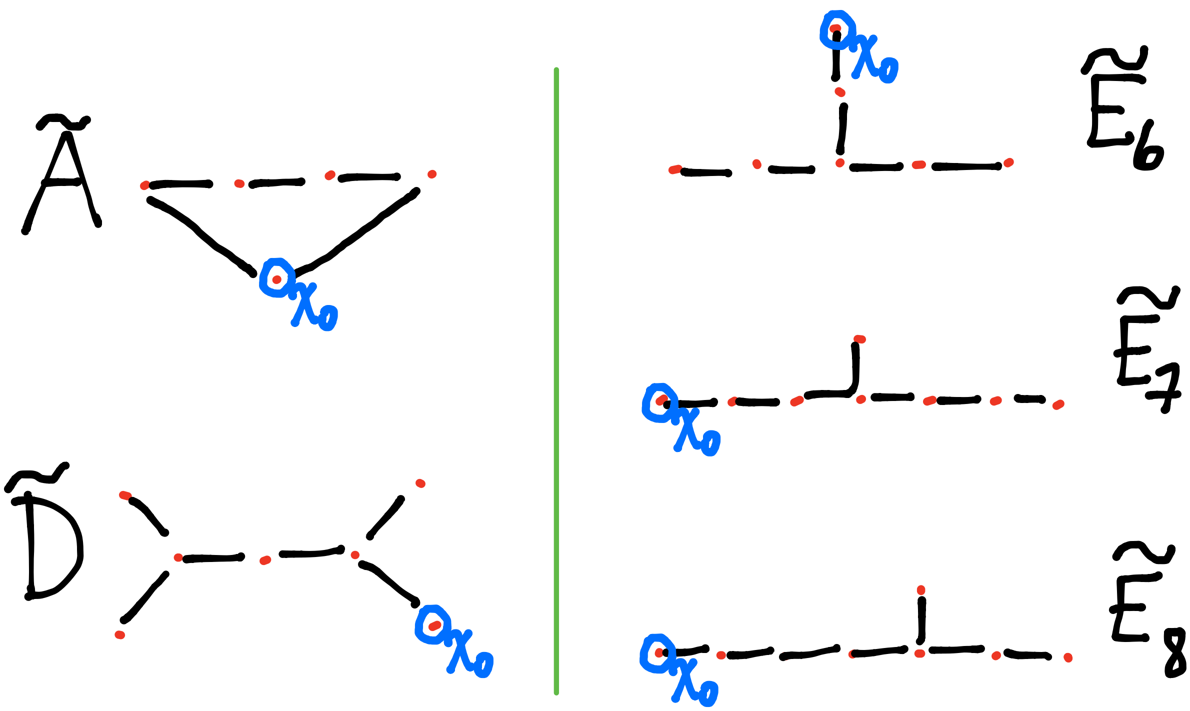
criticals

McKay's original theorem (1980)

When $G \xhookrightarrow{\rho} SL_2(\mathbb{C})$, then

$\overline{L}_\rho$, L_ρ are the Cartan, extended Cartan

matrices for a simply-laced root system Φ



THEOREM (Benkart-Klivans-R)

In McKay's $G \overset{\rho}{\hookrightarrow} SL_2(\mathbb{C})$ setting

$$K(\rho) \cong \underset{\substack{\uparrow \\ \text{abelianization} \\ \text{of } G}}{G^{ab}} = G/[G,G]$$

$$\left[\begin{array}{l} \cong \underset{\substack{\text{weight} \\ \text{lattice}}}{P(\Phi)} \Big/ \underset{\substack{\text{root} \\ \text{lattice}}}{Q(\Phi)} \\[2em] \cong \pi_1\left(\begin{array}{c}\text{adjoint form} \\ \text{of compact} \\ \text{Lie group} \\ \text{associated to } \Phi\end{array}\right) \\ \quad\quad \text{\underline{fundamental group} of } \Phi \end{array} \right]$$

Hopf algebras

Let A be a finite dimensional algebra over an algebraically closed field $\mathbb{F}$

$$\Longrightarrow \quad \underset{\text{left-regular } A\text{-module}}{A} \cong \bigoplus_{i=0}^{\ell} P_i^{\oplus \dim S_i}$$

where $S_0, S_1, \dots, S_\ell$ are the simple A-modules

$P_0, P_1, \dots, P_\ell$ the indecomposable projective A-modules

Now assume A is also a Hopf algebra:

- coproduct $A \xrightarrow{\Delta} A \otimes A$ — defines A-mod $V \otimes W$
- counit $A \xrightarrow{\epsilon} \mathbb{F}$ — Trivial A-mod S_0 on $\mathbb{F}$
- antipode $A \xrightarrow{\alpha} A$ — Left and right duals ${}^*V$, V^*

EXAMPLE $A = \mathbb{F}G$ = group algebra for a finite group G, with

- coproduct $g \mapsto g \otimes g$
- counit $g \mapsto 1$
- antipode $g \mapsto g^{-1}$

Instead of working with characters χ_V, work in Grothendieck group $G_0(A)$ where A-module sequences

$$0 \to U \to V \to W \to 0$$

give relations $[V] = [U] + [W]$.

Then $G_0(A) \cong \mathbb{Z}^{l+1}$ with $\mathbb{Z}$-basis $[S_0], [S_1], \ldots, [S_l]$

and $$[V] = \sum_{i=0}^{l} \underbrace{[V:S_i]}_{\substack{\text{composition} \\ \text{multiplicity of} \\ S_i \text{ in } V}} [S_i].$$

One has multiplication from

$$[V][W] := [V \otimes W].$$

DEFINITION: For an A-module V, let

- $\underset{\text{McKay matrix}}{M_V} \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$ express the map

$$\begin{array}{ccc} G_0(A) & \xrightarrow{(-)\cdot[V]} & G_0(A) \\ \cong & & \cong \\ \mathbb{Z}^{\ell+1} & & \mathbb{Z}^{\ell+1} \end{array}$$

that is, $(M_V)_{i,j} := [S_j \otimes V : S_i]$

- $L_V = nI_{\ell+1} - M_V$ where $n := \dim V$

- $\mathbb{Z} \oplus \underset{\text{sandpile group}}{K(V)} := \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_V} \mathbb{Z}^{\ell+1}\right)$

When is $K(V)$ finite?

Need to generalize $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ being faithful:

THEOREM (Grinberg-Huang-R)

$K(V)$ is finite

$\iff$ V is tensor-rich

every A-simple S_i occurs in at least one $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k\text{-fold}}$

$\iff$ $\overline{L}_V := L_V - \left\{ \begin{array}{c} \text{row, column} \\ \text{indexed by} \\ S_0 = \epsilon \end{array} \right\}$ is avalanche-finite

REMARK: For $\mathbb{C}G$-modules V

V tensor-rich $\underset{\text{Burnside}}{\iff}$ V faithful

REMARK: In general,

$$\mathbb{Z} \oplus \underbrace{K(V)}_{\substack{\not\cong \\ \operatorname{coker}(\overline{L}_V)}} = \operatorname{coker}(L_V)$$

unless A is semisimple as an algebra

$$\left(\text{e.g. } A = \mathbb{F}G \text{ with } \#G \in \mathbb{F}^{\times}\right).$$

But in the semisimple case, one can again compute in $K(V) = \operatorname{coker}(\overline{L}_V)$ via sandpiles in $\mathbb{N}^{\ell}$ and $\overline{L}_V$.

Nullvectors & eigenvectors

PROPOSITION

Let $\bar{s} := [s_0, s_1, \dots, s_\ell]^t$ where $s_i = \dim S_i$

$\bar{p} := [p_0, p_1, \dots, p_\ell]^t$ $p_i = \dim P_i$

Then $\bar{p}, \bar{s}$ are left, right nullvectors for L_V.

PROPOSITION For $A = \mathbb{F}G$, the Brauer character table columns

$\bar{s}(g) := [\chi_{S_0}(g), \dots, \chi_{S_\ell}(g)]^t$ for p-regular $g \in G$

and (permuted) indecomposable projective Brauer character table columns

$$\bar{p}(g) := [\chi_{{}^*P_0}(g), \dots, \chi_{{}^*P_\ell}(g)]^t$$

give left and right eigenbases for L_V.

For tensor-rich A-modules V,
what is $\#K(V)$?

A lemma of Lorenzini implies this:

PROPOSITION If L_V has eigenvalues $0=\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\ell$ then

$$\#K(V) = \frac{\gamma(A)}{\dim A} \lambda_1 \lambda_2 \cdots \lambda_\ell$$

where $\gamma(A) := \gcd\{\dim P_i\}_{i=0,1,\ldots,\ell}$

QUESTION:
Does $\gamma(A)$ have more meaning in terms of structure of A?

PROPOSITION: For $A = \mathbb{F}G$, with $\operatorname{char}\mathbb{F} = p$

$\gamma(A)$ = the size of any p-Sylow subgroup $= p^a$ where $\#G = p^a q$ with $\gcd(p,q)=1$

COROLLARY: For $A = \mathbb{F}G$, and an A-module V of dimension n,

$$\#K(V) = \frac{p^a}{\#G} \prod_{\substack{p\text{-regular}\\ G\text{-conj. classes}\\ [g]\neq\{e\}}} \left(n - \chi_V(g)\right)$$

Brauer character

The left regular A-module A itself is always tensor-rich.

THEOREM (Grinberg-Huang-R)

For any finite dim'l Hopf algebra A,

$$K(A) \cong \mathbb{Z}/\gamma\mathbb{Z} \oplus \left(\mathbb{Z}/d\mathbb{Z}\right)^{\ell-1}$$

where $\gamma := \gamma(A)$

$d := \dim A$

$\ell := \#\{$non-trivial simple A-modules $S_1, \ldots, S_\ell\}$

Various questions on finite-dimensional Hopf algebras arise...

Thanks for
your
attention!