Stirling numbers
and
Koszul algebras with symmetry

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Symmetry, Stability, and Interactions with Computation

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1. Stirling numbers $\underset{\text{1st kind}}{c(n,k)}$, $\underset{\text{2nd kind}}{S(n,k)}$

2. Algebras, Hilbert functions/series

3. Koszul algebras & duality

4. Representation theory results

1. Stirling numbers

\# k cycle permutations in $\mathfrak{S}_n$ $=: c(n,k)$ (signless) Stirling # of 1st kind

$c(4,4) = 1$

$(1)(2)(3)(4)$

$c(4,3) = 6$

$(12)(3)(4)$
$(13)(2)(4)$
$(14)(2)(3)$
$(23)(1)(4)$
$(24)(1)(3)$
$(34)(1)(2)$

$c(4,2) = 11$

$(123)(4)$ $(12)(34)$
$(132)(4)$ $(13)(24)$
$(124)(3)$ $(14)(23)$
$(142)(3)$
$(134)(2)$
$(143)(2)$
$(234)(1)$
$(243)(1)$

$c(4,1) = 6$

(1234)
(1243)
(1324)
(1342)
(1423)
(1432)

\# k block set partitions of $\{1,2,\ldots,n\}$ $=: S(n,k)$ Stirling # of 2nd kind

$S(4,4) = 1$

$1|2|3|4$

$S(4,3) = 6$

$12|3|4$ $23|1|4$
$13|2|4$ $24|1|3$
$14|2|3$ $34|1|2$

$S(4,2) = 7$

$123|4$ $12|34$
$124|3$ $13|24$
$134|2$ $14|23$
$234|1$

$S(4,1) = 1$

1234

Triangle recurrences

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

$c(n,k)$: k cycle permutations of $\{1,2,\dots,n-1,n\}$

$c(n-1,k-1)$: n is a singleton cycle

$(n-1)\cdot c(n-1,k)$: n is not a singleton cycle

n \ k =	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮	⋮					⋱

$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

$S(n,k)$: k block partitions of $\{1,2,\dots,n-1,n\}$

$S(n-1,k-1)$: n is a singleton block

$k\cdot S(n-1,k)$: n is not a singleton block

n \ k =	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮	⋮					⋱

Generating functions

$$\sum_{i=1}^{n} c(n,n-i)\, t^{i} = (1+t)(1+2t)\cdots(1+(n-1)t)$$

$$1+6t+11t^2+6t^3 = (1+t)(1+2t)(1+3t)$$

c(n,k)

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮						⋱

$$\sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^{i} = \frac{1}{(1-t)(1-2t)\cdots(1+(n-1)t)}$$

$$1+6t+25t^2+\dots = \frac{1}{(1-t)(1-2t)(1-3t)}$$

S(n,k)

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮				⋮		⋱

2. Algebras, Hilbert functions/series

$$A = \bigoplus_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \ldots \quad \text{with } A_i \cdot A_j \subset A_{i+j}$$

a graded associative $\Bbbk$-algebra

$\Bbbk$ a field

has Hilbert series

$$\mathrm{Hilb}(A,t) := \sum_{i=0}^{\infty} \underbrace{\dim_{\Bbbk}(A_i)}_{\text{called Hilbert function } h(A,i)} \cdot t^i$$

---

e.g. $\mathrm{Hilb}(\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}, t) = \sum_{i=0}^{n} \binom{n}{i} t^i = (1+t)^n$

$= \Lambda^{\bullet} V$ where $V = \mathrm{span}_{\Bbbk}\{x_1,\ldots,x_n\}$

exterior algebra $\quad x_i x_j = -x_j x_i, \ x_i^2 = 0$

---

e.g. $\mathrm{Hilb}(\Bbbk[y_1,\ldots,y_n], t) = \sum_{i=0}^{\infty} \binom{n+i-1}{i} t^i = \dfrac{1}{(1-t)^n}$

$= \mathrm{Sym}^{i}(V^*)$ where $V^* = \mathrm{span}_{\Bbbk}\{y_1,\ldots,y_n\}$

polynomial algebra (commutative) $\quad y_i y_j = y_j y_i$

$c(n,k)$ are also a Hilbert function...

... for two algebras A with $\mathrm{Hilb}(A,t) = \sum_{i=1}^{n} c(n,n-i)\,t^i = (1+t)(1+2t)\cdots(1+(n-1)t)$

exterior algebra

$$A = \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n}$$

$\cong H^{\bullet}(\mathrm{Conf}_n(\mathbb{C}), \Bbbk)$ — $\{(z_1,\ldots,z_n)\in\mathbb{C}^n : z_i\ne z_j \text{ for } i\ne j\}$

V.I. Arnold 1968

configuration space of n labeled points in $\mathbb{C}$ (or in $\mathbb{R}^d$, for $d=2,4,6,\ldots$ even)

$\cong$ group cohomology of pure braid group PB_n

(= Orlik-Solomon algebra of type A_{n-1} braid arrangement)

NOTE: $\deg(x_{ij}) = d-1$

OR

(commutative) polynomial algebra

$$A = \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ (x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n}$$

$\cong H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ for $d=3,5,7,\ldots$ odd

F. Cohen 1972

(= graded Varchenko-Gelfand ring of type A_{n-1} braid arrangement)

Why do they have $\mathrm{Hilb}(A,t) = (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{i=1}^{n} c(n,n-i)t^i$?

F. Cohen's proof shows these presentations

$$A = \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}})_{1\le i<j\le k\le n} \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ (\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}})_{1\le i<j\le k\le n} \end{cases}$$

are Gröbner basis presentations
(exterior, commutative)

with initial terms underlined in green, giving

standard monomial $\Bbbk$-bases for A

= squarefree products of at most one variable from these sets:

$$\{x_{12}\},\quad \{x_{13}, x_{23}\},\quad \{x_{14}, x_{24}, x_{34}\},\quad \ldots,\quad \{x_{1n}, x_{2n}, \ldots, x_{n-1,n}\}$$

$$(1+t)\quad\cdot\quad(1+2t)\quad\cdot\quad(1+3t)\quad\cdots\quad(1+(n-1)t)$$

Are the $S(n,k)$ also a Hilbert function?

Yes, $\dfrac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i = \mathrm{Hilb}(A^!, t)$

where $A^!$ is the Koszul dual algebra

for either of the quadratic algebras

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \big/ (x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 3,5,7,\ldots \end{cases}$$

3. Koszul algebras & their Koszul duals

DEFINITION: $A = \bigoplus_{i=0}^{\infty} A_i = \underbrace{A_0}_{=\Bbbk} \oplus A_1 \oplus A_2 \oplus \cdots$

a standard graded connected associative $\Bbbk$-algebra

means

$A = \underbrace{k\langle x_1, \dots, x_n\rangle}_{\substack{\text{free associative algebra} \\ \text{on } x_1,\dots,x_n}} / I$

or

tensor algebra $T^\bullet(V)$
on $V = \mathrm{span}_{\Bbbk}\{x_1, \dots, x_n\}$

for a two-sided ideal
$I \subset k\langle x_1, \dots, x_n\rangle$
which is homogeneous:

$$I = \bigoplus_{i=2}^{\infty} I_i$$

where $I_i := T^i(V) \cap I$

(Priddy 1970)

DEFINITION: A is a Koszul algebra if there exists a free A-resolution of $\Bbbk = A/\underbrace{A_+}_{A_1\oplus A_2\oplus A_3\oplus\cdots}$

having all linear maps:

i.e. all entries lie in A_1

$$\cdots \to A(-3)^{\beta_3} \xrightarrow{\begin{bmatrix}\cdots\\ \cdots\\ \cdots\end{bmatrix}} A(-2)^{\beta_2} \xrightarrow{\begin{bmatrix}\vdots & \vdots & \vdots\end{bmatrix}} A(-1)^{\beta_1} \xrightarrow{\begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}} A \to \Bbbk \to 0$$

$$\begin{matrix} e_1 & \longmapsto & x_1 \\ \vdots & & \vdots \\ e_n & \longmapsto & x_n \end{matrix}$$

$A = \Bbbk\langle x_1, \dots, x_n\rangle / I$ Koszul $\Longrightarrow$ I is quadratic:

$I = (I_2)$

is generated by $I_2 = I \cap T^2(V)$

THEOREM (Priddy 1970) When A is Koszul, its quadratic dual algebra $A^!$

defined by $A^! := \underbrace{\Bbbk\langle y_1, \ldots, y_n\rangle}_{T^\bullet(V^*)} / J$ where $J = (J_2)$

for $V^* = \mathrm{span}_{\Bbbk}\{y_1, \ldots, y_n\}$

with $(y_i, x_j) = \delta_{ij}$

with $J_2 := I_2^\perp \subset T^2(V^*) = V^* \otimes V^*$

gives an explicit linear free A-resolution of $\Bbbk$ built on $A \otimes_{\Bbbk} (A^!)^*$:

$$\cdots \to A \otimes_{\Bbbk} (A^!_3)^* \longrightarrow A \otimes_{\Bbbk} (A^!_2)^* \to A \otimes_{\Bbbk} (A^!_1)^* \to A \otimes_{\Bbbk} (A^!_0)^* \to \Bbbk \to 0$$

(now called Priddy's complex)

COROLLARY: $\mathrm{Hilb}(A, t) \cdot \mathrm{Hilb}(A^!, -t) = 1$ when A is Koszul.

i.e. $\mathrm{Hilb}(A^!, t) = \dfrac{1}{\mathrm{Hilb}(A, -t)}$

More generally, a group G of graded symmetries of A also acts on $A^!$, and has virtual (equivariant) G-character identities, recurrences:

$$\boxed{\mathrm{Hilb}_{eq}(A,t)\cdot\mathrm{Hilb}_{eq}((A^!)^*,-t)=1} \quad \text{in } \underbrace{R(G)}_{\substack{\text{ring of complex}\\ G\text{-characters}}}[[t]]$$

or equivalently

$$\boxed{(A_i^!)^* = A_1\otimes(A_{i-1}^!)^* - A_2\otimes(A_{i-2}^!)^* + A_3\otimes(A_{i-3}^!)^* - \cdots \pm A_i} \quad \text{in } R(G)$$

EXAMPLE

$$A = \underbrace{\Lambda_{\mathbb{k}}\{x_1, \dots, x_n\}}_{\Lambda^\bullet V} \cong \mathbb{k}\langle x_1, \dots, x_n\rangle / \underbrace{(x_i^2,\ x_i x_j + x_j x_i)}_{I = (I_2)}$$ is Koszul

$$A^! = \underbrace{\mathbb{k}[y_1, \dots, y_n]}_{\mathrm{Sym}^\bullet(V^*)} \cong \mathbb{k}\langle y_1, \dots, y_n\rangle / \underbrace{(y_i y_j - y_j y_i)}_{J = (J_2)}$$ is its Koszul dual where $J_2 = I_2^\perp \subset T^2(V^*)$

and Priddy's complex $=$ (usual) Koszul complex resolving $\mathbb{k}$ over $\mathbb{k}[\underline{y}]$:

$$0 \to \Lambda^n V \otimes_{\mathbb{k}} \mathrm{Sym}(V^*) \to \dots \to \Lambda^2 V \otimes_{\mathbb{k}} \mathrm{Sym}(V^*) \to V \otimes \mathrm{Sym}(V^*) \to \mathrm{Sym}(V^*) \to \mathbb{k} \to 0$$

$$x_1 \longmapsto y_1 \longmapsto 0$$

$$\vdots$$

$$x_n \longmapsto y_n \longmapsto 0$$

$$x_i \wedge x_j \longmapsto y_i x_j - y_j x_i$$

How to prove an algebra A is Koszul?

THEOREM (Folklore + Fröberg 1975 for monomial case): When A is commutative $\Bbbk[x_1,\ldots,x_n]/I$ or anticommutative $\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}/I$ and I has a quadratic Gröbner basis for some monomial order on $\Bbbk[x_1,\ldots,x_n]$ or $\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}$, then A is Koszul.

e.g. $A = H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ is Koszul

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=3,5,7,\ldots \end{cases}$$

$$A^! = \Bbbk\langle y_{ij}\rangle_{1\le i<j\le n} \Big/ \left([y_{ij}, y_{k\ell}]\right)_{\{i,j\}\cap\{k,\ell\}=\emptyset} + \left([y_{ij}, y_{ik}+y_{jk}]\right)_{1\le i<j<k\le n}$$

is its Koszul dual where $[a,b] := \begin{cases} ab-ba & \text{if } d \text{ even} \\ ab+ba & \text{if } d \text{ odd} \end{cases}$

REMARK: Supersolvable hyperplane arrangements are lurking here!

COROLLARY: $A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ (for d even or odd) have

$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A, t)}$$

$$= \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S((n-1)+i, n-1)\, t^i$$

i.e. $\boxed{\dim_{\Bbbk}\left(A^!_i\right) = S((n-1)+i, n-1)}$

	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
$\vdots$						$\ddots$

$n=4$

$\mathrm{Hilb}(A, t)$

$c(n,k)$

	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
$\vdots$				$\vdots$		$\ddots$

$n=4$

$\mathrm{Hilb}(A^!, t)$

$S(n,k)$

Topological REMARK:

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk) \text{ for } d \geq 3$$

has

$$A^! \cong H_\bullet(\Omega\,\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

(base pointed loop space)

Studied, e.g., by Cohen-Gitler 2002

who called its presentation infinitesimal braid relations

$[y_{ij}, y_{k\ell}] = 0$ for $\{i,j\} \cap \{k,\ell\} = \phi$

$[y_{ij}, y_{ik} + y_{jk}] = 0$ for $1 \leq i < j < k \leq n$

4. Representation theory

$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{k})$ carry actions of the symmetric group $\mathfrak{S}_n$ on $\{1,2,\ldots,n\}$.

Q: What do the $\mathfrak{S}_n$-representations on the graded components of A, $A^!$ look like?

Can one decompose them into the $\mathfrak{S}_n$-irreducible representations $\{S^\lambda\}$, indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of n?

$A = H^\cdot \mathrm{Conf}_n(\mathbb{R}^d)$ = Stirling reps of 1st kind have generating function formulas involving plethysms (Sundaram & Welker 1997)

- implemented in SAGE/cocalc by T. Karn

$n=4$ $\quad 1 + 6t + 11t^2 + 6t^3$

	A_0	A_1	A_2	A_3	total rep'n (ungraded)
$d=3,5,7,\ldots$ odd	$S^{(4)}$	$S^{(3,1)}$, $S^{(2,1,1)}$	$S^{(3,1)}$, $S^{(2,1,1)}$, $S^{(2,2)}$, $S^{(2,2)}$, $S^{(1,1,1,1)}$	$S^{(3,1)}$, $S^{(2,1,1)}$	$\Bbbk[\mathfrak{S}_4]$ = regular rep.
$d=2,4,6,\ldots$ even	$S^{(4)}$	$S^{(3,1)}$, $S^{(2,2)}$	$S^{(3,1)}$, $S^{(2,1,1)}$		2 copies of $\Bbbk[\mathfrak{S}_4/(\mathfrak{S}_2\times\mathfrak{S}_1\times\mathfrak{S}_1)]$
		$S^{(4)}$	$S^{(3,1)}$, $S^{(2,2)}$	$S^{(3,1)}$, $S^{(2,1,1)}$	

What about $A(n)^!$ for $A(n) = H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$?

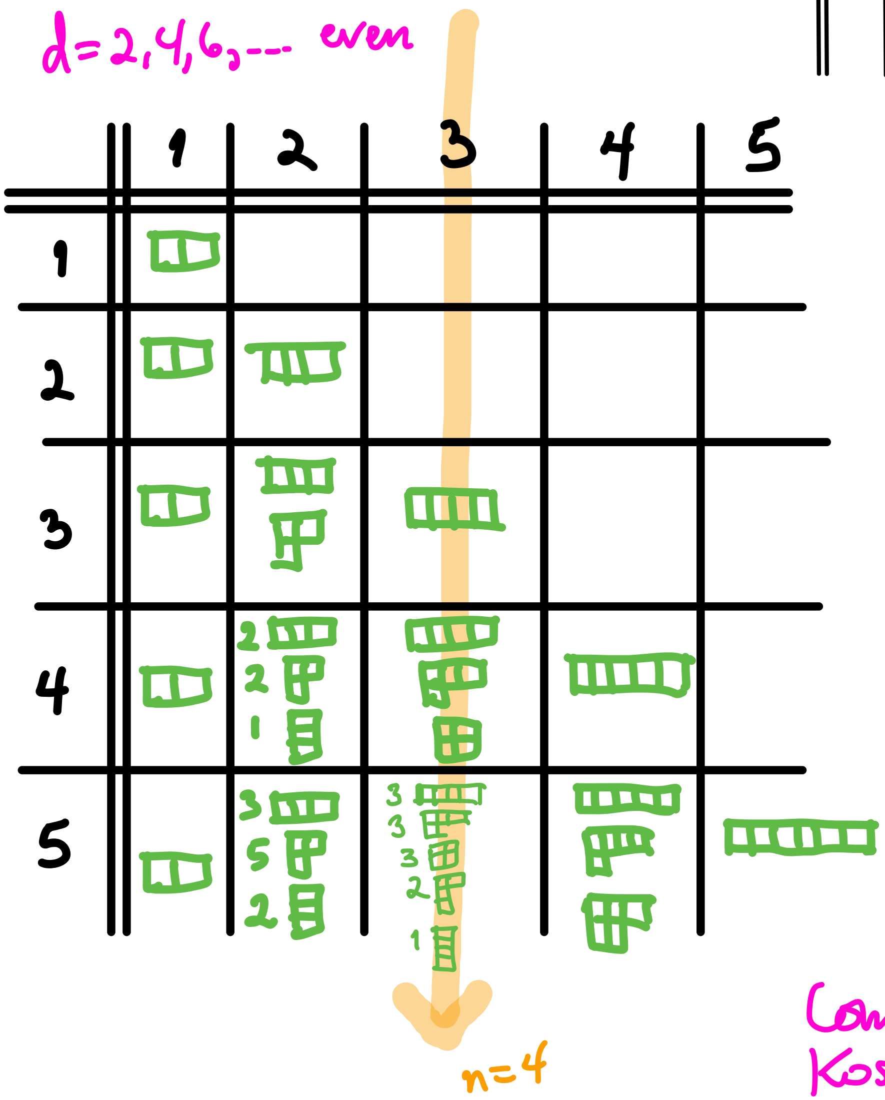
$S(n,k)$

$n=$ \ $k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

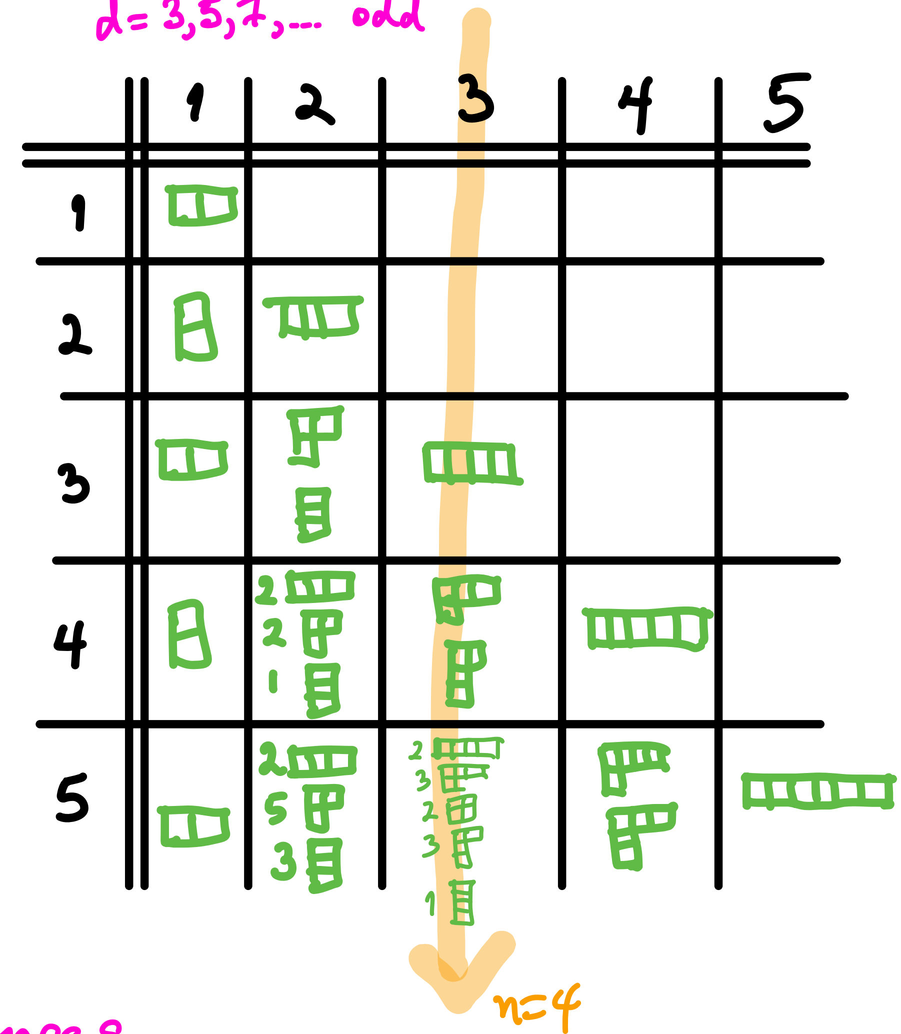
$n=4$

$$\dim A(n)^!_i = S((n-1)+i,\, n-1)$$

$d = 2, 4, 6, \ldots$ even



$d = 3, 5, 7, \ldots$ odd



Computed via Koszul recurrences.

THEOREM: The triangular Stirling recurrences

(Almousa–R.–Sundaram 2023+)

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

lift to short exact sequences of graded $\mathfrak{S}_{n-1}$-representations describing how $A(n)_i$ and $A(n)^!_i$ branch/restrict from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

defining permutation rep of $\mathfrak{S}_{n-1}$

$$0 \to A(n-1) \to A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to \chi^{(n-1)}_{\mathrm{def}} \otimes A(n-1)(-1) \to 0$$

$$0 \to \chi^{(n-1)}_{\mathrm{def}} \otimes \left(A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right)(-1) \to A(n)^!\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to A(n-1)^! \to 0$$

This reflects a general Koszul algebra branching relation ...

PROPOSITION: (ARS 2023+)

Given Koszul algebras $B \subset A$ with symmetries $H < G$ $\left(\text{e.g. } H^\bullet \mathrm{Conf}_{n-1}(\mathbb{R}^d) \subset H^\bullet \mathrm{Conf}_n(\mathbb{R}^d),\ \mathfrak{S}_{n-1} < \mathfrak{S}_n\right)$

and a $\Bbbk H$-module $\mathcal{U}$,

one has a sequence of character identities in $R(H)$

$$\boxed{A_i \downarrow^G_H = B_i + \mathcal{U} \otimes B_{i-1}} \quad \text{for } A$$

$$\Updownarrow$$

$$\boxed{A^!_i \downarrow^G_H = B^!_i + \mathcal{U}^* \otimes A^!_{i-1} \downarrow^G_H} \quad \text{for } A^!$$

Representation Stability

DEFINITION: (Church & Farb 2013) A sequence of $\mathfrak{S}_n$-representations $\{V_n\}_{n=1,2,3,\dots}$ are called representation-stable if

$\exists$ some N, and partitions $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(t)}$
and multiplicities $c_1, c_2, \dots, c_t$

such that $\forall\, n \geq N$, one has

$$V_n \cong \bigoplus_{j=1}^{t} \left(\mathbb{S}^{\begin{smallmatrix} \overbrace{\square\square\cdots\square}^{n-|\lambda^{(j)}|} \\ \lambda^{(j)} \end{smallmatrix}} \right)^{\oplus c_j}$$

---

e.g. **THEOREM:** ((Church & Farb 2013) Fixing $i \geq 0$, $\left\{ H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) \right\}_{n=1,2,\dots}$ is representation-stable.

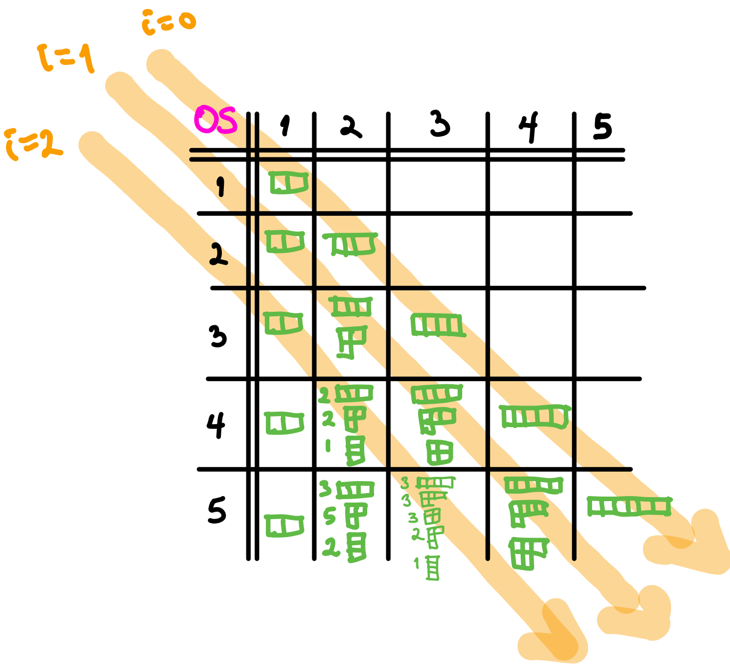
THEOREM: (Hersh & R. 2016) The above stability starts at $n = \begin{cases} 3i & \text{for } d=3,5,7,\dots \text{ odd} \\ 4i & \text{for } d=2,4,6,\dots \text{ even} \end{cases}$

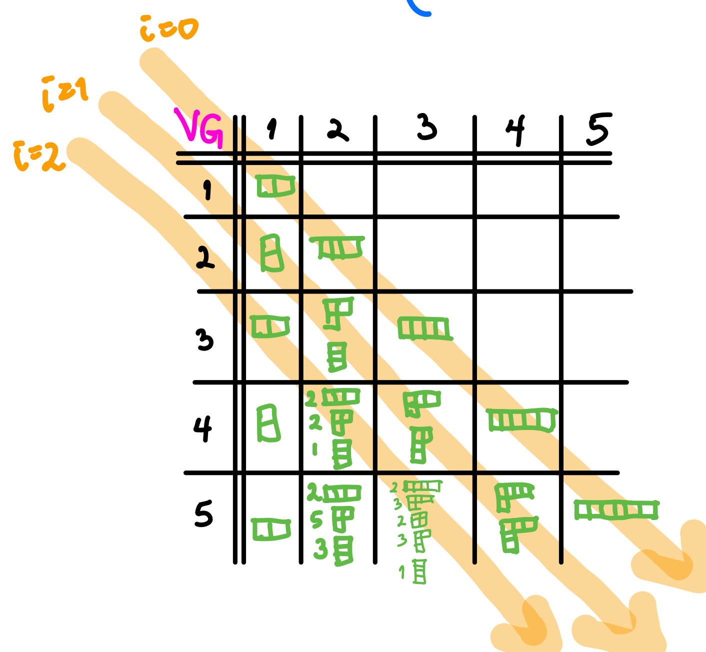
THEOREM (ARS 2023+): Assuming $\{A(n)\}_{n=1,2,\ldots}$ are Koszul, then

$$\{A(n)_i\}_{n=1,2,\ldots} \text{ rep-stable past } n = c \cdot i \implies \text{same for } \{A(n)^!_i\}_{n=1,2,\ldots}$$

COROLLARY (ARS 2023+): For $A(n) := H^{\bullet}\,\mathrm{Conf}_n(\mathbb{R}^d)$,

the $\{A(n)^!_i\}_{n=1,2,\ldots}$ are rep-stable past $n = \begin{cases} 3i & \text{for } d=3,5,7,\ldots \text{ odd} \\ 4i & \text{for } d=2,4,6,\ldots \text{ even} \end{cases}$





THEOREM:
(ARS 2023^+)

- $\mathrm{Hilb}_{eq}(H^\cdot \mathrm{Conf}_n(\mathbb{R}^d), t)$ is divisible by $1+t$ for $d=2,4,6,\ldots$ even

  because multiplication by $x_1+x_2+\ldots+x_n$ makes $H^\cdot \mathrm{Conf}_n(\mathbb{R}^d) =: A$
  a G-equivariant exact cochain complex

  $$0 \to H^0 \to H^1 \to H^2 \to \cdots \to H^{n-1} \to 0$$

  (Yuzvinsky 2001)

- $\mathrm{Hilb}_{eq}(A^!, t)$ is divisible by $1+t+t^2+\ldots = \frac{1}{1-t}$ for $d=2,4,6,\ldots$ even

  because multiplication on the right by $y_1+y_2+\ldots+y_n$
  gives G-equivariant injective maps

  $$A^!_0 \hookrightarrow A^!_1 \hookrightarrow A^!_2 \hookrightarrow \cdots$$

Permutation representations

The $\mathfrak{S}_n$-representations on $A_i := H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d)$ are not permutation representations.

But when $d = 2, 4, 6, \ldots$ even,
A_i turned out be permutation representations surprisingly often:

- for $i = 0, 1$ (and $\frac{1}{2}$ a perm rep for $i = 2$!)
- for $n = 1, 2, 3, 4, 5$

(but failed for $n = 9$ with $i = 3$,
$n = 6$ with $i = 5$)

checked with T. Karn's Burnside Solver

We really don't understand why ?

Thanks for your attention!

$S(n,k)$

$n= \backslash k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$A = H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d)$

$\dim A^!_i = S((n-1)+i,\, n-1)$