Sandpiles and Representation Theory

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OUTLINE

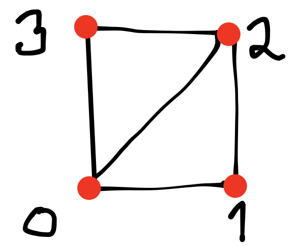
Laplacian & sandpile group for a...

- ... graph
- ... group representation
- ... module over a Hopf algebra

Graphs

$\Gamma = (V, E)$ an undirected (multi-) graph

$V = \{0, 1, 2, \ldots, \ell\}$



$L_\Gamma := D_\Gamma - A_\Gamma$

graph Laplacian

diagonal matrix of vertex degrees

adjacency matrix

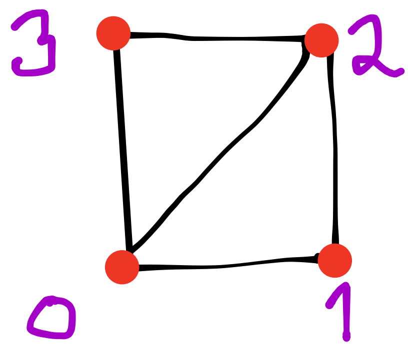
$$(L_\Gamma)_{i,j} = \begin{cases} \deg_\Gamma(i) & \text{if } i = j \\ -\#\{\text{edges } i \text{ to } j\} & \text{if } i \neq j \end{cases}$$

$$\underbrace{L_\Gamma}_{\text{graph Laplacian}} := \underbrace{D_\Gamma}_{\text{diagonal matrix of vertex degrees}} - \underbrace{A_\Gamma}_{\text{adjacency matrix}}$$

$$(L_\Gamma)_{i,j} = \begin{cases} \deg_\Gamma(i) & \text{if } i=j \\ -\#\{\text{edges } i \text{ to } j\} & \text{if } i \neq j \end{cases}$$

EXAMPLE

$\Gamma =$



$$L_\Gamma = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

The graph Laplacian L_Γ

- is positive semidefinite

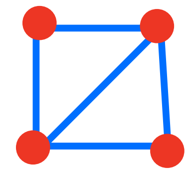
$$\left(L_\Gamma = \partial \partial^T \text{ where } \begin{array}{ccc} \mathbb{R}^E & \xrightarrow{\partial} & \mathbb{R}^V \\ \parallel & & \parallel \\ C_1(\Gamma, \mathbb{R}) & & C_0(\Gamma, \mathbb{R}) \end{array} \right)$$

- has $\mathbb{R}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \subseteq \ker(L_\Gamma)$

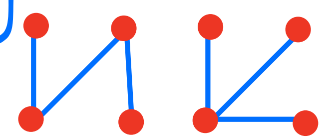
- equality $\boxed{\text{here}}$ $\Leftrightarrow$ Γ connected

- From spectrum (=eigenvalues) of L_Γ

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_\ell$$



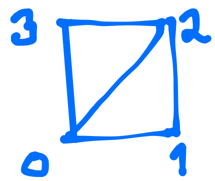
one can count the spanning trees in Γ:

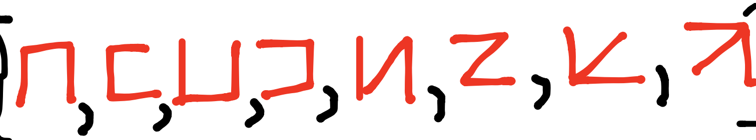


$$\underbrace{\tau(\Gamma)}_{\substack{\#\text{spanning} \\ \text{trees in } \Gamma}} = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell + 1}$$

- Alternatively,

$$\tau(\Gamma) \underset{\substack{\text{Kirchhoff's} \\ \text{Matrix-Tree} \\ \text{Theorem} \\ (1845)}}{=} \det\Big(\underbrace{L_\Gamma - \left\{ \begin{matrix} 0^{\text{th}} \text{ row}, \\ 0^{\text{th}} \text{ column} \end{matrix} \right\}}_{\substack{\text{reduced Laplacian} \\ \overline{L_\Gamma}}} \Big)$$

EXAMPLE $\Gamma =$  has

$\tau(\Gamma) = \#\{$  $\} = 8$

$$L_\Gamma = \begin{array}{cc} & \begin{matrix}0&1&2&3\end{matrix} \\ \begin{matrix}0\\1\\2\\3\end{matrix} & \begin{bmatrix}3&-1&-1&-1\\-1&2&-1&0\\-1&-1&3&-1\\-1&0&-1&2\end{bmatrix}\end{array}$$

has eigenvalues

$$\underset{\lambda_0}{0} \le \underset{\lambda_1}{2} \le \underset{\lambda_2}{4} \le \underset{\lambda_3}{4}$$

So $$\tau(\Gamma) = \frac{\lambda_1\lambda_2\lambda_3}{4} = \frac{2\cdot4\cdot4}{4} = 8 \checkmark$$

Or, $$\tau(\Gamma) = \det \bar{L}_\Gamma = \det\begin{bmatrix}2&-1&0\\-1&3&-1\\0&-1&2\end{bmatrix}$$

$$= 2(3\cdot2-1)+(-1\cdot2-0)$$

$$= 10-2 = 8 \checkmark$$

REMARK:

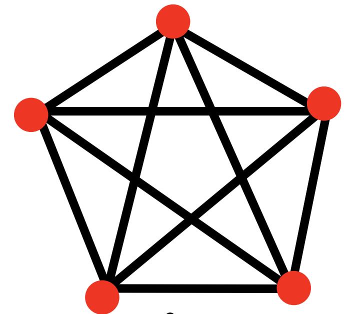
Eigenvalues of L_Γ are known for several families of graphs, letting one compute $\tau(\Gamma)$:

usually graphs with large symmetry or with inductive structure

- complete graphs, complete multipartite graphs
- cubes, Cartesian products
- distance-regular graphs
- threshold graphs, co-graphs

EXAMPLE

Complete graphs K_n



K_5

have L_{K_n} eigenvalues

$$\lambda_0 < \lambda_1 = \lambda_2 = \dots = \lambda_{n-1}$$

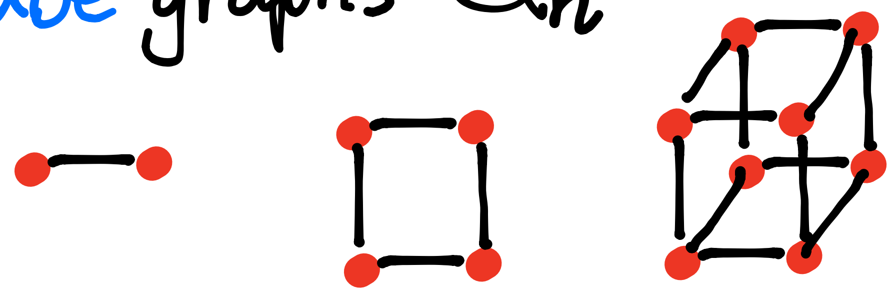
$$(0, n, n, \dots, n)$$

COROLLARY

$$\tau(K_n) = \frac{n^{n-1}}{n} = n^{n-2}$$

Cayley 1889
Borchardt 1860

EXAMPLE n-dimensional cube graphs Q_n



Q_1 $\qquad$ Q_2 $\qquad$ Q_3

have L_{Q_n} eigenvalues

λ	0	2	4	$\cdots$	$2n-2$	$2n$
mult.	1	$\binom{n}{1}$	$\binom{n}{2}$	$\cdots$	$\binom{n}{n-1}$	$\binom{n}{n}$

COROLLARY

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

What about the Laplacian L_Γ considered as a map $R^V \xrightarrow{L_\Gamma} R^V$ for other rings R, e.g. what is $\operatorname{rank}(L_\Gamma)$ when reduced mod p?

To answer this, one can work with $R = \mathbb{Z}$ and compute

$$\operatorname{coker}\left(\mathbb{Z}^V \xrightarrow{L_\Gamma} \mathbb{Z}^V\right)$$
$$:= \mathbb{Z}^V / \operatorname{im}(L_\Gamma)$$
$$\cong \mathbb{Z} \oplus K(\Gamma)$$

critical group
or sandpile group

Alternatively, one can show that

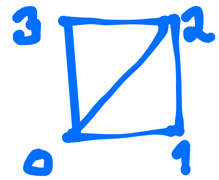
$$K(\Gamma) = \mathrm{coker}\left(\mathbb{Z}^\ell \xrightarrow{L_\Gamma} \mathbb{Z}^\ell\right)$$

with

$K(\Gamma)$ finite $\iff$ Γ connected

Kirchhoff's Thm. then implies

$$\# K(\Gamma) = \tau(\Gamma) = \#\text{spanning trees in } \Gamma$$

EXAMPLE $\Gamma =$ 

has $L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 3 & -1 \\ 3 & -1 & 0 & -1 & 2 \end{array}$ with

$$\operatorname{coker}\left(\mathbb{Z}^4 \xrightarrow{L_\Gamma} \mathbb{Z}^4\right) \cong \mathbb{Z} \oplus \underbrace{\mathbb{Z}/8\mathbb{Z}}_{K(\Gamma)}$$

because one can compute L_Γ has

Smith normal form

$$P L_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for some $P, Q \in GL_4(\mathbb{Z})$

Alternatively, using the reduced Laplacian $\overline{L}_\Gamma$

$$K(\Gamma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/8\mathbb{Z}$$

via (equivalent) Smith form calculation

$$P\overline{L}_\Gamma Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

For example, given p prime

$$\operatorname{rank}_{\mathbb{F}_p}(L_\Gamma) = \begin{cases} 2 & \text{if } p = 2 \\ 3 & \text{if } p \neq 2 \end{cases}$$

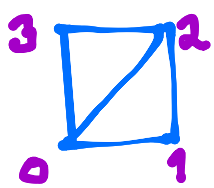
Why the name sandpile group?

The reduced Laplacian $\overline{L}_\Gamma$ is an avalanche-finite matrix:

- entries in $\mathbb{Z}$
- off-diagonal entries ≤ 0
- invertible,
  with inverse entries ≥ 0

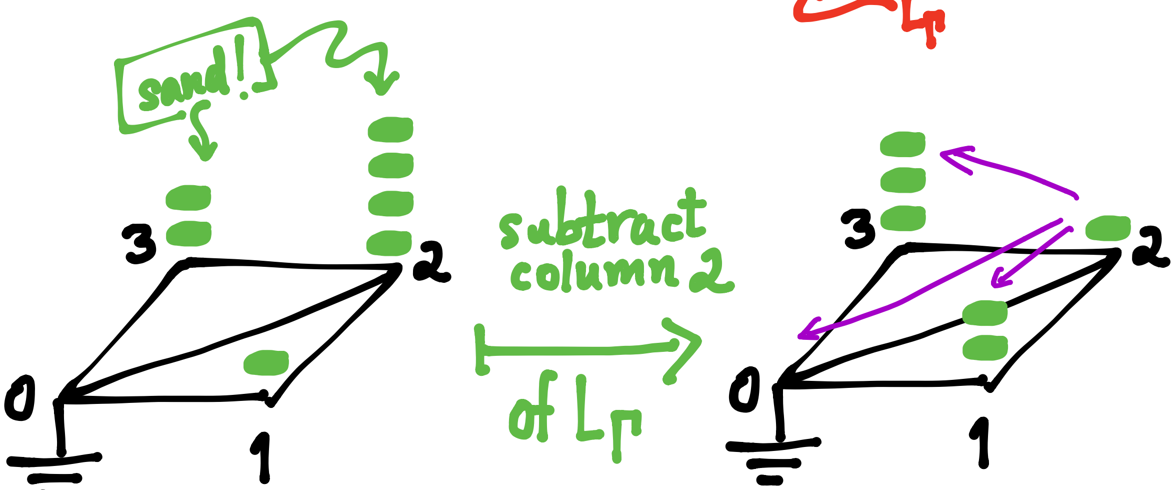
(Also known as
nonsingular M-matrices)

This implies every vector $x \in \mathbb{N}^{\ell}$ can be brought via a finite sequence of steps that subtract columns of $\overline{L}_\Gamma$, keeping it in $\mathbb{N}^{\ell}$, until no such subtraction is possible; x is stable.

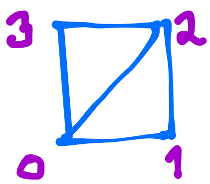
$\Gamma =$ 

$$L_\Gamma = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \end{array}$$

$\overline{L}_\Gamma$

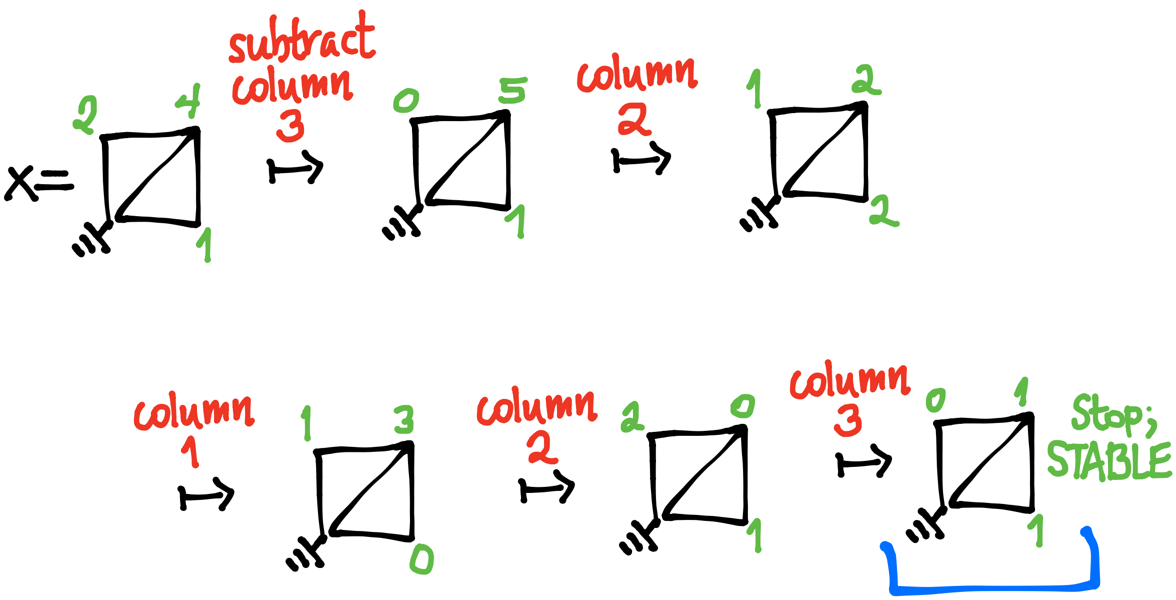


EXAMPLE

$\Gamma =$ 

$$L_\Gamma = \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & -1 & -1 & -1 \\ 1 & -1 & 2 & -1 & 0 \\ 2 & -1 & -1 & 3 & -1 \\ 3 & -1 & 0 & -1 & 2 \end{array}$$

$\tilde{L}_\Gamma$

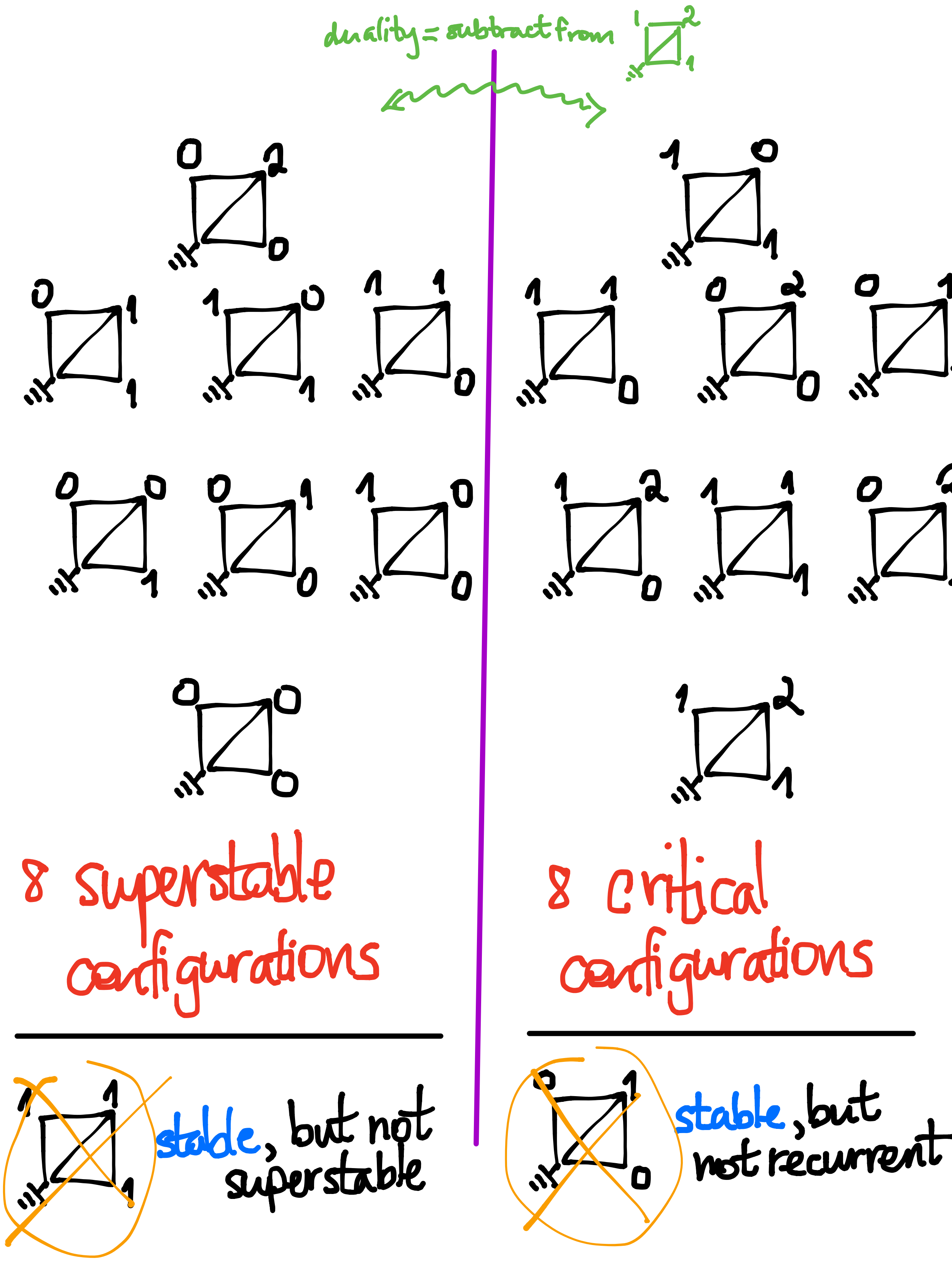


The stabilization is unique, independent of choices of firings.

Leads to two interesting classes of coset representatives in $\mathbb{N}^\ell$ for $K(\Gamma) = \mathbb{Z}^\ell / \operatorname{im} L_\Gamma$

- critical configurations (= stable + recurrent)
- superstable configurations

1987 Bak-Tang-Wiesenfeld
1990 Dhar
1991 Lorenzini
1993 Gabrielov
2007 Baker-Norine
2009 Shokrieh
2012 Levine-Pegden-Smart
2013 Holroyd-Levine-Mészáros-Peres-Propp-Wilson
⋮

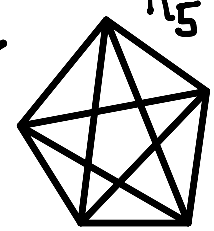

duality = subtract from
8 superstable configurations
8 critical configurations
stable, but not superstable
stable, but not recurrent

The exact structure of the sandpile group $K(\Gamma)=\mathbb{Z}^\ell/\mathrm{im}\,\overline{L}_\Gamma$ is known for very few graphs Γ, even when eigenvalues and eigenvectors and $\tau(\Gamma)=\#K(\Gamma)$ are easy.

(easy)

EXAMPLE Complete graphs K_n

K_5

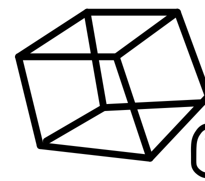


have $\tau(K_n) = n^{n-2}$

and $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$

(frustrating!)

EXAMPLE n-dimensional cubes Q_n



Q_3

have L_{Q_n} eigenspaces easy

and

$$\tau(K_n) = \frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}$$

The p-primary/p-Sylow structure of $K(Q_n)$ is known for p odd

$$\mathrm{Syl}_p K(Q_n) \cong \mathrm{Syl}_p \bigoplus_{k=1}^{n} (\mathbb{Z}/k\mathbb{Z})^{\binom{n}{k}}$$

but for $p=2$

$\mathrm{Syl}_2 K(Q_n)$ is an unknown mess!

Now for (ordinary)

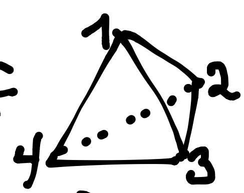
Finite group representations

- G a finite group
- irreducible/simple complex G-representations / $\mathbb{C}G$-modules

$$\underbrace{\mathbb{1}_G}_{\text{trivial } G\text{-rep}} = S_0, S_1, S_2, \ldots, S_\ell$$

- characters $\chi_0, \chi_1, \ldots, \chi_\ell$

EXAMPLE

$G = A_4$ = rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$\mathbb{1}_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

$\omega = e^{2\pi i/3}$

DEFINITION:

Given a representation

$$G \xrightarrow{\rho} GL_n(\mathbb{C})$$

define its McKay matrix $M_\rho = (m_{ij})$ via

$$\left(\chi_{S_i \otimes \rho} = \right) \chi_i \chi_\rho = \sum_{j=0}^{\ell} m_{ij} \chi_j$$

or

$$S_i \otimes \rho = \bigoplus_{j=0}^{\ell} S_j^{\oplus m_{ij}}$$

$$(\chi_{s_i \otimes \rho^{=}})\,\chi_i\,\chi_\rho = \sum_{j=0}^{\ell} m_{ij}\chi_j \quad \text{defines } M_\rho$$

Then...

- $L_\rho := nI_{\ell+1} - M_\rho$

- $\overline{L}_\rho := L_\rho - \begin{Bmatrix} \chi_0 \text{ row} \\ \chi_0 \text{ column} \end{Bmatrix}$

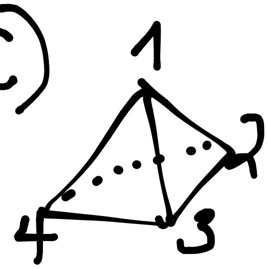
- $\underset{\text{sandpile group}}{K(\rho)} := \mathrm{coker}\left(\mathbb{Z}^\ell \xrightarrow{\overline{L}_\rho} \mathbb{Z}^\ell\right)$

or

$$\mathbb{Z} \oplus K(\rho) = \mathrm{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_\rho} \mathbb{Z}^{\ell+1}\right)$$

EXAMPLE

$G = \mathcal{A}_4 \overset{\rho}{\hookrightarrow} SO_3(\mathbb{R}) \subseteq GL_3(\mathbb{C})$

via rotational symmetries of 

	e	(123)	(132)	$(12)(34)$
$1_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\chi_\rho = \chi_3$	3	0	0	-1

$$\chi_0\chi_\rho = \chi_1\chi_\rho = \chi_2\chi_\rho = \chi_\rho = 1\chi_3$$

$$\chi_3\chi_\rho = 1\chi_0 + 1\chi_1 + 1\chi_2 + 2\chi_3$$

$$\rightsquigarrow \quad M_\rho = \begin{array}{c|c} & \begin{matrix} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{matrix} \\ \hline \begin{matrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

$$M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \end{array}$$

McKay matrix

$$L_\rho = 3I_4 - M_\rho = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \end{array}$$

$$\tilde{L}_\rho = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \xrightarrow[\text{normal form}]{\text{Smith}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$K(\rho) = \operatorname{coker} \tilde{L}_\rho \cong \mathbb{Z}/3\mathbb{Z}$$

Sandpile group

Why is $\operatorname{coker} L_\rho = \mathbb{Z} \oplus \underbrace{\operatorname{coker} \overline{L}_\rho}_{K(\rho)}$?

L_ρ has

$$\bar{s} = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix} = \begin{bmatrix} 1 \\ \dim(S_1) \\ \vdots \\ \dim(S_\ell) \end{bmatrix}$$

as right- and left-nullvector.

EXAMPLE

$$L_\rho \bar{s} = \begin{bmatrix} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

THEOREM

The inclusion $\mathbb{R}\bar{s} \subseteq \ker L_\rho$ is an equality

$\iff G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ is faithful

(analogous of Γ connected)

More generally, the columns of the character table

$$\bar{s}(g) := \begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix},$$

and re-ordered columns

$$\bar{s}^*(g) := \begin{bmatrix} \chi_0^*(g) \\ \chi_1^*(g) \\ \vdots \\ \chi_\ell^*(g) \end{bmatrix}$$

give right- and left-eigenbases for M_ρ, L_ρ:

$$\sum_{j=0}^{\ell} m_{ij} \chi_j(g) = \chi_i(g)\, \chi_\rho(g)$$

$$\Rightarrow \quad M_\rho\, \bar{s}(g) = \chi_\rho(g)\, \bar{s}(g)$$

$$L_\rho\, \bar{s}(g) = \underbrace{\left(n - \chi_\rho(g)\right)}_{\text{eigenvalues of } L_\rho} \bar{s}(g)$$

THEOREMS & EXAMPLES

THEOREM (Benkart-Klivans-R)

For faithful G-reps ρ,
$\overline{L}_\rho$ is an avalanche-finite matrix,
so one can compute in
$K(\rho) = \mathrm{coker}(\overline{L}_\rho)$ via toppling
with superstable or critical
coset representatives in $\mathbb{N}^\ell$

EXAMPLE (continued)

$$\overline{L}_\rho = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array}\begin{array}{c} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \end{array} \\ \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \end{array} \quad \text{with } K(\rho) = \mathbb{Z}/3\mathbb{Z}$$

χ_1	χ_2	χ_3
[0	0	0]
[1	0	0]
[0	1	0]

superstables

χ_1	χ_2	χ_3
[2	2	0]
[1	2	0]
[2	1	0]

criticals

Some (di-)graph Laplacians do re-appear...

THEOREM (Benkart-Klivans-R.)

For faithful abelian group reps $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$

$$K(\rho) = \underbrace{K(\Gamma)}_{\text{(di-)graph sandpile group}}$$

where Γ = Cayley digraph for

the dual group $G^{\vee} = \mathrm{Hom}(G, \mathbb{C}^{\times}) = \{\chi_0, \chi_1, \ldots, \chi_\ell\}$

with respect to generators $\{\chi_{i_1}, \ldots, \chi_{i_n}\}$,

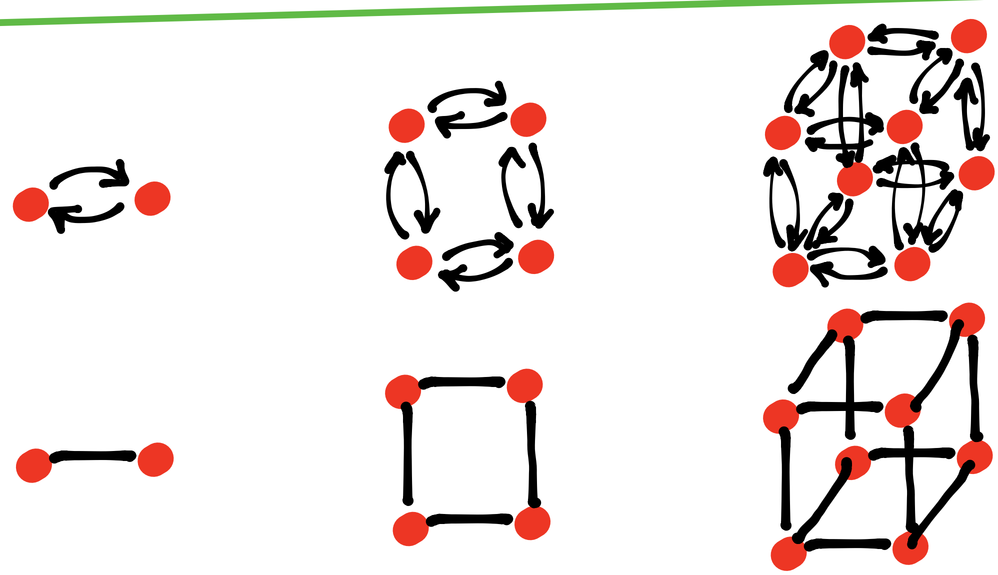
if $\chi_\rho = \chi_{i_1} + \cdots + \chi_{i_n}$.

EXAMPLE

$$G = (\mathbb{Z}/2\mathbb{Z})^n \xrightarrow{\rho} GL_n(\mathbb{C})$$

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{bmatrix} \longmapsto \begin{bmatrix} (-1)^{\epsilon_1} & & & 0 \\ & (-1)^{\epsilon_2} & & \\ & & \ddots & \\ 0 & & & (-1)^{\epsilon_n} \end{bmatrix}$$

has $K(\rho) = K\left(\overset{n\text{-cube}}{Q_n}\right)$



The analogue of $\#K(\Gamma) = \tau(\Gamma) = \frac{\lambda_1 \lambda_2 \cdots \lambda_\ell}{\ell+1}$ is

THEOREM (Gaetz) For any faithful representation ρ of G,

$$\#K(\rho) = \frac{1}{\#G} \cdot \prod_{\substack{G\text{-conj. classes} \\ [g] \neq \{e\}}} (n - \chi_\rho(g))$$

EXAMPLE $G = \mathfrak{S}_4 \xhookrightarrow{\rho} SO_3(\mathbb{R}) \subset GL_3(\mathbb{C})$

had $K(\rho) = \mathbb{Z}/3\mathbb{Z}$

	e	(123)	(132)	$(12)(34)$
χ_ρ	3	0	0	-1

$$\#K(\rho) = \frac{1}{12}(3-0)(3-0)(3-(-1)) = 3$$

THEOREM (Goetz) If $n = \# G$, **regular representation** of G

$$G \xhookrightarrow{\mathrm{reg}_G} GL_n(\mathbb{C}) \quad \text{has}$$

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{\#\left(\substack{G\text{-conjugacy}\\ \text{classes}}\right)-2}$$

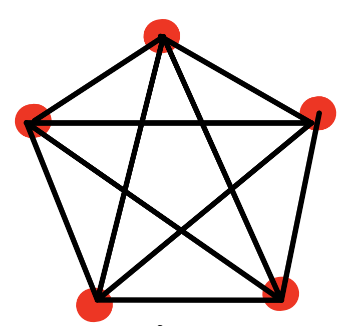
$\Big\downarrow$ G abelian

$$K(\mathrm{reg}_G) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

$\Big\downarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

$$K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$$

complete graph



K_5

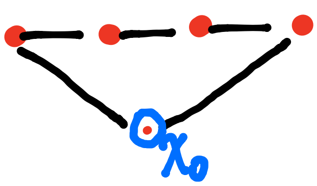
McKay's original theorem (1980)

When $G \xhookrightarrow{\rho} SL_2(\mathbb{C})$, then

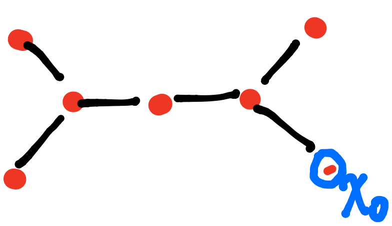
$\overline{L}_\rho, L_\rho$ are the Cartan, extended Cartan

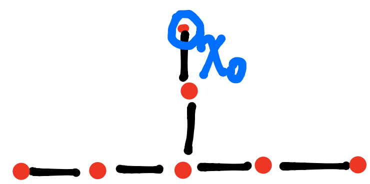
matrices for a simply-laced root system Φ

$\tilde{A}$

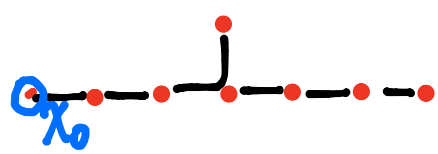


$\tilde{D}$

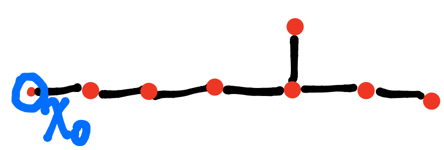




$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$

THEOREM (Benkart–Klivans–R)

In McKay's $G \xhookrightarrow{\rho} SL_2(\mathbb{C})$ setting

$$K(\rho) \cong \mathrm{Hom}(G, \mathbb{C}^\times)$$
$$= \text{1-dim'l characters } \chi_i \text{ of } G$$

$$\left(\overset{\text{Pontrjagin dual}}{\cong} \underbrace{G^{ab}}_{\text{abelianization of } G} = G/[G,G] \right)$$

$$\left(\cong \underbrace{P(\Phi)}_{\text{weight lattice}} / \underbrace{Q(\Phi)}_{\text{root lattice}} \cong \underbrace{\pi_1 \left(\begin{array}{c}\text{adjoint form} \\ \text{of compact} \\ \text{Lie group} \\ \text{associated to } \Phi\end{array} \right)}_{\text{fundamental group of } \Phi} \right)$$

THEOREM (Benkart-Klivans-R)

More generally, when $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$

one has a surjection

$$K(\rho) \twoheadrightarrow \mathrm{Hom}(G, \mathbb{C}^\times)$$

THEOREM (Gaetz)

When $G \overset{\rho}{\hookrightarrow} SL_n(\mathbb{C})$,

$$\left\{ \begin{matrix} \text{1-dimensional} \\ \text{characters } \chi_i \end{matrix} \right\} \hookrightarrow \left\{ \begin{matrix} \text{superstable} \\ \text{configurations} \\ \text{for } \overline{L}_\rho \end{matrix} \right\}$$

$$\|$$

$$\mathrm{Hom}(G, \mathbb{C}^\times)$$

Do we really need complex G-representations

that is, $\mathbb{C}G$-modules ?

Why not representations

$$G \xrightarrow{\rho} GL_n(\mathbb{F}),$$

that is, $\mathbb{F}G$-modules ?

Much of it works

replacing $A = \mathbb{F}G$ with...

Hopf algebras

Let A be a

finite dimensional Hopf algebra

over an algebraically closed field $\mathbb{F}$

so it has product $A \otimes A \xrightarrow{\mu} A$

and A-modules V,

but also ...

- coproduct $A \xrightarrow{\Delta} A \otimes A$
- counit $A \xrightarrow{\epsilon} \mathbb{F}$
- antipode $A \xrightarrow{\alpha} A$

defines

$V \otimes W$

Trivial A-mod S_0 on $\mathbb{F}$

Left and right duals ${}^*V$, V^*

EXAMPLE

$A = \mathbb{F}G$ = group algebra for a finite group G, with

- coproduct $g \overset{\Delta}{\longmapsto} g \otimes g$
- counit $g \overset{\epsilon}{\longmapsto} 1$
- antipode $g \overset{\alpha}{\longmapsto} g^{-1}$

$\mathbb{Z}^{\ell+1} \cong$ virtual characters of G

$\mathbb{Z}^{\ell+1} \cong G_0(A) =$ Grothendieck group of A-modules

$\left(\begin{array}{l} 0 \to U \to V \to W \to 0 \\ \quad \Rightarrow [V] = [U] + [W] \end{array} \right)$

with $\mathbb{Z}$-basis $[S_0], [S_1], \ldots, [S_\ell]$

where $S_0, S_1, \ldots, S_\ell$ are the simple A-mods

ϵ"

and $[V] = \sum_{i=0}^{\ell} [V : S_i][S_i]$.

composition multiplicity of S_i in V

$G_0(A)$ has multiplication from

$$[V][W] := [V \otimes W].$$

DEFINITION: For an A-module V, let

- $\underset{\text{McKay matrix}}{M_V} \in \mathbb{Z}^{(\ell+1)\times(\ell+1)}$ express the map

$$\begin{array}{ccc} G_0(A) & \xrightarrow{(-)\cdot[V]} & G_0(A) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z}^{\ell+1} & & \mathbb{Z}^{\ell+1} \end{array}$$

that is, $(M_V)_{i,j} := [S_j \otimes V : S_i]$

- $L_V = nI_{\ell+1} - M_V$ where $n := \dim V$

- $\mathbb{Z} \oplus \underset{\text{sandpile group}}{K(V)} := \operatorname{coker}\left(\mathbb{Z}^{\ell+1} \xrightarrow{L_V} \mathbb{Z}^{\ell+1}\right)$

When is $K(V)$ finite, generalizing $G \overset{\rho}{\hookrightarrow} GL_n(\mathbb{C})$ being faithful ?

THEOREM (Grinberg-Huang-R)

$K(V)$ is finite

$\iff$ V is tensor-rich

every A-simple S_i occurs in at least one $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k\text{-fold}}$

$\iff$ $\overline{L}_V := L_V - \left\{\begin{matrix}\text{row, column}\\ \text{indexed by}\\ S_0 = \epsilon\end{matrix}\right\}$ avalanche-finite

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REMARK: For $\mathbb{C}G$-modules V

V tensor-rich $\underset{\text{Burnside}}{\iff}$ V faithful

REMARK: In general,

$$\operatorname{coker}(L_V) = \mathbb{Z} \oplus \underbrace{K(V)}_{\substack{\not\cong \\ \operatorname{coker}(\overline{L}_V)}}$$

unless A is semisimple as an algebra

But in the semisimple case, one can again compute in

$$K(V) = \operatorname{coker}(\overline{L}_V)$$

via sandpiles in $\mathbb{N}^\ell$ and $\overline{L}_V$.

Recall

$$A \cong \bigoplus_{i=0}^{\ell} P_i^{\oplus \dim S_i}$$

left-regular A-module

where $P_0, P_1, \ldots, P_\ell$ are the indecomposable projective A-modules (so P_i = projective cover of S_i)

PROPOSITION

Let $\bar{s} := [s_0, s_1, \cdots, s_\ell]^t$ where $s_i := \dim S_i$

$\bar{p} := [p_0, p_1, \cdots, p_\ell]^t$ $p_i := \dim P_i$

Then $\bar{p}, \bar{s}$ are left, right nullvectors for L_V.

PROPOSITION For $A=\mathbb{F}G$ one knows **all** the eigenspaces:

the Brauer character table columns

$$\bar{s}(g) := [\chi_{S_0}(g), \ldots, \chi_{S_\ell}(g)]^t \text{ for } p\text{-regular } g \in G$$

and

(permuted) indecomposable projective Brauer character table columns

$$\bar{p}(g) := [\chi_{{}^*P_0}(g), \ldots, \chi_{{}^*P_\ell}(g)]^t$$

give left and right eigenbases for L_V.

For tensor-rich A-modules V, what is $\#K(V)$?

A lemma of Lorenzini implies this:

PROPOSITION If L_V has eigenvalues $0=\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_\ell$ then

$$\#K(V) = \frac{\gamma(A)}{\dim A} \lambda_1 \lambda_2 \cdots \lambda_\ell$$

where $\gamma(A) := \gcd\{\dim P_i\}_{i=0,1,\ldots,\ell}$

QUESTION:
Does $\gamma(A)$ have more meaning in terms of structure of A?

PROPOSITION: For $A = \mathbb{F}G$, with $\operatorname{char}\mathbb{F} = p$

$\gamma(A)$ = the size of any p-Sylow subgroup

$= p^a$ where $\#G = p^a q$

with $\gcd(p,q) = 1$

COROLLARY: For $A = \mathbb{F}G$,

and an A-module V of dimension n,

$$\#K(V) = \frac{p^a}{\#G} \prod_{\substack{p\text{-regular} \\ G\text{-conj. classes} \\ [g] \neq \{e\}}} \left(n - \underset{\text{Brauer character}}{\chi_V}(g)\right)$$

The left regular A-module A itself is always tensor-rich.

THEOREM (Grinberg-Huang-R)
For any finite dim'l Hopf algebra A,

$$K(A) \cong \mathbb{Z}/\gamma\mathbb{Z} \oplus \left(\mathbb{Z}/d\mathbb{Z}\right)^{\ell-1}$$

where $\gamma := \gamma(A)$
$d := \dim A$
$\ell := \#\{$non-trivial simple A-modules $S_1, \ldots, S_\ell\}$

Questions on finite-dimensional Hopf algebras naturally arose, some now answered in work of
Benkart-Diaconis-Liebeck-Tiep 2018
Burciu 2018

Thanks for your attention!