Stirling numbers and Koszul algebras with symmetry

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1. Stirling numbers $c(n,k)$, $S(n,k)$
   1st kind, 2nd kind

2. Algebras, Hilbert functions/series

3. Koszul algebras & duality

4. Representation theory results

1. Stirling numbers

\# k cycle permutations in $\mathfrak{S}_n$ $=: c(n,k)$ (signless) Stirling # of 1^{st} kind

$c(4,4) = 1$

(1)(2)(3)(4)

$c(4,3) = 6$

(12)(3)(4)
(13)(2)(4)
(14)(2)(3)
(23)(1)(4)
(24)(1)(3)
(34)(1)(2)

$c(4,2) = 11$

(123)(4)
(132)(4)
(124)(3)
(142)(3)
(134)(2)
(143)(2)
(234)(1)
(243)(1)
(12)(34)
(13)(24)
(14)(23)

$c(4,1) = 6$

(1234)
(1243)
(1324)
(1342)
(1423)
(1432)

\# k block set partitions of $\{1,2,\dots,n\}$ $=: S(n,k)$ Stirling # of 2^{nd} kind

$S(4,4) = 1$

1|2|3|4

$S(4,3) = 6$

12|3|4
13|2|4
14|2|3
23|1|4
24|1|3
34|1|2

$S(4,2) = 7$

123|4
124|3
134|2
234|1
12|34
13|24
14|23

$S(4,1) = 1$

1234

Triangle recurrences

$$c(n,k) = \underbrace{c(n-1,k-1)}_{\text{n is a singleton cycle}} + \underbrace{(n-1)\cdot c(n-1,k)}_{\text{n is not a singleton cycle}}$$

$c(n,k)$: k cycle permutations of $\{1,2,\dots,n-1,n\}$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮						⋱

$$S(n,k) = \underbrace{S(n-1,k-1)}_{\text{n is a singleton block}} + \underbrace{k\cdot S(n-1,k)}_{\text{n is not a singleton block}}$$

$S(n,k)$: k block partitions of $\{1,2,\dots,n-1,n\}$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮						⋱

Generating functions

$$1+6t+11t^2+6t^3=(1+t)(1+2t)(1+3t)$$

$$\sum_{i=1}^{n} c(n,n-i)\,t^i=(1+t)(1+2t)\cdots(1+(n-1)t)$$

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮						⋱

$c(n,k)$

$$1+6t+25t^2+\dots=\frac{1}{(1-t)(1-2t)(1-3t)}$$

$$\sum_{i=0}^{\infty} S((n-1)+i,\,n-1)\,t^i=\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮				⋮		⋱

$S(n,k)$

2. Algebras, Hilbert functions/series

$$A = \bigoplus_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \dots$$

with $A_i \cdot A_j \subset A_{i+j}$

a graded associative $\mathbb{k}$-algebra

$\mathbb{k}$ a field

has **Hilbert series**

$$\mathrm{Hilb}(A,t) := \sum_{i=0}^{\infty} \underbrace{\dim_{\mathbb{k}}(A_i)}_{\text{called Hilbert function } h(A,i)} \cdot t^i$$

EXAMPLES:

$$\mathrm{Hilb}\left(\Lambda_{\mathbb{k}}\{x_1,\dots,x_n\},t\right) = \sum_{i=0}^{n}\binom{n}{i}t^i = (1+t)^n$$

$= \Lambda^\bullet V$ where $V = \mathrm{span}_{\mathbb{k}}\{x_1,\dots,x_n\}$

exterior algebra

$x_i x_j = -x_j x_i$

$x_i^2 = 0$

$$\mathrm{Hilb}\left(\mathbb{k}[y_1,\dots,y_n],t\right) = \sum_{i=0}^{\infty}\binom{n+i-1}{i}t^i = \frac{1}{(1-t)^n}$$

$= \mathrm{Sym}^\bullet(V^*)$ where $V^* = \mathrm{span}_{\mathbb{k}}\{y_1,\dots,y_n\}$

polynomial algebra (commutative)

$y_i y_j = y_j y_i$

$c(n,k)$ are also a Hilbert function ...

... for two related cohomology algebras A, both with

$$\mathrm{Hilb}(A,t) = \sum_{i=1}^{n} c(n,n-i)\,t^i = (1+t)(1+2t)\cdots(1+(n-1)t)$$

THEOREM: $A := H^\bullet(\mathrm{Conf}_n(\mathbb{C}), \Bbbk)$

V.I. Arnold 1968

$\{(z_1,\ldots,z_n)\in\mathbb{C}^n : z_i\neq z_j \text{ for } i\neq j\}$

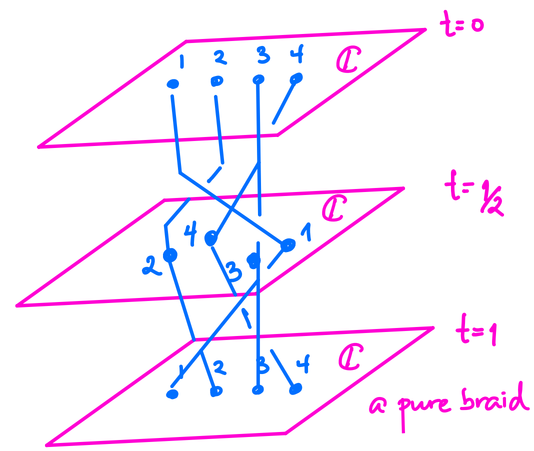
configuration space of n labeled points in $\mathbb{C}$

exterior algebra

$$\cong \Lambda_{\Bbbk}\{x_{ij}\}_{1\leq i<j\leq n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}\right)_{1\leq i<j\leq k\leq n}$$

$\cong$ group cohomology of pure braid group PB_n

$$PB_n = \ker(\underset{\text{braid group}}{B_n} \longrightarrow \underset{\text{symmetric group}}{\mathfrak{S}_n})$$



$\cong$ Orlik-Solomon algebra of type A_{n-1} reflection arrangement

THEOREM: Same presentation works for $\mathrm{Conf}(\mathbb{R}^d)$, $d=2,4,6,\ldots$ even
F. Cohen
1972

(not just $\mathbb{C}=\mathbb{R}^2$)

and similarly, for $d=3,5,7,\ldots$ odd, one has

$$A := H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), k)$$

(commutative) polynomial algebra

$$\cong k[x_{ij}]_{1\le i<j\le n} \Big/ \big(x_{ij}^2,\ \underbrace{x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}}_{\text{same!}}\big)_{1\le i<j\le k\le n}$$

$\cong$ graded Varchenko-Gelfand ring of type A_{n-1} reflection arrangement

{ Varchenko-Gelfand 1987
deLongueville-Schultz 2001
Moseley 2017 }

NOTE: We will rescale the grading on both algebras A to divide by $d-1$, making $\deg(x_{ij})=1$ rather than $x_{ij} \in H^{d-1}(\mathrm{Conf}_n(\mathbb{R}^d))$

Why do both have $\mathrm{Hilb}(A,t) = (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{i=1}^{n} c(n,n-i)t^i$?

F. Cohen's proof shows both presentations

$$A = \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}} \right)_{1\le i<j\le k\le n} \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}} \right)_{1\le i<j\le k\le n} \end{cases}$$

are Gröbner basis presentations
(exterior, commutative)

with initial terms underlined in green, giving

standard monomial $\Bbbk$-bases for A

= squarefree products of at most one variable from these sets:

$$\{x_{12}\},\quad \{x_{13}, x_{23}\},\quad \{x_{14}, x_{24}, x_{34}\},\quad \ldots,\quad \{x_{1n}, x_{2n}, \ldots, x_{n-1,n}\}$$

$$(1+t)\quad\cdot\quad(1+2t)\quad\cdot\quad(1+3t)\quad\cdots\quad(1+(n-1)t)$$

Are the $S(n,k)$ also a Hilbert function?

Yes, $$\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i = \mathrm{Hilb}(A^!, t)$$

where $A^!$ is the Koszul dual algebra

for either of the quadratic algebras

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \big/ (x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 3,5,7,\ldots \end{cases}$$

3. Koszul algebras & their Koszul duals

DEFINITION: $A = \bigoplus_{i=0}^{\infty} A_i = \underbrace{A_0}_{=\Bbbk} \oplus A_1 \oplus A_2 \oplus \cdots$

a standard graded connected associative $\Bbbk$-algebra

means

$A = \underbrace{k\langle x_1, \ldots, x_n \rangle}_{\text{free associative algebra}} / I$ for a two-sided ideal $I \subset k\langle x_1, \ldots, x_n \rangle$

free associative algebra on $x_1, \ldots, x_n \in A_1$

or

tensor algebra $T^{\bullet}(V)$ on $V = \mathrm{span}_{\Bbbk}\{x_1, \ldots, x_n\} = A_1$

which is homogeneous:

$$I = \bigoplus_{i=2}^{\infty} I_i$$

where $I_i := T^i(V) \cap I$

(Priddy 1970)

DEFINITION: A is a Koszul algebra if there exists a free A-resolution of $\Bbbk = A/\underbrace{A_+}_{A_1 \oplus A_2 \oplus A_3 \oplus \cdots}$

having all linear maps:

i.e. all entries lie in A_1

$$\cdots \to A(-3)^{\beta_3} \xrightarrow{\left[\begin{smallmatrix} \cdots \\ \cdots \\ \cdots \end{smallmatrix}\right]} A(-2)^{\beta_2} \xrightarrow{\left[\begin{smallmatrix} \vdots & \vdots \\ \vdots & \vdots \end{smallmatrix}\right]} A(-1)^{\beta_1} \xrightarrow{[x_1 \cdots x_n]} A \to \Bbbk \to 0$$

$$e_1 \longmapsto x_1$$
$$\vdots \qquad \vdots$$
$$e_n \longmapsto x_n$$

$A = \Bbbk\langle x_1, \ldots, x_n\rangle / I$ Koszul $\Longrightarrow$ I is quadratic:

$I = (I_2)$

is generated by $I_2 = I \cap T^2(V)$

THEOREM (Priddy 1970) When A is Koszul, its quadratic dual algebra $A^!$

defined by $A^! := \underbrace{\mathbb{k}\langle y_1, \ldots, y_n \rangle}_{T^\bullet(V^*)} / J$ where $J = (J_2)$

for $V^* = \mathrm{span}_{\mathbb{k}}\{y_1, \ldots, y_n\}$ with $(y_i, x_j) = \delta_{ij}$

with $J_2 := I_2^{\perp} \subset V^* \otimes V^*$

gives an explicit linear free A-resolution of $\mathbb{k}$ built on $A \otimes_{\mathbb{k}} (A^!)^*$:

$$\cdots \to A \otimes_{\mathbb{k}} (A^!_3)^* \longrightarrow A \otimes_{\mathbb{k}} (A^!_2)^* \to A \otimes_{\mathbb{k}} (A^!_1)^* \to A \otimes_{\mathbb{k}} (A^!_0)^* \to \mathbb{k} \to 0$$

(now called Priddy's complex)

COROLLARY: $\boxed{\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!, -t) = 1}$ when A is Koszul.

i.e. $\mathrm{Hilb}(A^!, t) = \dfrac{1}{\mathrm{Hilb}(A, -t)}$

More generally, a group G of graded symmetries of A also acts on $A^!$,

and has virtual G-character (equivariant) identities, recurrences:

$$\boxed{\mathrm{Hilb}_{eq}(A,t)\cdot\mathrm{Hilb}_{eq}((A^!)^*,-t)=1} \quad \text{in } \underbrace{R(G)}_{\substack{\text{ring of complex}\\ G\text{-characters,}\\ \text{or Grothendieck ring of}\\ kG\text{-modules}}}[[t]]$$

or equivalently

$$\boxed{(A_i^!)^*=A_1\otimes(A_{i-1}^!)^*-A_2\otimes(A_{i-2}^!)^*+A_3\otimes(A_{i-3}^!)^*-\cdots\pm A_i} \quad \text{in } R(G)$$

Koszul recurrence for $\{A_i^!\}$ in terms of $\{A_i\}$

EXAMPLE

$$A = \underbrace{\Lambda_{\Bbbk}\{x_1,\dots,x_n\}}_{\Lambda^\bullet V} \cong \Bbbk\langle x_1,\dots,x_n\rangle \big/ \underbrace{(x_i^2,\ x_ix_j + x_jx_i)}_{I = (I_2)}$$ is Koszul

$$A^! = \underbrace{\Bbbk[y_1,\dots,y_n]}_{Sym^\bullet(V^*)} \cong \Bbbk\langle y_1,\dots,y_n\rangle \big/ \underbrace{(y_iy_j - y_jy_i)}_{J = (J_2)}$$ is its Koszul dual, where $J_2 = I_2^\perp \subset T^2(V^*)$

and Priddy's complex = (usual) Koszul complex resolving $\Bbbk$ over $\Bbbk[\underline{y}]$:

$$0 \to \Lambda^n V \otimes_{\Bbbk} Sym(V^*) \to \dots \to \Lambda^2 V \otimes_{\Bbbk} Sym(V^*) \to V \otimes Sym(V^*) \to Sym(V^*) \to \Bbbk \to 0$$

$$\begin{aligned} x_1 &\longmapsto y_1 \longmapsto 0 \\ &\ \ \vdots \\ x_n &\longmapsto y_n \longmapsto 0 \end{aligned}$$

$$x_i \wedge x_j \longmapsto y_ix_j - y_jx_i$$

How to prove an algebra A is Koszul?

THEOREM (Folklore + Fröberg 1975 for monomial case): When A is commutative $\Bbbk[x_1,\ldots,x_n]/I$ or anticommutative $\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}/I$ and I has a quadratic Gröbner basis for some monomial order on $\Bbbk[x_1,\ldots,x_n]$ or $\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}$, then A is Koszul.

e.g. $A = H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ is Koszul

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=3,5,7,\ldots \end{cases}$$

$$A^! = \Bbbk\langle y_{ij}\rangle_{1\le i<j\le n} \Big/ \left([y_{ij}, y_{k\ell}]\right)_{\{i,j\}\cap\{k,\ell\}=\emptyset} + \left([y_{ij}, y_{ik}+y_{jk}]\right)_{1\le i<j<k\le n}$$

is its Koszul dual where $[a,b] := \begin{cases} ab-ba & \text{if } d \text{ even} \\ ab+ba & \text{if } d \text{ odd} \end{cases}$

REMARK: Supersolvable hyperplane arrangements are lurking here!

COROLLARY: $A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ (for d even or odd) have

$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A,t)} = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

$$= \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i$$

i.e. $\boxed{\dim_{\Bbbk}(A^!_i) = S((n-1)+i,\, n-1)}$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
$\vdots$						$\ddots$

$n=4$ $\mathrm{Hilb}(A,t)$

$c(n,k)$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
$\vdots$				$\vdots$		$\ddots$

$n=4$ $\mathrm{Hilb}(A^!,t)$

$S(n,k)$

Topological REMARK:

$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{k})$ for $d \geq 3$

has $A^! \cong H_\bullet(\Omega\,\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{k})$

(base-pointed) loop space

Studied, e.g., by Cohen-Gitler 2002

who called the relations "infinitesimal braid relations"

$[y_{ij}, y_{k\ell}] = 0$ for $\{i,j\} \cap \{k,\ell\} = \phi$

$[y_{ij}, y_{ik} + y_{jk}] = 0$ for $1 \leq i < j < k \leq n$

QUESTION: Can this help us better understand the $\mathfrak{S}_n$-reps on $A^!$?

4. Representation theory

$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ carry actions of the symmetric group $\mathfrak{S}_n$ on $\{1,2,\ldots,n\}$.

Q: What do the $\mathfrak{S}_n$-representations on the graded components of A, $A^!$ look like?

Can one decompose them into the $\mathfrak{S}_n$-irreducible representations $\{S^\lambda\}$, indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of n?

$A = H^\cdot \mathrm{Conf}_n(\mathbb{R}^d)$ = Stirling reps of 1st kind have generating function formulas involving plethysms (Sundaram & Welker 1997)

– implemented in SAGE/cocalc by T. Karn

$n=4$

	1 +	$6t$ +	$11t^2$ +	$6t^3$	total rep'n (ungraded)
	A_0	A_1	A_2	A_3	
$d=3,5,7\ldots$ odd	$S^{(4)}$	$S^{(3,1)}$ $S^{(2,1,1)}$	$S^{(3,1)}$ $S^{(2,1,1)}$ $S^{(2,2)}$ $S^{(2,2)}$ $S^{(1,1,1,1)}$	$S^{(3,1)}$ $S^{(2,1,1)}$	$\Bbbk[\mathfrak{S}_4]$ = regular rep.
$d=2,4,6,\ldots$ even	$S^{(4)}$	$S^{(3,1)}$ $S^{(2,2)}$ $S^{(4)}$	$S^{(3,1)}$ $S^{(2,1,1)}$ $S^{(3,1)}$ $S^{(2,2)}$	$S^{(3,1)}$ $S^{(2,1,1)}$	2 copies of $\Bbbk[\mathfrak{S}_4/(\mathfrak{S}_2\times\mathfrak{S}_1\times\mathfrak{S}_1)]$

What about $A(n)^!$ for $A(n) = H^* \mathrm{Conf}_n(\mathbb{R}^d)$?

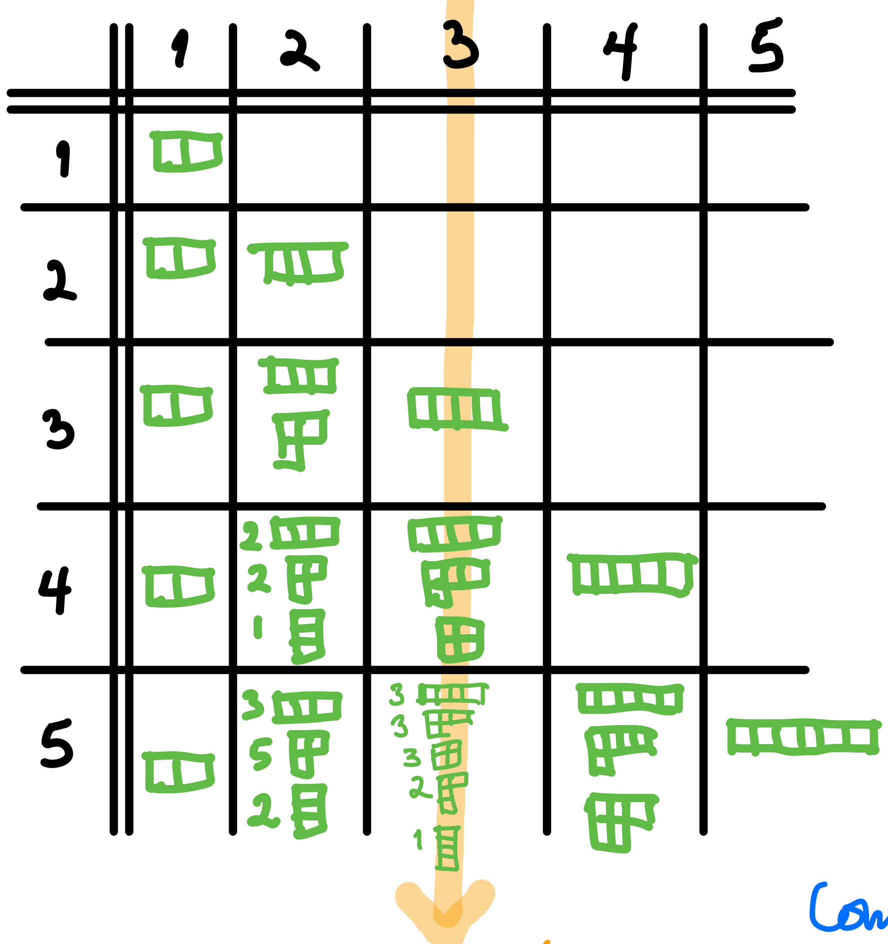
$S(n,k)$

$n= \backslash\ k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

n=4

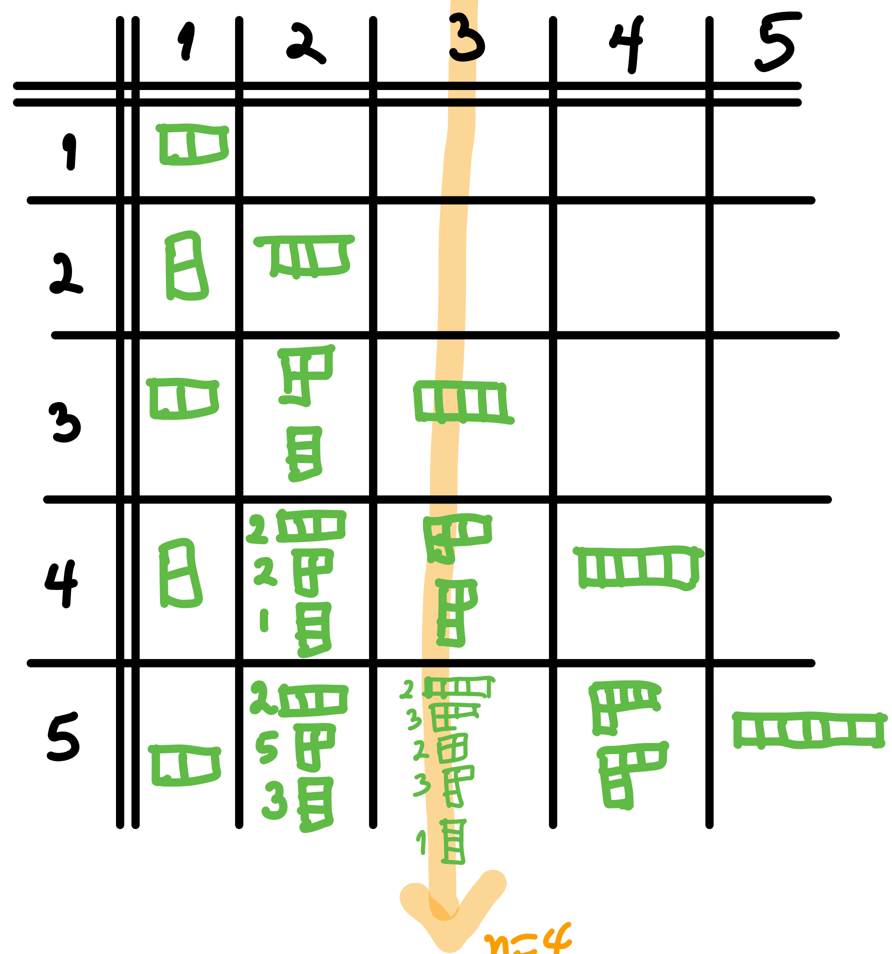
$$\dim A(n)^!_i = S((n-1)+i,\, n-1)$$

d=2,4,6,--- even



n=4

d=3,5,7,--- odd



n=4

Computed via Koszul recurrences.

THEOREM: (Almousa–R.–Sundaram 2023+) The triangular Stirling recurrences

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$
$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

lift to short exact sequences of graded $\mathfrak{S}_{n-1}$-representations describing how $A(n)_i$ and $A(n)^!_i$ branch/restrict from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

---

defining permutation rep of $\mathfrak{S}_{n-1}$

$$0 \to A(n-1) \to A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to \chi^{(n-1)}_{def} \otimes A(n-1)(-1) \to 0$$

$$0 \to \chi^{(n-1)}_{def} \otimes \left(A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right)(-1) \to A(n)^!\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to A(n-1)^! \to 0$$

This reflects a general Koszul algebra branching relation ...

PROPOSITION: (ARS 2023⁺)

Given Koszul algebras $B \subset A$ with symmetries $H < G$ (e.g. $H^\cdot \mathrm{Conf}_{n-1}(\mathbb{R}^d) \subset H^\cdot \mathrm{Conf}_n(\mathbb{R}^d)$ with symmetries $\mathfrak{S}_{n-1} < \mathfrak{S}_n$)

and a $\Bbbk H$-module U,

one has a sequence of character identities in $R(H)$

$$\boxed{A_i \downarrow_H^G = B_i + U \otimes B_{i-1}} \quad \text{for } A$$

$$\Updownarrow$$

$$\boxed{A_i^! \downarrow_H^G = B_i^! + U^* \otimes A_{i-1}^! \downarrow_H^G} \quad \text{for } A^!$$

Representation Stability

DEFINITION: (Church & Farb 2013) A sequence of $\mathfrak{S}_n$-representations $\{V_n\}_{n=1,2,3,\ldots}$ are called representation-stable if

$\exists$ some N, and partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t)}$
and multiplicities $c_1, c_2, \ldots, c_t$

such that $\forall\, n \geq N$, one has

$$V_n \cong \bigoplus_{j=1}^{t} \left(S^{\left(\overbrace{\quad\quad}^{n-|\lambda^{(j)}|},\ \lambda^{(j)}\right)} \right)^{\oplus c_j}$$

---

e.g. **THEOREM:** (Church & Farb 2013) Fixing $i \geq 0$, $\left\{ H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) \right\}_{n=1,2,\ldots}$ is representation-stable.

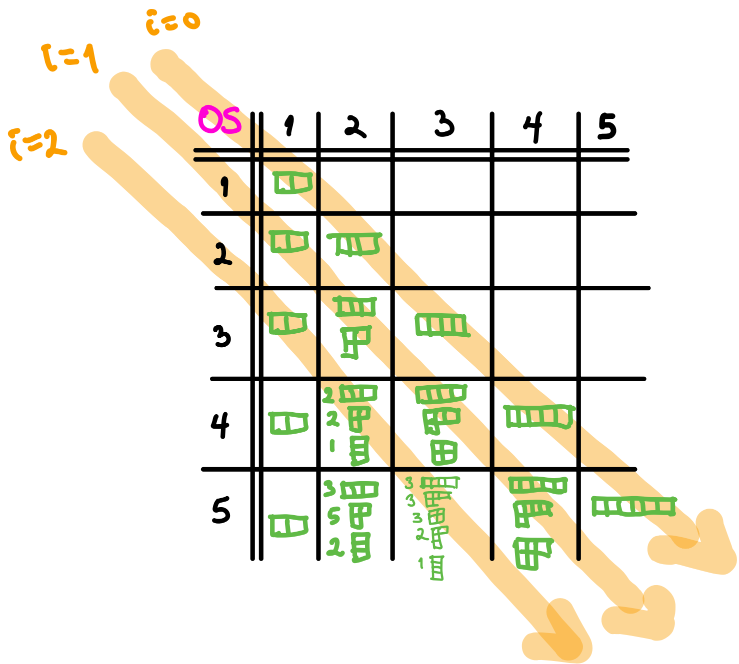
THEOREM: (Hersh & R. 2016) The above stability starts at $n = \begin{cases} 3i & \text{for } d=3,5,7,\ldots \text{ odd} \\ 4i & \text{for } d=2,4,6,\ldots \text{ even} \end{cases}$

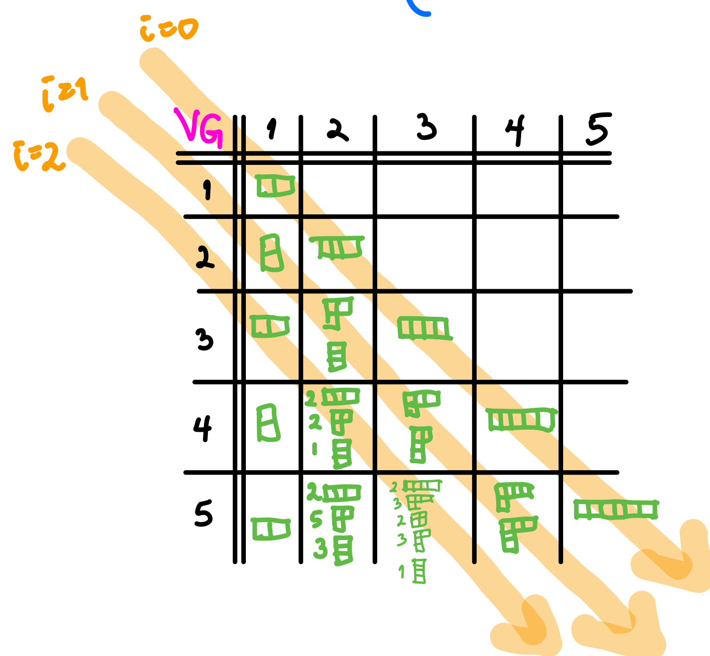
THEOREM: (ARS 2023+) Assuming $\{A(n)\}_{n=1,2,\ldots}$ are Koszul, then

$\{A(n)_i\}_{n=1,2,\ldots}$ rep-stable past $n = c\cdot i$ $\Rightarrow$ same for $\{A(n)^!_i\}_{n=1,2,\ldots}$

COROLLARY: (ARS 2023+) For $A(n) := H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$,

the $\{A(n)^!_i\}_{n=1,2,\ldots}$ are rep-stable past $n = \begin{cases} 3i & \text{for } d=3,5,7,\ldots \text{ odd} \\ 4i & \text{for } d=2,4,6,\ldots \text{ even} \end{cases}$





THEOREM: For $d = 2, 4, 6, \ldots$ even,
(ARS 2023+)

- $\mathrm{Hilb}_{eq}(H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d), t)$ is divisible by $1+t$ for $d = 2, 4, 6, \ldots$ even

  because multiplication by $x_1 + x_2 + \ldots + x_n$ makes $H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d) =: A$

  a G-equivariant exact cochain complex

  $$0 \to H^0 \to H^1 \to H^2 \to \cdots \to H^{n-1} \to 0$$

  (Yuzvinsky 2001)

- $\mathrm{Hilb}_{eq}(A^!, t)$ is divisible by $1 + t + t^2 + \ldots = \frac{1}{1-t}$ for $d = 2, 4, 6, \ldots$ even

  because multiplication on the right by $y_1 + y_2 + \ldots + y_n$

  gives G-equivariant injective maps

  $$A^!_0 \hookrightarrow A^!_1 \hookrightarrow A^!_2 \hookrightarrow \ldots$$

Permutation representations

The $\mathfrak{S}_n$-representations on $A_i := H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d)$ are not permutation representations.

But when $d = 2, 4, 6, \ldots$ even,

A_i turned out be permutation representations surprisingly often:

- for $i = 0, 1$ (and $\frac{1}{2}$ a perm rep for $i = 2$!)
- for $n = 1, 2, 3, 4, 5$.

(but failed for $n = 9$ with $i = 3$,
$n = 6$ with $i = 5$) ← checked with T. Karn's Burnside Solver

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QUESTION: Is there a reason why this occurs?

Thanks for your attention!

$S(n,k)$

$n= \backslash k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$A = H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d)$$

$$\dim A_i \overset{!}{=} S((n-1)+i,\, n-1)$$