Koszulity and Stirling representations

– a preliminary report

Ayah Almousa
Vic Reiner
} U. Minnesota
Sheila Sundaram

Various Guises of Reflection Arrangements
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1. Stirling numbers $\overset{1^{st}\text{ kind}}{c(n,k)}$, $\overset{2^{nd}\text{ kind}}{S(n,k)}$

2. (Familiar) algebras of the 1^{st} kind

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5. (Koszul dual) algebras of the 2^{nd} kind

6. Properties and Questions

1. Stirling numbers $c(n,k)$, $S(n,k)$

(Signless) Stirling number of the 1^{st} kind

$$c(n,k) := \# \text{ permutations in } \mathfrak{S}_n \text{ with } k \text{ cycles}$$

for $1 \le k \le n$

$c(4,4) = 1$	$c(4,3) = 6$	$c(4,2) = 11$		$c(4,1) = 6$
$e = (1)(2)(3)(4)$	(12)	(123)	$(12)(34)$	(1234)
	(13)	(132)	$(13)(24)$	(1243)
	(14)	(124)	$(14)(23)$	(1324)
	(23)	(142)		(1342)
	(24)	(134)		(1423)
	(34)	(143)		(1432)
		(234)		
		(243)		

Generating function definition:

$$\sum_{k=1}^{n} c(n,k)\, t^{n-k} = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

$n=4$:

$$(1+t)(1+2t)(1+3t) = 1 + 6t + 11t^2 + 6t^3$$

$c(4,4)$ $c(4,3)$ $c(4,2)$ $c(4,1)$

(Signless) Stirling number of the 2^{nd} kind

$S(n,k) :=$ # partitions of $\{1,2,\dots,n\}$ with k blocks

for $1 \le k \le n$

$S(4,4) = 1$	$S(4,3) = 6$	$S(4,2) = 7$	$S(4,1) = 1$
1|2|3|4	12|3|4	123|4	1234
	13|2|4	124|3	
	14|2|3	134|2	
	23|1|4	234|1	
	24|1|3	12|34	
	34|1|2	13|24	
		14|23	

Generating function definition:

$$\sum_{n=k}^{\infty} S(n,k)\,t^n = \frac{t^k}{(1-t)(1-2t)(1-3t)\cdots(1-k\cdot t)}$$

Rewritten for later purposes:

$$\sum_{i=0}^{\infty} S(n-1+i,\,n-1)\,t^i = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

$$\left(\text{cf. } \sum_{k=1}^{n} c(n,k)\,t^{n-k} = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)\right)$$

Triangles and recursions

$c(n,k)$

$n= \backslash k=$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
5	24	50	35	10	1

$S(n,k)$

$n= \backslash k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1)$$

$$S(n,k) = k\cdot S(n-1,k) + S(n-1,k-1)$$

2. (Familiar) algebras of the 1st Kind

$$A(n) := H^{\bullet}\,\mathrm{Conf}(n,\mathbb{R}^d)$$

(ordered) configuration space of n distinct points $(p_1, p_2, \dots, p_n)$ in $\mathbb{R}^d$, $p_i \neq p_j$

$$\cong \begin{cases} A_{OS}(n) = \overbrace{\Lambda[e_{ij}]}^{\text{skew-commutative exterior algebra}} \Big/ (e_{ij}e_{ik} - e_{ij}e_{jk} + e_{ik}e_{jk}) & d \text{ even} \\ A_{VG}(n) = \underbrace{\Bbbk[x_{ij}]}_{\text{commutative polynomial ring}} \Big/ (x_{ij}^2 \text{ and } x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}) & d \text{ odd} \end{cases}$$

Arnol'd 1968

for $1 \le i < j < k \le n$

F. Cohen 1972

rescale cohomological grading by $d-1$

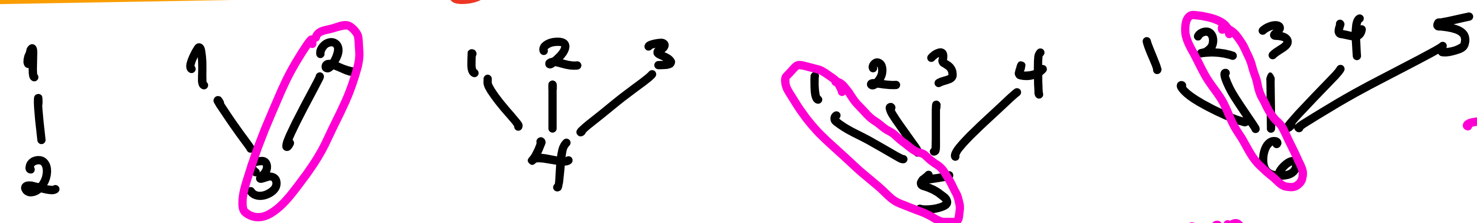
Both $A(n) = A_{OS}(n)$, $A_{VG}(n)$ have

$$\mathrm{Hilb}(A(n),t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

since those were Gröbner bases for the ideals, with initial terms $e_{ik}e_{jk}$ and $x_{ik}x_{jk}, x_{ij}^2$, leading to standard monomial NBC bases = "at most one finger from each hand"

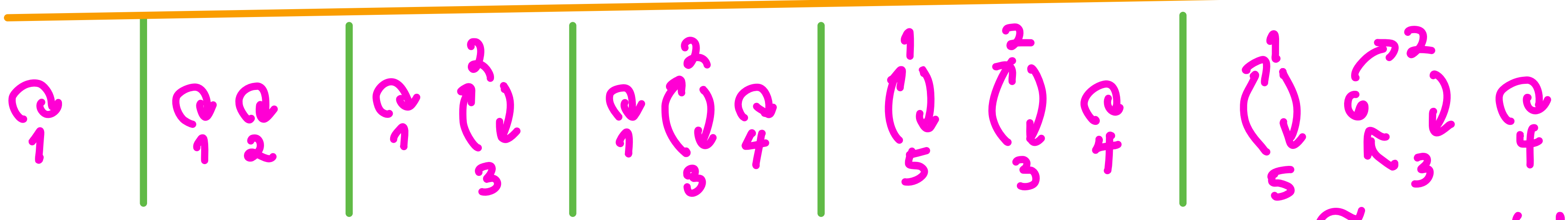
Barcelo 1988

$n=6$:



$\rightsquigarrow e_{23}e_{15}e_{26}$

NBC monomial of deg 3



$w \in \mathfrak{S}_6$ counted by $c(6,6-3) = c(6,3)$

As $\mathfrak{S}_n$-representations, both $A(n)=A_{OS}(n)$, $A_{VG}(n)$ are well-studied, but not completely understood.

$n=4$	1 + A_0	$6t$ + A_1	$11t^2$ + A_2	$6t^3$ A_3	total rep'n (ungraded)
$A_{VG}(4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$ $S_{(2,2)}$ $S_{(2,2)}$ $S_{(1,1,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$\Bbbk[\mathfrak{S}_4]$ = regular rep.
$A_{OS}(4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,2)}$	$S_{(3,1)}$ $S_{(2,1,1)}$		2 copies of $\Bbbk[\mathfrak{S}_4/(\mathfrak{S}_2\times\mathfrak{S}_1\times\mathfrak{S}_1)]$
		$S_{(4)}$	$S_{(3,1)}$ $S_{(2,2)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	

THEOREM (Sundaram-Welker 1997) As $\mathfrak{S}_n$-representations,

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n} \mathrm{ch}\, A(n)_{n-k}\, t^k =$$

$$\begin{cases} \displaystyle\sum_{\lambda = 1^{m_1} 2^{m_2} \cdots} t^{\ell(\lambda)} \cdot \prod_{j=1}^{\infty} h_{m_j}[\mathrm{Lie}_j] = \prod_{m=1}^{\infty} (1-p_m)^{-a_m(t)} & \text{for VG} \\ \displaystyle\sum_{\lambda = 1^{m_1} 2^{m_2} \cdots} t^{\ell(\lambda)} \prod_{j \text{ odd}} h_{m_j}[\pi_j] \cdot \prod_{j \text{ even}} e_{m_j}[\pi_j] = \prod_{m=1}^{\infty} (1+(-1)^m p_m)^{a_m(-t)} & \text{for OS} \end{cases}$$

plethysm formulas vs. product generating functions

where $a_m(t) = \frac{1}{m} \sum_{d \mid m} \mu(d)\, t^{m/d}$

Many results by Lehrer-Solomon, Whitehouse, Douglass-Pfeiffer-Röhrle, ...

Branching for $A(n)$

The recurrence

$$c(n,k) = (n-1)\cdot c(n-1,k) + c(n-1,k-1)$$

lifts easily (via the generating functions):

$c(n,k)$

$n=$ \ $k=$	1	2	3	4	5
1	1				
2	1	1			
3	2	3	1		
4	6	11	6	1	
5	24	50	35	10	1

PROPOSITION: (Sundaram 1994, 2020) Both $A(n) = A_{OS}(n), A_{VG}(n)$ have these branching rules for restriction $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

$$A(n)_i \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong \chi^{(n-1)}_{def} \otimes A(n-1)_{i-1} \oplus A(n-1)_i$$

defining rep. of $\mathfrak{S}_{n-1}$ as permutation matrices

Better phrasing:

PROPOSITION: $\mathcal{A}(n) = \left\{ \begin{matrix} A_{OS}(n) \\ A_{VG}(n) \end{matrix} \right\}$ have $\mathfrak{S}_{n-1}$-equivariant s.e.s.

$$0 \to \mathcal{A}(n-1) \longrightarrow \mathcal{A}(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \longrightarrow \left[\chi^{(n-1)}_{def} \otimes \mathcal{A}(n-1) \right](-1) \to 0$$

This generalizes to $\left\{ \begin{matrix} \text{Orlik-Solomon} \\ \text{Varchenko-Gelfand} \end{matrix} \right\}$ algebras

$$A_{OS}(\mathcal{A}) := \Lambda\{e_H\}_{H \in \mathcal{A}} \Big/ \left(\partial e_C \ : \text{circuits } C \right)$$

$$A_{VG}(\mathcal{A}) := \Bbbk[x_H]_{H \in \mathcal{A}} \Big/ \left(\partial x_C \ : \text{circuits } C \right)$$

where for a circuit $C = \{H_1, \ldots, H_p\}$ with $\sum_{j=1}^{p} c_j \alpha_j = \underline{0}$ if $H_j := \ker(\alpha_j)$

$$\partial e_C := \sum_{j=1}^{p} (-1)^j e_{H_1} \wedge \cdots \wedge \hat{e}_{H_j} \wedge \cdots \wedge e_{H_p}$$

$$\partial x_C := \sum_{j=1}^{p} \operatorname{sgn}(c_j)\, x_{H_1} \cdots \hat{x}_{H_j} \cdots x_{H_p}$$

PROPOSITION: (Orlik–Terao 1992, ARS 2023) Assume the arrangement $\mathcal{A}$ has symmetries W and X a modular coatom (=line) with W-stabilizer N_X.

'if $H, H' \not\supset X$ then $\exists H'' \supset X$ with $H'' \supset H \cap H'$'

Then both algebras $A(\mathcal{A}) = A_{OS}(\mathcal{A}), A_{VG}(\mathcal{A})$ have an N_X-equivariant s.e.s.

$$0 \to A(\mathcal{A}_X) \to A(\mathcal{A}) \Big\downarrow^{W}_{N_X} \xrightarrow{\bigoplus_{H : H \not\supset X} j_H} \left[\bigoplus_{H : H \not\supset X} A(\mathcal{A}^H) \right](-1) \to 0$$

The maps j_H come from addition-deletion sequences

$$0 \to A(\mathcal{A} - \{H\}) \to A(\mathcal{A}) \xrightarrow{j_H} A(\mathcal{A}^H) \to 0$$

$$e_{H_1} \wedge \cdots \wedge e_{H_p} \longmapsto 0 \text{ if } H \notin \{H_1, \ldots, H_p\}$$

$$e_H \wedge e_{H_1} \wedge \cdots \wedge e_{H_p} \longmapsto e_{H \cap H_1} \wedge \cdots \wedge e_{H \cap H_p}$$

3. Koszulity review

Let $A = \bigoplus_{d=0}^{\infty} A_d$ be a standard graded associative $\Bbbk$-algebra, generated by $A_1 =: V$

$$= \Bbbk \oplus \underbrace{A_1}_{\substack{\Bbbk\text{-basis} \\ x_1, \ldots, x_n \text{ for } V = A_1}} \oplus A_2 \oplus \cdots$$

$$\cong \underbrace{\Bbbk\langle x_1, \ldots, x_n \rangle}_{T(V)} / I$$

where the 2-sided ideal I is homogeneous:

$I = \bigoplus_{d=0}^{\infty} I_d$ where

$$I_d = I \cap \underbrace{T^d(V)}_{= \Bbbk\langle x_1, \ldots, x_n \rangle_d}$$

DEFINITION: A is Koszul if $\exists$ an A-free resolution of $\mathbb{k} = A/A_+$ which is linear:

$$0 \leftarrow \mathbb{k} \leftarrow A \xleftarrow{d_1} A(-1)^{n} \xleftarrow{d_2} A(-2)^{\beta_2} \xleftarrow{d_3} A(-3)^{\beta_3} \leftarrow \cdots$$

$$\begin{array}{ccc} 0 & \mapsfrom & x_1 \\ & & \vdots \\ 0 & \mapsfrom & x_n \end{array} \qquad [x_1 \cdots x_n] \qquad \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \qquad \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

all entries lie in A_1

Equivalently, $\mathrm{Tor}_i^A(\mathbb{k},\mathbb{k})_j = 0$ unless $i=j$

(or same for $\mathrm{Ext}_A^i(\mathbb{k},\mathbb{k})_j$)

When A is Koszul, one can write down a beautiful explicit resolution of $\Bbbk$ called the Priddy complex ...

A is necessarily a quadratic algebra

i.e. $A = k\langle x_1, \ldots, x_n\rangle / I$ with $I = (I_2)$, $I_2 \subset V \otimes V$

so one can define its quadratic dual algebra

$A^! := k\langle \underbrace{y_1, \ldots, y_n}_{V^* \text{ basis dual to } x_1, \ldots, x_n} \rangle / J$ where $J := (I_2^\perp)$

perp with respect to $(x \otimes x', y \otimes y') := (x, y) \cdot (x', y')$

The Priddy complex is $A \otimes (A^!)^*$, linearly resolving $\Bbbk$:

$$0 \leftarrow \Bbbk \leftarrow A \otimes (A^!_0)^* \xleftarrow{d_1} A \otimes (A^!_1)^* \xleftarrow{d_2} A \otimes (A^!_2)^* \xleftarrow{d_3} \cdots$$

with d_i = mult. by $\sum_{j=1}^{n} x_j \otimes (y_j)^*$

Exactness of the Priddy complex shows

$$\mathrm{Hilb}((A^!)^*, -t) \cdot \mathrm{Hilb}(A,t) = 1 \qquad [= \mathrm{Hilb}(\mathbb{k},t)]$$

so $$\mathrm{Hilb}(A^!,t) = \frac{1}{\mathrm{Hilb}(A,-t)}$$

When W acts on A via graded $\mathbb{k}$-algebra automorphisms, exactness in degree d shows virtual character identities

$$(A^!_d)^* - A_1 \otimes (A^!_{d-1})^* + A_2 \otimes (A^!_{d-2})^* - \cdots \pm (A^!_1)^* \otimes A_{d-1} \mp A_d = 0$$

expressing $(A^!_d)^*$ (and hence also $A^!_d$)

recursively in terms of $A_0, A_1, \ldots, A_d$

A useful sufficient condition for Koszulity of A:

PROPOSITION:

When $A = \Lambda(e_1, \ldots, e_n)/I$
or
$\Bbbk[x_1, \ldots, x_n]/I$

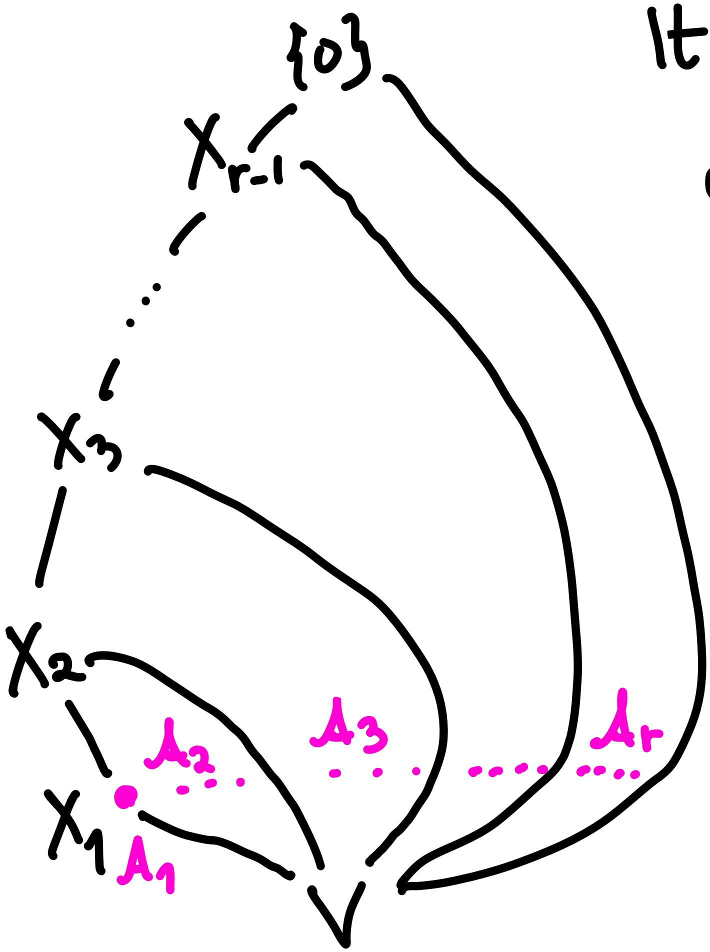
and $\exists$ a monomial order $\prec$ on $\Lambda(e_1, \ldots, e_n)$
$\Bbbk[x_1, \ldots, x_n]$

for which I has a quadratic Gröbner basis
i.e. $\mathrm{in}_{\prec}(I)$ is a quadratic (monomial) ideal,

then A is Koszul.

4. Supersolvability

DEFINITION: A central arrangement $\mathcal{A}$ is **supersolvable** if it has an **M-chain** := maximal flag of **modular flats** $V \supset X_1 \supset X_2 \supset \dots \supset X_{r-1} \supset \{\underline{0}\}$



It partitions $\mathcal{A} = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \dots \sqcup \mathcal{A}_r$

where $\mathcal{A}_i := \mathcal{A}_{X_i} - \mathcal{A}_{X_{i-1}} = \left\{ H \in \mathcal{A} : \begin{array}{l} H \supset X_i \\ H \not\supset X_{i-1} \end{array} \right\}$

and defines **exponents** $e_1, e_2, \dots, e_r$ by

$$e_i := |\mathcal{A}_i|$$

THEOREM (Björner 1990, Björner-Ziegler 1991, Shelton-Yuzvinsky 1997, Peeva 2003, Dorpalen-Barry 2021)

Either of the rings $A(\mathcal{A}) = A_{OS}(\mathcal{A})$, $A_{VG}(\mathcal{A})$ has a term order $\prec$ on $\Lambda(e_1, \ldots, e_n)$ or $\Bbbk[x_1, \ldots, x_n]$ for which its defining ideal has a quadratic Gröbner basis

$\iff$ $\mathcal{A}$ is supersolvable

In this case, the term orders $\prec$ which work are those having $x_{H_i} \prec x_{H_j}$ if $\begin{cases} H_i \in \mathcal{A}_i \\ H_j \in \mathcal{A}_j \end{cases}$ with $i < j$.

Furthermore, the initial terms look like $x_H x_{H'}$ for $H, H' \in \mathcal{A}_i$ so the standard monomial NBC-basis is the set of monomials with at most one "finger" x_H from each "hand" $\mathcal{A}_i$

$\Rightarrow \mathrm{Hilb}(A(\mathcal{A}), t) = (1+e_1 t)(1+e_2 t) \cdots (1+e_r t)$

COROLLARY: When $\mathcal{A}$ is supersolvable,
both $A(\mathcal{A}) = A_{OS}(\mathcal{A})$, $A_{VG}(\mathcal{A})$ are Koszul,
with $\mathrm{Hilb}(A^!, t) = \dfrac{1}{\mathrm{Hilb}(A, -t)} = \dfrac{1}{(1-e_1 t)(1-e_2 t)\cdots(1-e_r t)}$

COROLLARY: (ARS 2023?) For $\mathcal{A}$ supersolvable, $A(\mathcal{A})^! = \begin{cases} A_{OS}(\mathcal{A})^! \\ A_{VG}(\mathcal{A})^! \end{cases}$
have (noncommutative) quadratic Gröbner basis
with initial terms $y_{H_j} y_{H_i}$ for $\begin{cases} H_i \in \mathcal{A}_i \\ H_j \in \mathcal{A}_j \end{cases}$ with $i<j$.
and standard monomial basis
$\left\{ m^{(1)} \cdot m^{(2)} \cdot \ldots \cdot m^{(r)} : \begin{array}{l} m^{(i)} \text{ noncommutative} \\ \text{monomial in } \{y_H\}_{H \in \mathcal{A}_i} \end{array} \right\}$

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i.e. "no revisiting an earlier hand".
(or "all-you-can-eat salad bar, but keep it moving!")

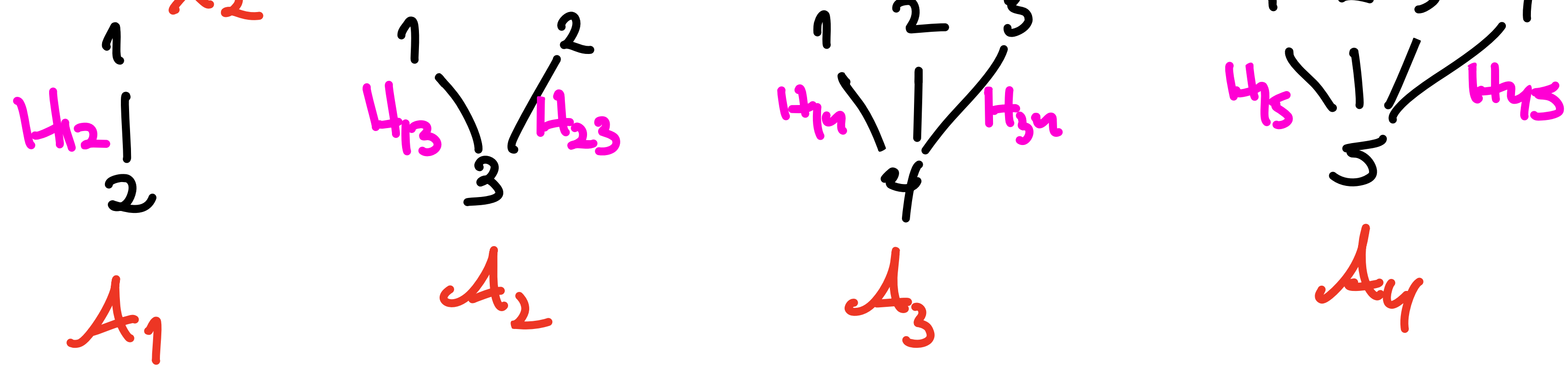
5. (Koszul dual) algebras of the 2^{nd} kind

EXAMPLE Type A reflection arrangement in $V = \mathbb{R}^n / \mathbb{R}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$ for $W = \mathfrak{S}_n$ has hyperplanes $\{H_{ij} := \{x_i = x_j\} : 1 \le i < j \le n\}$,

supersolvable, M-chain:

$$V \supset \underbrace{\{x_1 = x_2\}}_{X_1} \supset \underbrace{\{x_1 = x_2 = x_3\}}_{X_2} \supset \cdots \supset \underbrace{\{x_1 = x_2 = \cdots = x_{n-1}\}}_{X_3} \supset \underbrace{\{x_1 = x_2 = \cdots = x_n\}}_{X_{n-1}} = \{0\}$$

"hands" =

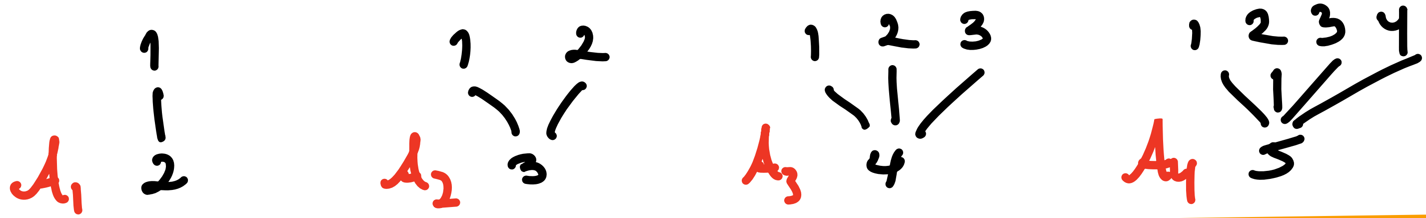


COROLLARY:

$$\mathrm{Hilb}(A(n)^!, t) = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S(n-1+i,\, n-1)\, t^i$$

EXAMPLE

$A(\mathfrak{S}_5)^!$ has standard monomials $m^{(1)}\, m^{(2)}\, m^{(3)}\, m^{(4)}$ where
$m^{(i)}$ picks noncommutative monomials in i^{th} hand $\mathcal{A}_i$:



e.g. $\quad m^{(1)} \quad\cdot\quad m^{(2)} \quad\cdot\quad m^{(3)} \quad\cdot\quad m^{(4)}$

$$= y_{12}y_{12} \cdot y_{13}y_{23}y_{23}y_{13} \cdot y_{34}y_{14} \cdot y_{45}y_{25} \in \underbrace{A(\mathfrak{S}_5)^!_{10}}_{\substack{\dim \\ = S(14,4)}}$$

$14 = 5-1+10$, $\quad 4 = 5-1$

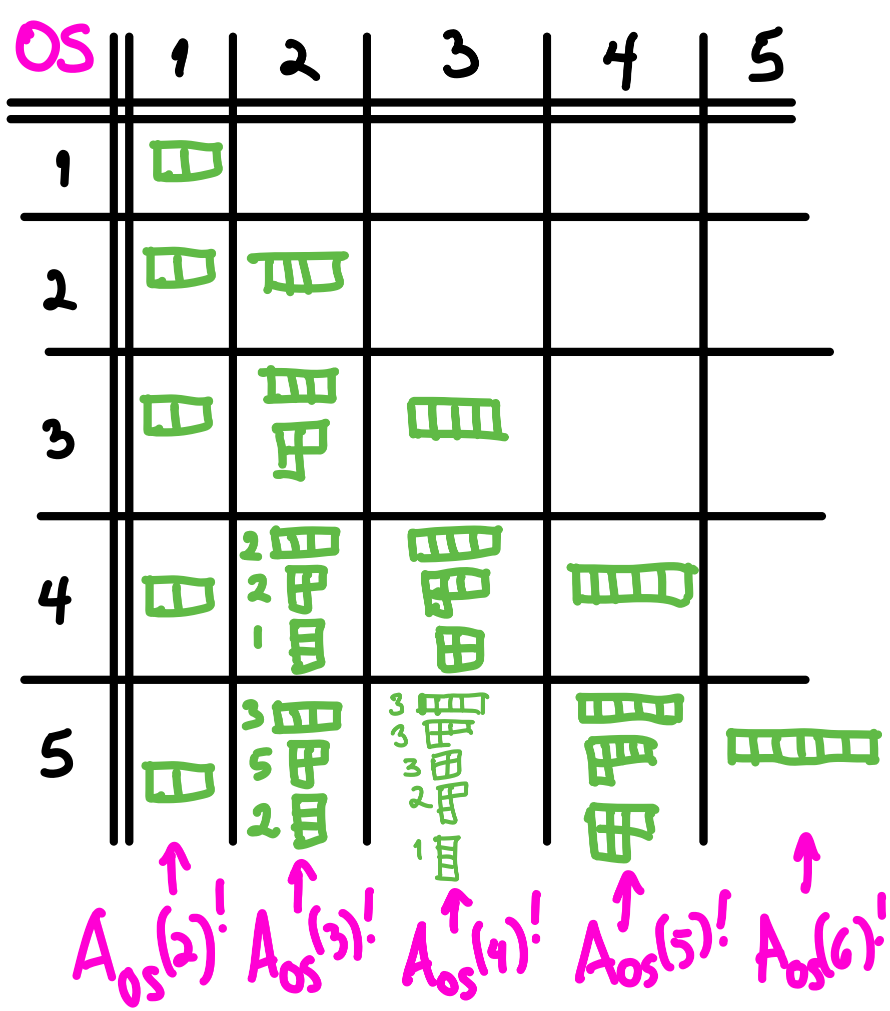
to get a set partition $\left\{\begin{array}{l}\text{insert spacers} \\ 1 \;\_\_\_\; 2 \;\_\_\_\; 3 \;\_\_\_\; 4 \;\_\_\_\; 5 \\ \text{and record } 1^{st} \text{ indices } i \text{ in } y_{ij} \\ \text{to get restricted growth function}\end{array}\right.$

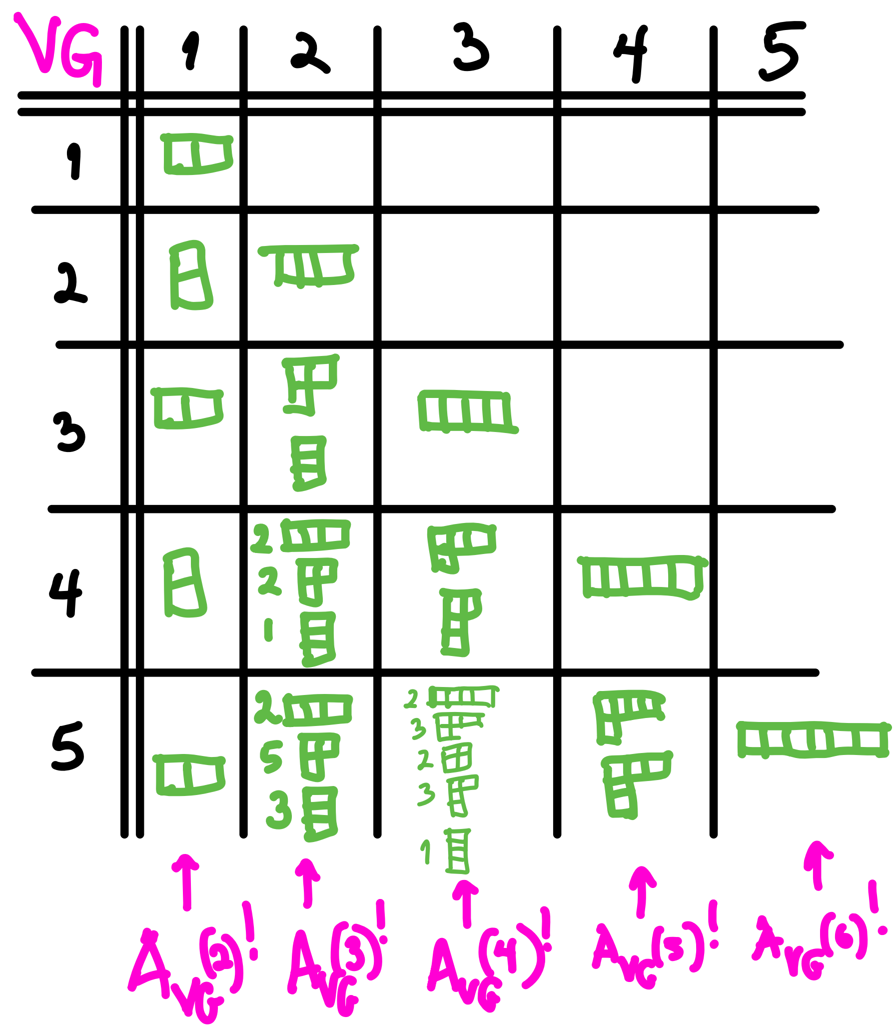
1	1	1	2	1	2	2	1	3	3	1	4	4	2	5
1	2	3	4	5	6	7	8	9	10	11	12	13	14	

$\rightsquigarrow$ $\overset{\text{block}}{1}\{1,2,3,5,8,11\}$, $\overset{\text{block}}{2}\{4,6,7,14\}$, $\overset{\text{block}}{3}\{9,10\}$, $\overset{\text{block}}{4}\{12,13\}$

S(n,k)

n= \ k=	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1





6. Properties and Questions

- **Branching rule**

$S(n,k) = k \cdot S(n-1,k) + S(n-1,k-1)$ generalizes to...

PROPOSITION: Both $A(n)^!_{} = A_{OS}(n)^!,\ A_{VG}(n)^!$ have these branching rules for restriction $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

$$A(n)^!_i \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \cong \chi^{(n-1)}_{def} \otimes A(n)^!_{i-1} \Big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \oplus A(n-1)^!_i$$

Better phrasing:

PROPOSITION: (ARS 2023) $A(n)^! = \begin{cases} \mathrm{Aos}(n)^! \\ \mathrm{AvG}(n)^! \end{cases}$ have $\mathfrak{S}_{n-1}$-equivariant s.e.s.

$$0 \leftarrow A(n-1)^! \leftarrow A(n)^! \big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \longleftarrow \left[\chi^{(n-1)}_{\mathrm{def}} \otimes A(n)^! \right](-1) \leftarrow 0$$

and more generally ...

PROPOSITION: (ARS 2023) For $\mathcal{A}$ supersolvable, X a modular coatom, one has an N_X-equivariant s.e.s.

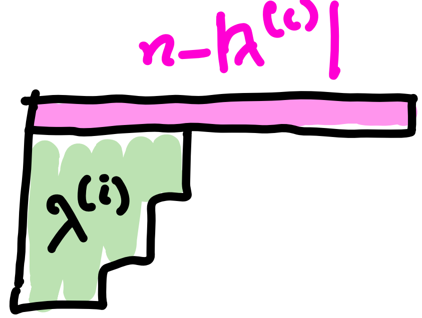
$$0 \leftarrow A(\mathcal{A}_X)^! \longleftarrow A(\mathcal{A})^! \big\downarrow^{W}_{N_X} \xleftarrow{\bigoplus_{H:\, H \not\supset X} i_H} \left[\bigoplus_{H:\, H \not\supset X} A(\mathcal{A})^! \right](-1) \leftarrow 0$$

$$0 \xleftarrow[\text{if } H \not\supset X]{} y_H \qquad a y_H \longleftarrow a$$

• Representation stability

DEFINITION (Church–Farb 2005) A sequence $\{V_n\}_{n=1,2,\dots}$ of S_n-reps is representation-stable if $\exists\ \lambda^{(1)}, \dots, \lambda^{(t)}$ and constants $c_1, \dots, c_t \in \mathbb{N}$ such that $\forall\ n \gg 0$, the S_n-irreducible decomposition looks like

$$V_n \cong \bigoplus_{i=1}^{t} \Big[\chi$$



$$\Big]^{\oplus c_i}$$

THEOREM: For fixed $i = 0, 1, 2, \ldots$
(Church-Farb)

both $A(n) = \begin{cases} A_{OS}(n) \\ A_{VG}(n) \end{cases}$

have $\{A(n)_i\}$ representation-stable

... and more generally,

$\{H^i\, \mathrm{Conf}(n, X)\}$ is rep-stable

for certain kinds of manifolds X.

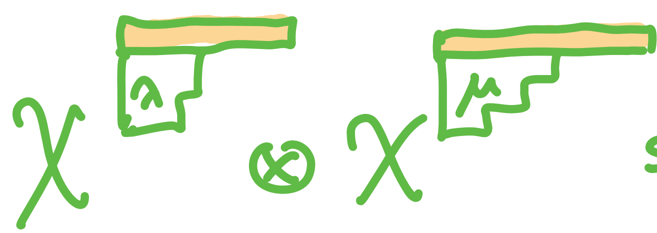
COROLLARY (ARS 2023) For fixed $i = 0, 1, 2, \ldots$

$\{A(n)^!_i\}$ are also representation-stable

Proof: Induct on i. Virtually we have

$$A(n)^!_i = \sum_{j=1}^{i} \underbrace{A(n)_j}_{\substack{\text{rep-stable} \\ \text{by Church-Farb}}} \otimes \underbrace{A(n)^!_{i-j}}_{\substack{\text{rep-stable} \\ \text{by induction}}}$$

rep-stable by Murnaghan's Stability Thm:



stabilizes for large n. $\blacksquare$

QUESTION:

Can we use more FI-module theory to bound the stable range for $n >> 0$?

This could help approach ...

QUESTION:

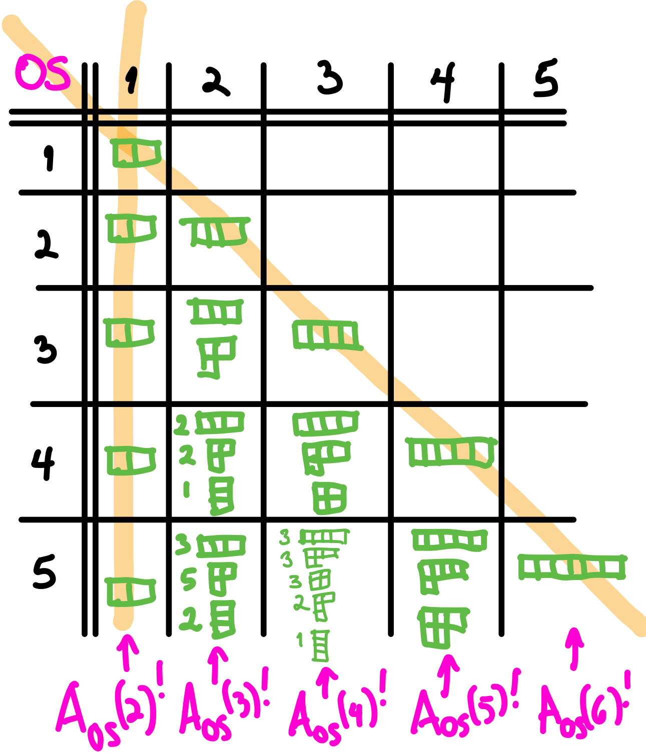
Can we find exact formulas for $A(n)^!_i$?

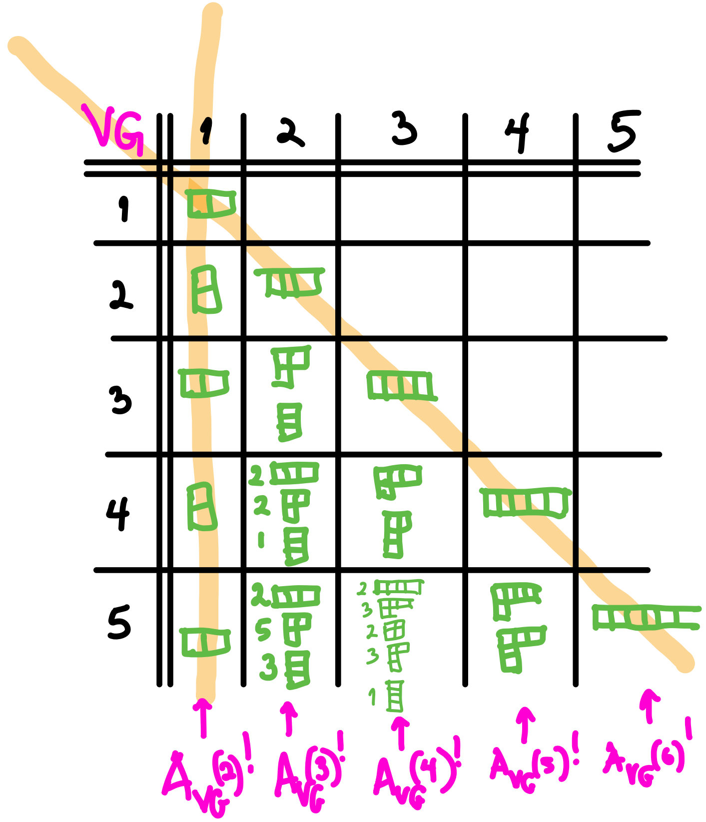
e.g., generating functions for $\sum_{n,i} \operatorname{ch} A(n)^!_i \cdot t^i$

involving plethysm

or infinite products?

- **Boundary cases**

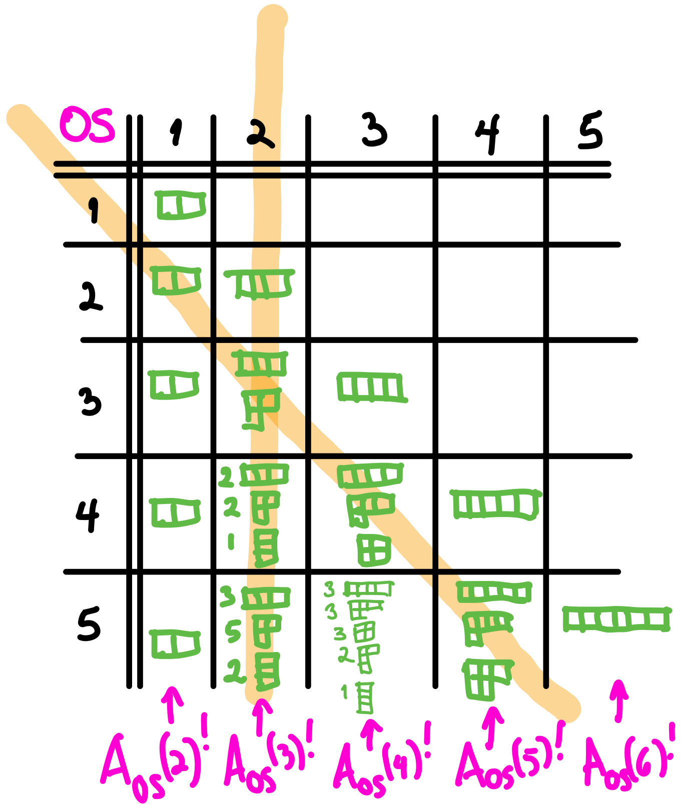


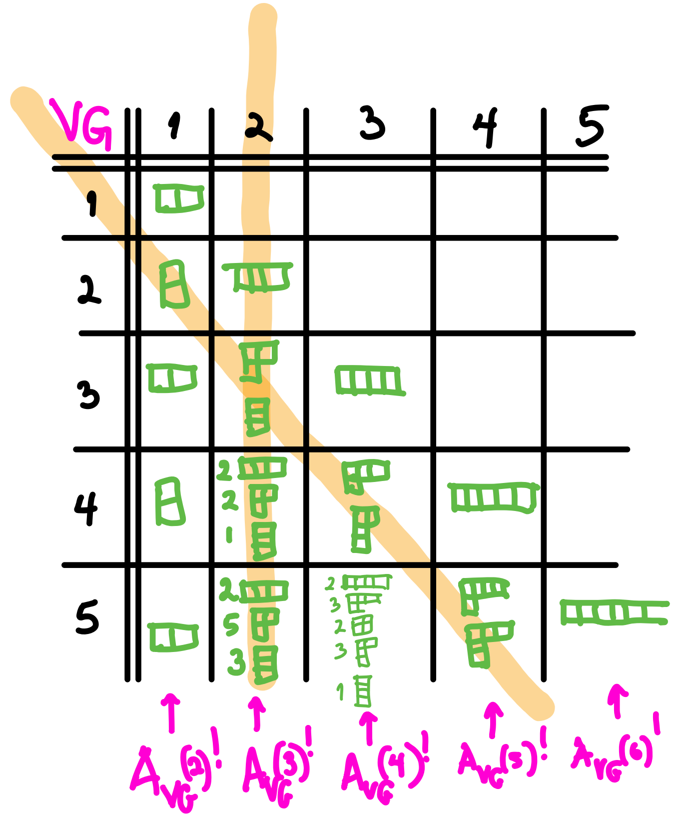


$$S(n-1,\, n-1) = 1 \rightsquigarrow \mathrm{ch}\, A(n)^{!}_{n-1} = s_{\square\square\square\square\square}$$

$$S(m,\, 1) = 1 \rightsquigarrow \mathrm{ch}\, A_{OS}(2)^{!}_{i} = s_{\square\square}$$

$$\mathrm{ch}\, A_{VG}(2)^{!}_{i} = \begin{cases} s_{\square\square} & i \text{ even} \\ s_{\substack{\square\\\square}} & i \text{ odd} \end{cases}$$





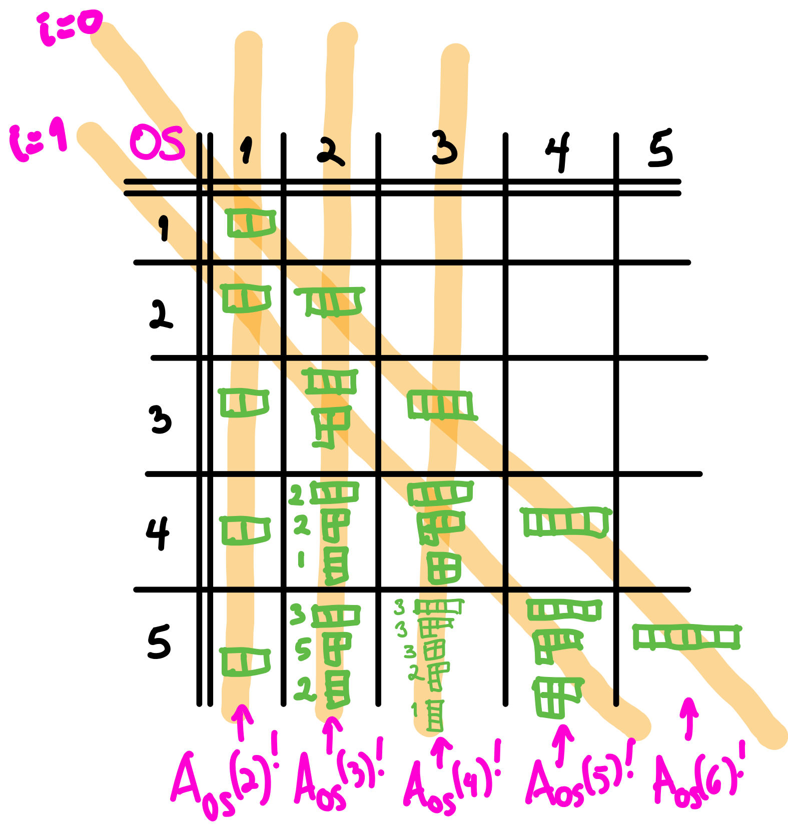
$$S(n,n-1) = \binom{n}{2} \rightsquigarrow \operatorname{ch} A(n)^!_{n-1} = \begin{cases} s_{\square\square}\, s_{\underbrace{\square\square\cdots\square}_{n-2}} & \text{for OS} \\ s_{\substack{\square\\\square}}\, s_{\underbrace{\square\square\cdots\square}_{n-2}} & \text{for VG} \end{cases}$$

$$S(n,2) = 2^{n-1} - 1 \rightsquigarrow A_{OS}(3)^!_i = \mathbb{1} + \chi^{(2,1)} + \chi^{(2,1)} \otimes \chi^{(2,1)} + \chi^{(2,1)} \otimes \chi^{(2,1)} \otimes \chi^{(2,1)}$$

$$= 1 + 2 + 2^2 + \dots + 2^{n-2} \qquad + \dots + \left(\chi^{(2,1)}\right)^{\otimes i}$$

Most mysterious …

CONJECTURE: $A_{OS}(n)^!_i$ is always an $\mathfrak{S}_n$-permutation representation (!)



Verified for $n = 2, 3, 4$
$i = 0, 1$

(The $\mathfrak{S}_n$-orbit stabilizers are not all parabolic subgroups $\mathfrak{S}_{\alpha_1} \times \cdots \times \mathfrak{S}_{\alpha_\ell}$)

- Homotopy Lie algebras (see A. Suciu's talk?)

$\exists$ a graded Lie (super)algebra $\mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d$ with

$$A(n)^! = \mathrm{Ext}^{A(n)}(\Bbbk,\Bbbk) \cong \mathcal{U}(\mathcal{L})$$

universal enveloping algebra

$$\underset{\text{PBW Thm.}}{\cong} \mathrm{Sym}^{\pm}(\mathcal{L})$$

graded polynomial algebra

$$\cong \begin{cases} \displaystyle\bigoplus_{\lambda=1^{m_1}2^{m_2}\cdots} \bigotimes_{i=1}^{\infty} \mathrm{Sym}^{m_i}(\mathcal{L}_i) & \text{for } A_{OS}(n) \\ \displaystyle\bigoplus_{\lambda=1^{m_1}2^{m_2}\cdots} \bigotimes_{i \text{ odd}} \Lambda^{m_i}(\mathcal{L}_i) \otimes \bigotimes_{i \text{ even}} \mathrm{Sym}^{m_i}(\mathcal{L}_i) & \text{for } A_{VG}(n) \end{cases}$$

So if we understood the $\mathfrak{S}_n$-representations on

$$\mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d$$

it would help us understand those on $A(n)$:

Sadly, calculations for $A_{OS}(n) = \mathrm{Sym}(\mathcal{L})$

show that these $\mathcal{L}_d$

are not $\mathfrak{S}_n$-permutation reps!

Thanks ICMS,

and thank you for

your attention!