Cyclic symmetry

Vic Reiner
(Univ. of Minnesota)

Smith College
Thursday Lunch Talk

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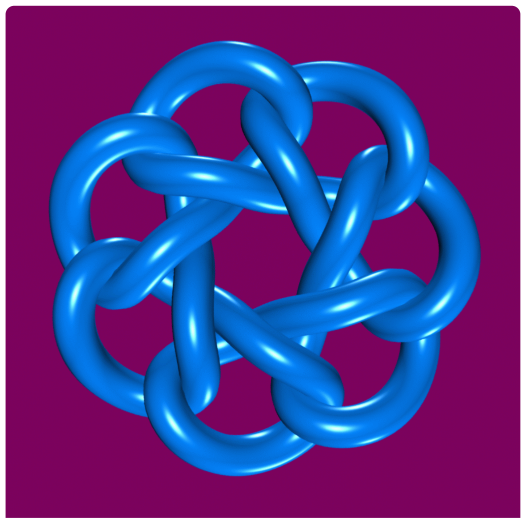
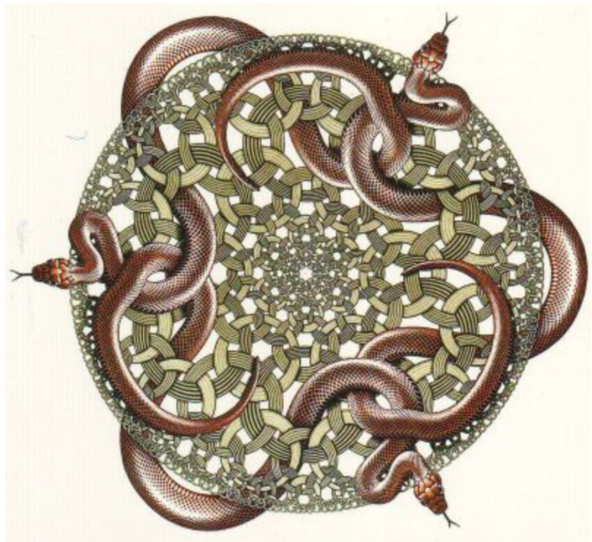
Everyone enjoys



symmetry!

3-fold



5-fold

I like it cyclic.

Better yet, I like counting formulas for objects with cyclic symmetry, particularly when they come from formulas that were already there ...

Let's count

k-element subsets

of $\{1, 2, \ldots, n\}$

EXAMPLE: $k=2$
$n=4$

There are

$$\binom{n}{k} = \binom{4}{2} = 6$$

of them:

$\{1,2\}, \{1,3\}, \{1,4\},$

$\{2,3\}, \{2,4\}, \{3,4\}$

$\binom{n}{k}$ = the binomial coefficient

$$= \frac{n!}{k!\,(n-k)!}$$

where $n! = n(n-1)\cdots 3\cdot 2\cdot 1$

e.g. $\binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 2\cdot 1} = 6$ ✓

Pascal:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Pascal's triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

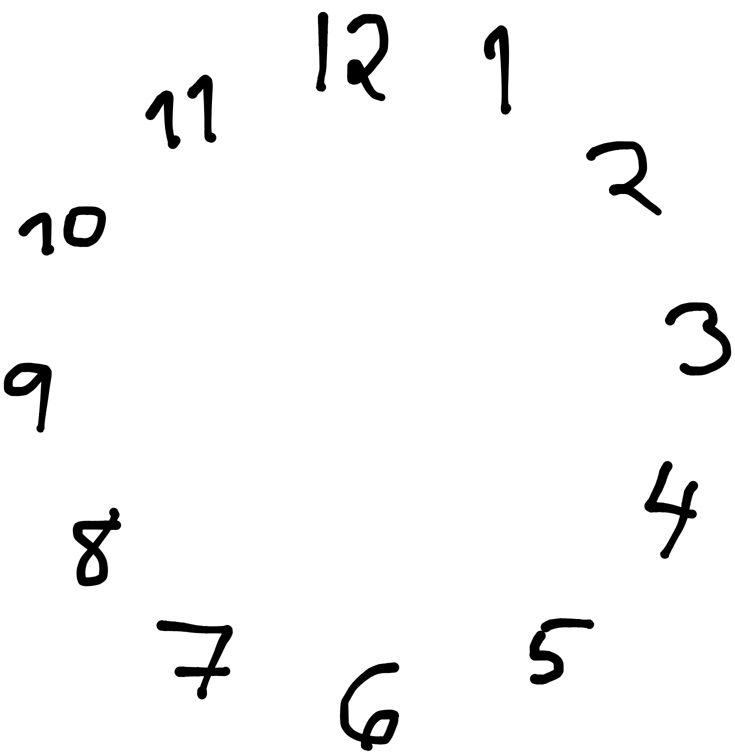
⋰ ⋮ ⋱

$\binom{3}{1} + \binom{3}{2}$

$= \binom{4}{2}$

Fine, but can subsets have cyclic symmetry?

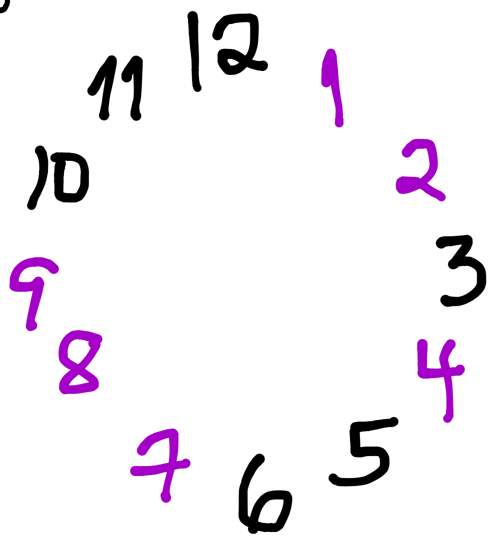
Yes, if we place $\{1,2,\ldots,n\}$ on a circle.....



$n=12$

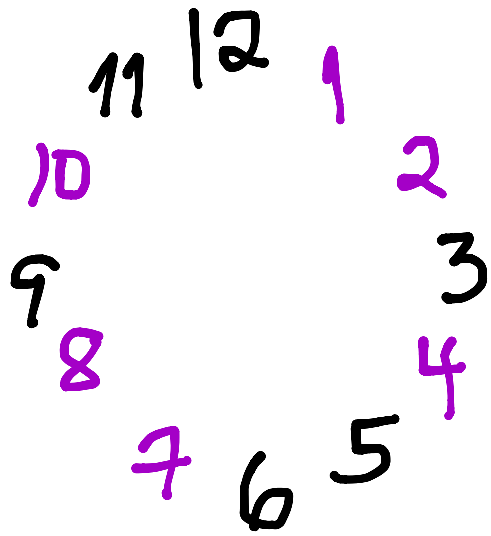
Every 6-element subset of $\{1,2,\ldots,12\}$ has 1-fold symmetry.

e.g. $\{1,2,4,7,8,9\}$

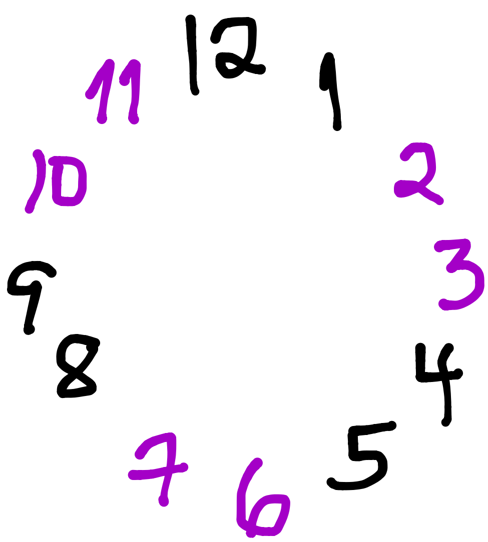


Some have 2-fold symmetry.

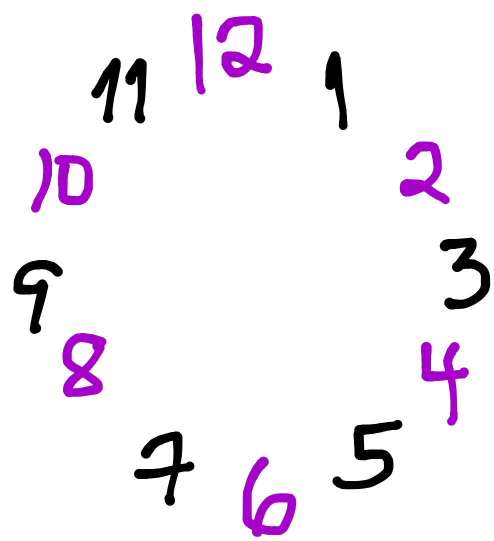
e.g. $\{1,2,4,7,8,10\}$



Some have 3-fold symmetry:



Some even have 6-fold symmetry:



So how many k-element subsets of $\{1,2,\ldots,n\}$ have d-fold symmetry?

THEOREM (Stanton-White-R. 2004)

It's what you get from the *q*-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q$$

when you plug in for q a primitive complex d^{th} root-of-unity!

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \overset{\text{DEF.}}{=} \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

where $[n]!_q \overset{\text{DEF}}{=} [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q$

$$[n]_q \overset{\text{DEF}}{=} 1+q+q^2+\dots+q^{n-1} = \frac{1-q^n}{1-q}$$

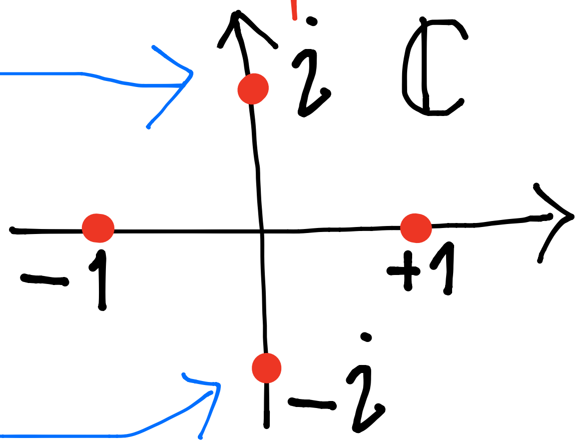
EXAMPLE: $k=2,\ n=4$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q\,[2]!_q} = \frac{[4]_q[3]_q[2]_q[1]_q}{[2]_q[1]_q\cdot[2]_q[1]_q}$$

$$= \frac{(1+q+q^2+q^3)(1+q+q^2)\cancel{(1+q)}\cancel{(1)}}{(1+q)\cancel{(1)}\cdot\cancel{(1+q)}\cancel{(1)}}$$

$$= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4$$

Remember complex roots-of-unity?

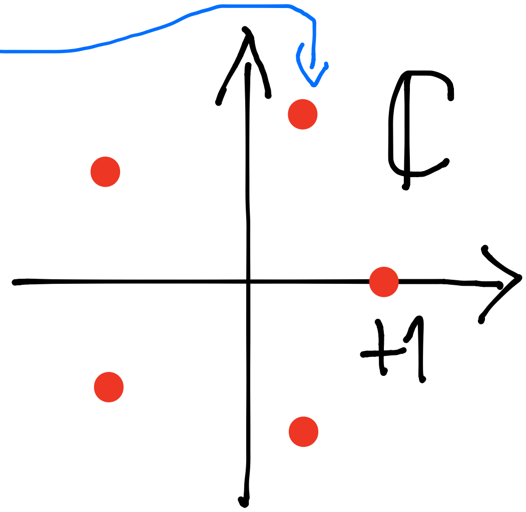


The 4^{th} roots-of-unity

$\pm i$ are the primitive 4^{th} roots-of-unity

-1 is a primitive 2^{nd} root-of-unity

$e^{\frac{2\pi i}{5}}$



The 5^{th} roots-of-unity

So we plug in $q = +1, -1, \pm i$

1st root, 2nd root, 4th roots

in the q-binomial coefficient

$$\begin{bmatrix}4\\2\end{bmatrix}_q = 1+q+2q^2+q^3+q^4$$

$q=+1$		$q=-1$	$q=+i$
$1+1+2+1+1=6$		$1-1+2-1+1=2$	$1+i-2-i+1 = 0$
1 4 **2** 3	**1** 4 2 **3**	**1** 4 2 **3**	
1 **4** 2 3	1 4 **2** **3**	2-fold symmetric	4-fold symmetric (none!)
1 **4** **2** 3	1 **4** 2 **3**	1 **4** **2** 3	

ALL!

The theory of q-analogues...

$$[n]_q = 1+q+q^2+\dots+q^{n-1} \xrightarrow[q=1]{} n$$

$$[n]!_q = [n]_q \cdots [2]_q [1]_q \xrightarrow[q=1]{} n!$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q} \xrightarrow[q=1]{} \binom{n}{k}$$

is a very well-studied, well-developed and fascinating subject.

The q-binomials $\begin{bmatrix} n \\ k \end{bmatrix}_q$ have beautiful properties and many interpretations, e.g.,

- $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in q, with nonnegative coefficients

  e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = 1+q+2q^2+q^3+q^4$

- $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$ (q-Pascal!)

  e.g. $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q + q^{4-2}\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q$

  $= 1+q+q^2+q^2(1+q+q^2)$

  $= 1+q+2q^2+q^3+q^4$

- $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has interpretations in geometry, topology, representation theory

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- When you plug in $q = p^m$ a prime power there is a finite field $\mathbb{F}_q$ with q elements, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts (!) the set $\left\{ k\text{-dimensional } \mathbb{F}_q\text{-linear subspaces inside } \mathbb{F}_q^{\,n} \right\}$

EXAMPLE: $q = 3 = p^1$

$\mathbb{F}_q = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$

$= \{\text{integers modulo } 3\}$

e.g. $\bar{1} + \bar{1} + \bar{1} = \bar{0} = \bar{1} + \bar{2}$

$\bar{2} \cdot \bar{2} = \bar{4} = \bar{1}$ in $\mathbb{F}_3$

So taking $k=1$ and $n=2$,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = \frac{[2]_q [1]_q}{[1]_q \cdot [1]_q} = \frac{(1+q)(1)}{(1)\cdot(1)} = 1+q$$

$$\xrightarrow[\text{plug in } q=3]{} 1+3 = 4$$

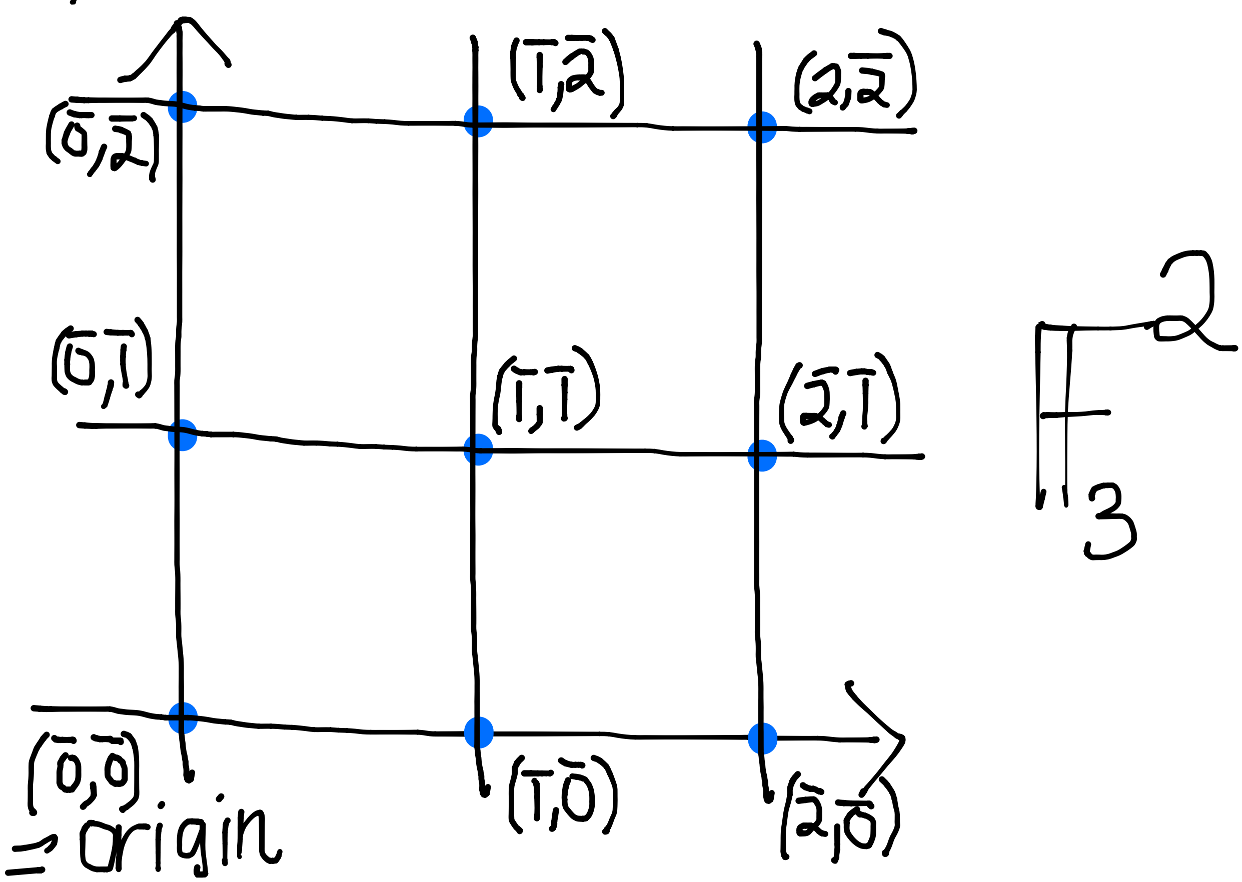
Since $\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \underset{q=3}{\rightsquigarrow} 4$,

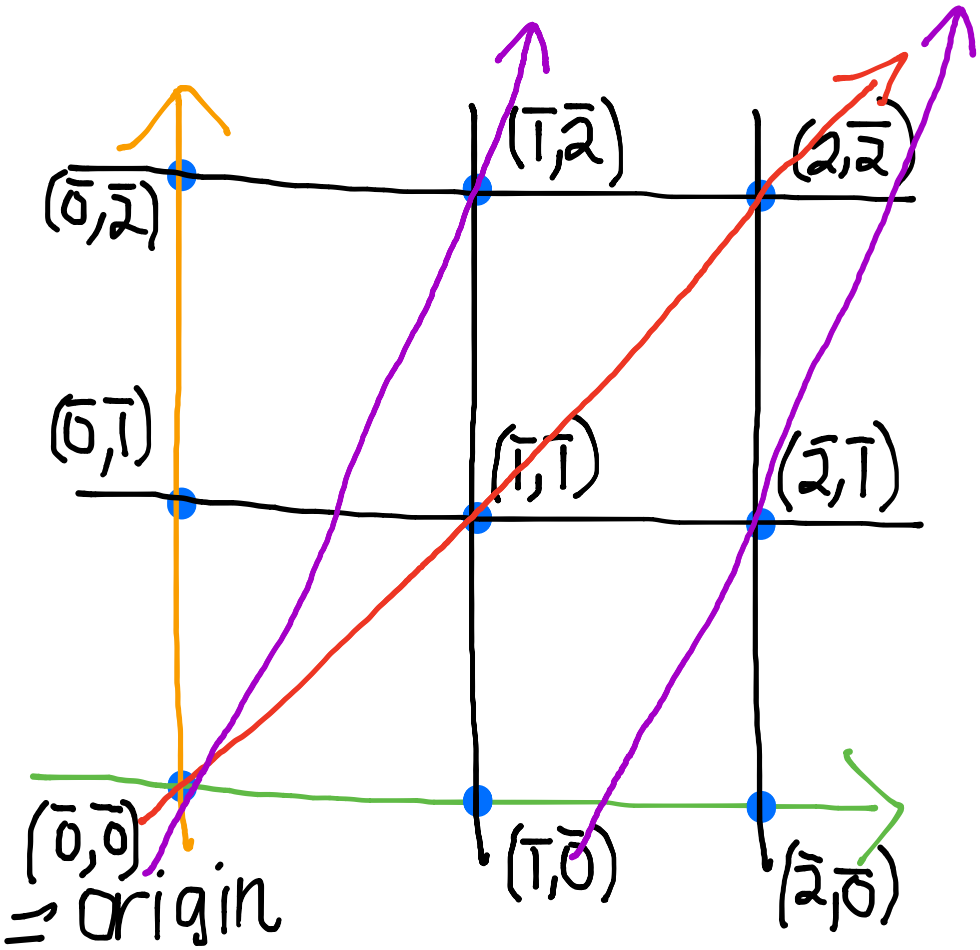
there should be exactly 4

1-dimensional $\mathbb{F}_3$-linear subspaces

(= lines through the origin)

in the 2-dimensional space $\mathbb{F}_3^2$:





slope $\bar{0}$ = x-axis

slope $\bar{1}$ = diagonal

slope $\bar{2}$ = $\{(0,0), (\bar{1},\bar{2}), (\bar{2},\bar{1})\}$

slope ∞ = y-axis

4 lines ✓

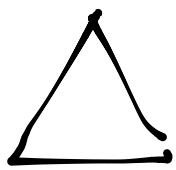
The fact that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ can count cyclically symmetric k-element subsets of $\{1, 2, \ldots, n\}$ has many proofs.

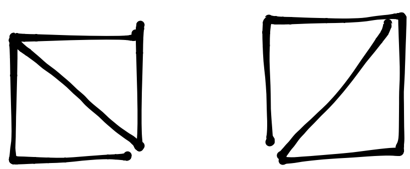
- Some use the connections to representation theory.
- Others are more direct, but perhaps less illuminating.

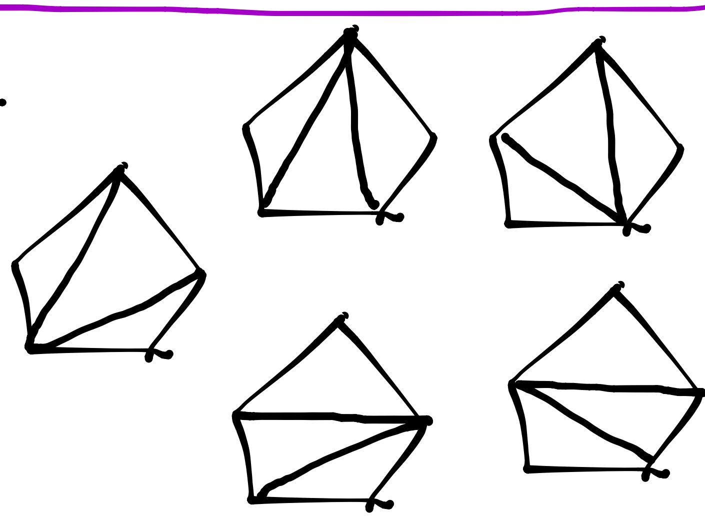
We have many, many examples where a poynomial in q counts cyclically symmetric objects when we plug in a root-of-unity for q.
(The "cyclic sieving phenomenon")

Here is one that I like very much, but feel we understand poorly...

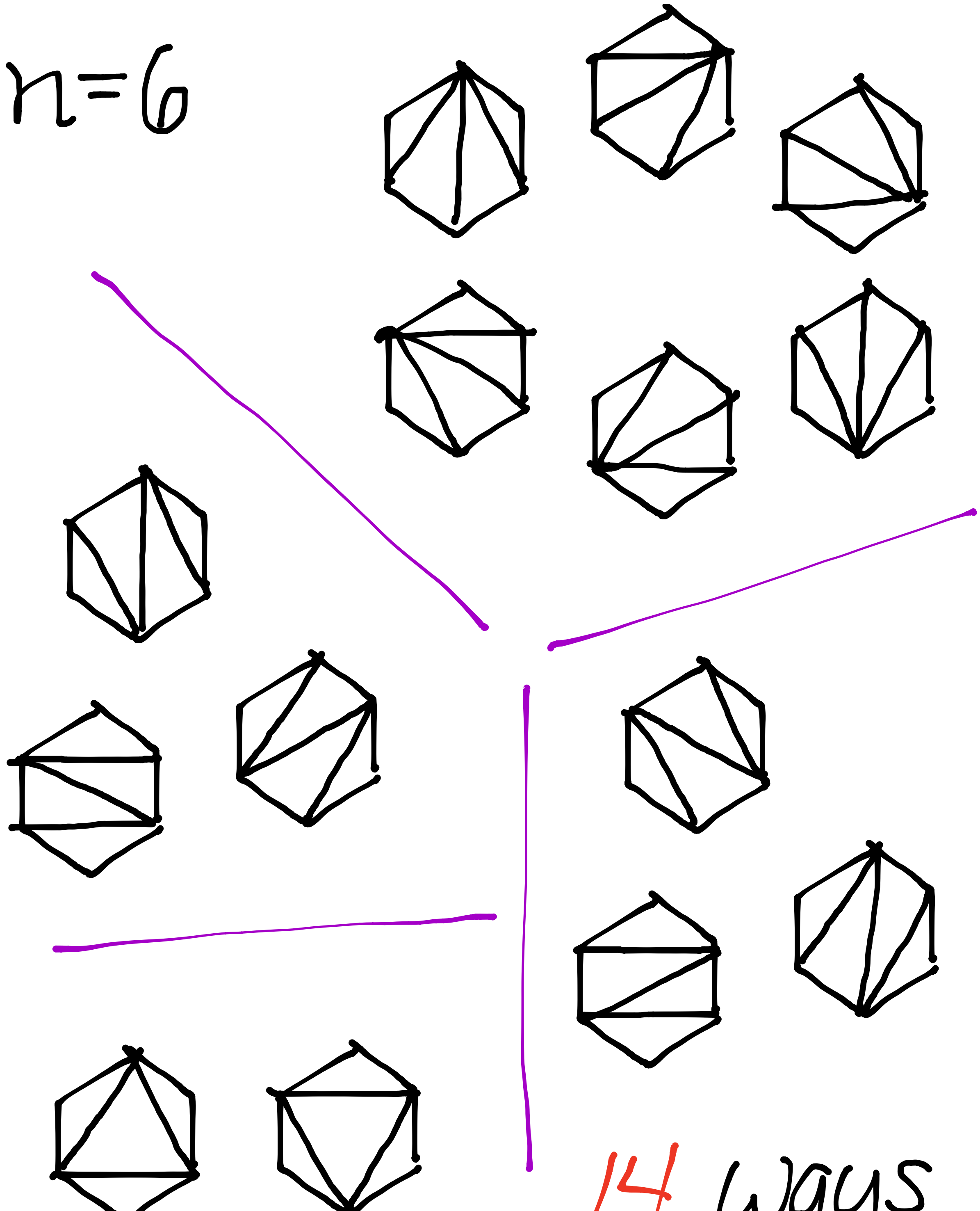
An old counting problem:
How many ways to
triangulate (cut into triangles)
a convex n-sided polygon?

$n=3$  1 way.

$n=4$  2 ways.

$n=5$  5 ways.

$n=6$



14 ways

THEOREM (Euler, Segner, Goldbach 1750's):

There are

$$\frac{1}{n-1}\binom{2(n-2)}{n-2} = \frac{n\,(n+1)\cdots(2n-4)}{2\cdot 3\cdots n-2}$$

Catalan number

ways to triangulate the n-sided polygon

n	$\frac{1}{n}\binom{2(n-2)}{n-2}$
3	(empty product) = 1
4	$\frac{4}{2} = 2$
5	$\frac{5\cdot 6}{2\cdot 3} = 5$
6	$\frac{6\cdot 7\cdot 8}{2\cdot 3\cdot 4} = 14$

THEOREM (RSW 2004):

The d-fold cyclically symmetric triangulations of a (regular) n-sided convex polygon are counted by plugging in a primitive complex d^{th} root-of-unity for q in this:

$$\underbrace{\frac{1}{[n-1]_q}\begin{bmatrix}2(n-2)\\ n-2\end{bmatrix}_q}_{q\text{-Catalan}} = \frac{[n]_q[n+1]_q\cdots[2n-4]_q}{[2]_q[3]_q\cdots[n-2]_q}$$

EXAMPLE: $n=6$

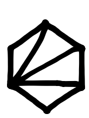
$$\frac{1}{[n-1]_q}\begin{bmatrix} 2(n-2) \\ n-2 \end{bmatrix}_q = \frac{[6]_q\,[7]_q\,[8]_q}{[2]_q\,[3]_q\,[4]_q}$$

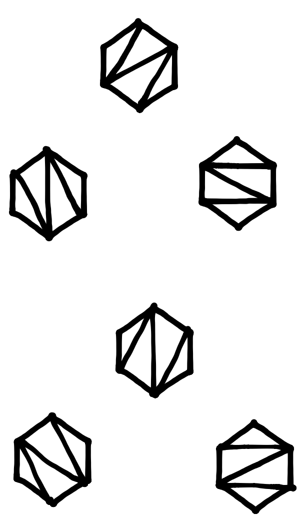
$$= 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

Skipped algebra!

$q=1$	$q=-1$	$q=e^{\frac{2\pi i}{3}}$	$q=e^{\frac{2\pi i}{6}}$
$1+1+1+2+1+2+1+2+1+1+1$	$1+1-1+2-1+2-1+2-1+1+1$	(Skipped summing roots-of-unity)	(Skipped summing roots-of-unity)
$=14$	$=6$	$=2$	$=0$

All 14 of them, e.g.





None of them

1-fold

2-fold

3-fold

6-fold

The q-Catalan $\frac{1}{[n-1]_q}\begin{bmatrix} 2(n-2) \\ n-2 \end{bmatrix}_q$ has many properties and interpetations:

- It is again a polynomial in q, with nonnegative coefficients.
- It again has meaning in geometry and in representation theory.
- It again counts something if we plug in $q = p^m$ a prime power:

$$\left\{ \text{orbits of } \mathbb{F}_{q^{2n-3}}^{\times} \text{ acting on } (n-1)\text{-dimensional } \mathbb{F}_q\text{-linear subspaces inside } \mathbb{F}_{q^{2n-3}} \right\}$$

Thanks

for

coming!