Stirling numbers, Koszulity, representations, and supersolvability

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1. 4 counts: 2 easier, 2 harder
$\binom{n}{k}$ $\left(\!\binom{n}{k}\!\right)$ $c(n,k)$, $S(n,k)$

2. Algebras, Hilbert functions/series

3. Koszul algebras & duality

4. Supersolvable hyperplane arrangements

5. Representation theory results

1. Four counts – 2 easier

$\#$ k-element subsets of $\{1,2,\dots,n\} =: \binom{n}{k} = \frac{n!}{k!(n-k)!}$

binomial coefficients

$\binom{4}{0}$	$\binom{4}{1}$	$\binom{4}{2}$	$\binom{4}{3}$	$\binom{4}{4}$
$=1$	$=4$	$=6$	$=4$	$=1$
$\emptyset$	1 2 3 4	12 13 14 23 24 34	123 124 134 234	1234

$\#$ k-element multisubsets of $\{1,2,\dots,n\} =: \left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$

"multichoose"?

$\left(\!\binom{3}{0}\!\right)$	$\left(\!\binom{3}{1}\!\right)$	$\left(\!\binom{3}{2}\!\right)$	$\left(\!\binom{3}{3}\!\right)$	$\left(\!\binom{3}{4}\!\right)$	…
$=1$	$=3$	$=6$	$=10$	$=15$	
$\emptyset$	1 2 3	11 12 13 22 23 33	111 112 113 122 123 133 222 223 233 333	1111 1112 1113 1122 1123 1133 1222 1223 1233 1333 2222 2223 2233 2333 3333	

2 harder

\# k cycle permutations in $\mathfrak{S}_n$ $=: c(n,k)$ (signless) Stirling # of 1st kind

$c(4,4) = 1$

$(1)(2)(3)(4)$

$c(4,3) = 6$

$(12)(3)(4)$
$(13)(2)(4)$
$(14)(2)(3)$
$(23)(1)(4)$
$(24)(1)(3)$
$(34)(1)(2)$

$c(4,2) = 11$

$(123)(4)$ $(12)(34)$
$(132)(4)$ $(13)(24)$
$(124)(3)$ $(14)(23)$
$(142)(3)$
$(134)(2)$
$(143)(2)$
$(234)(1)$
$(243)(1)$

$c(4,1) = 6$

(1234)
(1243)
(1324)
(1342)
(1423)
(1432)

\# k block set partitions of $\{1,2,\ldots,n\}$ $=: S(n,k)$ Stirling # of 2nd kind

$S(4,4) = 1$

$1|2|3|4$

$S(4,3) = 6$

$12|3|4$ $23|1|4$
$13|2|4$ $24|1|3$
$14|2|3$ $34|1|2$

$S(4,2) = 7$

$123|4$ $12|34$
$124|3$ $13|24$
$134|2$ $14|23$
$234|1$

$S(4,1) = 1$

1234

Triangles, recurrences

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

k element subsets of $\{1,2,\ldots,n-1,n\}$	not containing n	containing n

$$\binom{n+k-1}{k} \qquad \binom{n+k-2}{k} \qquad \binom{n+k-2}{k-1}$$

$$\| \qquad\qquad \| \qquad\qquad \|$$

$$\left(\!\!\binom{n}{k}\!\!\right) = \left(\!\!\binom{n-1}{k}\!\!\right) + \left(\!\!\binom{n}{k-1}\!\!\right)$$

k element multi-subsets of $\{1,2,\ldots,n-1,n\}$	not containing n	containing n

PASCAL'S TRIANGLE

	k = 0	1	2	3	4	5
n = 0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1
⋮	⋮					⋱

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

k cycle permutations of $\{1,2,\dots,n-1,n\}$

n is a singleton cycle

n is not a singleton cycle

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
$\vdots$	$\vdots$					$\ddots$

$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

k block partitions of $\{1,2,\dots,n-1,n\}$

n is a singleton block

n is not a singleton block

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
$\vdots$	$\vdots$					$\ddots$

Generating functions

$$\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n \qquad 1+4t+6t^2+4t^3+t^4 = (1+t)^4$$

$$\sum_{k=0}^{\infty} \underbrace{\left(\!\!\binom{n}{k}\!\!\right)}_{\binom{n+k-1}{k}} t^k = \frac{1}{(1-t)^n} \qquad 1+4t+10t^2+20t^3+35t^4+\dots = \frac{1}{(1-t)^4}$$

$$\sum_{i=1}^{n} c(n,n-i)\, t^i = (1+t)(1+2t)\cdots(1+(n-1)t) \qquad 1+6t+11t^2+6t^3 = (1+t)(1+2t)(1+3t)$$

$$\sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i = \frac{1}{(1-t)(1-2t)\cdots(1+(n-1)t)} \qquad 1+6t+25t^2+\dots = \frac{1}{(1-t)(1-2t)(1-3t)}$$

2. Algebras, Hilbert functions/series

$$A = \bigoplus_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \cdots \quad \text{with } A_i \cdot A_j \subset A_{i+j}$$

a graded associative $\mathbb{k}$-algebra

$\mathbb{k}$ a field

has Hilbert series

$$\mathrm{Hilb}(A,t) := \sum_{i=0}^{\infty} \underbrace{\dim_{\mathbb{k}}(A_i)}_{\substack{\text{called Hilbert function} \\ h(A,i)}} \cdot t^i$$

$$\mathrm{Hilb}(\wedge_{\mathbb{k}}\{x_1,\ldots,x_n\}, t) = \sum_{i=0}^{n} \binom{n}{i} t^i = (1+t)^n$$

$= \wedge^{\bullet} V$ where $V = \mathrm{span}_{\mathbb{k}}\{x_1,\ldots,x_n\}$

exterior algebra $\quad x_i x_j = -x_j x_i, \; x_i^2 = 0$

$$\mathrm{Hilb}(\mathbb{k}[y_1,\ldots,y_n], t) = \sum_{i=0}^{\infty} \left(\!\binom{n}{i}\!\right) t^i = \frac{1}{(1-t)^n}$$

$= \mathrm{Sym}(V^*)$ where $V^* = \mathrm{span}_{\mathbb{k}}\{y_1,\ldots,y_n\}$

polynomial algebra (commutative) $\quad y_i y_j = y_j y_i$

$c(n,k)$ are also a Hilbert function ...

... for two algebras A with $\mathrm{Hilb}(A,t) = \sum_{i=1}^{n} c(n,n-i)t^i = (1+t)(1+2t)\cdots(1+(n-1)t)$

exterior algebra

$$A = \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}\right)_{1\le i<j\le k\le n}$$

$$\cong H^\bullet(\mathrm{Conf}_n(\mathbb{C}), \Bbbk)$$

V.I. Arnold 1968

$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1,\ldots,z_n)\in\mathbb{C}^n : z_i \ne z_j \text{ for } i\ne j\}$

configuration space of n labeled points in $\mathbb{C}$ (or in $\mathbb{R}^d$, for $d=2,4,6,\ldots$ even)

$\cong$ group cohomology of pure braid group PB_n

OR

commutative polynomial algebra

$$A = \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}\right)_{1\le i<j\le k\le n}$$

$$\cong H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk) \text{ for } d=3,5,7,\ldots \text{ odd}$$

F. Cohen 1972

Why do they have $\mathrm{Hilb}(A,t) = (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{i=1}^{n} c(n,n-i)\, t^i$?

F. Cohen's proof shows these presentations

$$A = \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} \end{cases}$$

are Gröbner basis presentations (for certain monomial orders)

(exterior, commutative)

with initial terms underlined in green,

giving standard monomial $\Bbbk$-bases for A

= squarefree products of at most one variable from these sets:

$\{x_{12}\}$, $\{x_{13}, x_{23}\}$, $\{x_{14}, x_{24}, x_{34}\}$, $\ldots$, $\{x_{1n}, x_{2n}, \ldots, x_{n-1,n}\}$

$(1+t) \cdot (1+2t) \cdot (1+3t) \cdots (1+(n-1)t)$

Are the $S(n,k)$ also a Hilbert function?

Yes, $\dfrac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\,t^i = \mathrm{Hilb}(A^!, t)$

where $A^!$ is the Koszul dual algebra

to either of the quadratic algebras

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ (x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 3,5,7,\ldots \end{cases}$$

3. Koszul algebras & their Koszul duals

DEFINITION: $A = \bigoplus_{i=0}^{\infty} A_i = \underbrace{A_0}_{=\Bbbk} \oplus A_1 \oplus A_2 \oplus \cdots$

a standard graded connected associative $\Bbbk$-algebra

means

$A = \underbrace{k\langle x_1, \dots, x_n \rangle}_{\text{free associative algebra on } x_1,\dots,x_n} / I$

or

tensor algebra $T^{\bullet}(V)$ on $V = \mathrm{span}_{\Bbbk}\{x_1, \dots, x_n\}$

for a two-sided ideal

$I \subset k\langle x_1, \dots, x_n \rangle$

which is homogeneous:

$I = \bigoplus_{i=2}^{\infty} I_i$

where $I_i := T^i(V) \cap I$

(Priddy 1970)

DEFINITION: A is a Koszul algebra if there exists a free A-resolution of $\Bbbk = A/\underbrace{A_+}_{A_1 \oplus A_2 \oplus A_3 \oplus \cdots}$

having all linear maps:

i.e. all entries lie in A_1

$$\cdots \to A(-3)^{\beta_3} \xrightarrow{\begin{bmatrix} \cdots \end{bmatrix}} A(-2)^{\beta_2} \xrightarrow{\begin{bmatrix} \vdots \end{bmatrix}} A(-1)^{\beta_1} \xrightarrow{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}} A \to \Bbbk \to 0$$

$$e_1 \longmapsto x_1$$
$$\vdots \qquad\qquad \vdots$$
$$e_n \longmapsto x_n$$

$A = \Bbbk\langle x_1, \ldots, x_n\rangle / I$ Koszul $\Longrightarrow$ I is quadratic:

$$I = (I_2)$$

is generated by $I_2 = I \cap T^2(V)$

THEOREM (Priddy 1970) A is Koszul $\iff$ its quadratic dual algebra defined by

$$A^! := \underbrace{\Bbbk\langle y_1, \dots, y_n\rangle}_{T^\bullet(V^*)} / J \quad \text{where } J = (J_2)$$

for $V^* = \mathrm{span}_{\Bbbk}\{y_1, \dots, y_n\}$ with $(y_i, x_j) = \delta_{ij}$

and $J_2 = I_2^\perp \subset T^2(V^*) = V^* \otimes V^*$

gives rise to an explicit linear free A-resolution of $\Bbbk$

built on $A \otimes_{\Bbbk} (A^!)^*$:

$$\cdots \to A \otimes_{\Bbbk} (A^!_3)^* \longrightarrow A \otimes_{\Bbbk} (A^!_2)^* \to A \otimes_{\Bbbk} (A^!_1)^* \to A \otimes_{\Bbbk} (A^!_0)^* \to \Bbbk \to 0$$

(now called Priddy's complex)

Exactness of Priddy's complex $A \otimes_{\Bbbk} (A^!)^*$ resolving $\Bbbk$ $\Rightarrow$

COROLLARY: $\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!,-t) = 1$

i.e. $\mathrm{Hilb}(A^!,t) = \dfrac{1}{\mathrm{Hilb}(A,-t)}$

More generally, a group G of graded symmetries of A also acts on $A^!$, and has (equivariant) virtual G-character identities, recurrences:

$$\mathrm{Hilb}_{eq}(A,t) \cdot \mathrm{Hilb}_{eq}((A^!)^*,-t) = 1 \text{ in } \underbrace{R(G)}_{\text{ring of complex } G\text{-characters}}[[t]]$$

$$(A_i^!)^* = A_1 \otimes (A_{i-1}^!)^* - A_2 \otimes (A_{i-2}^!)^* + A_3 \otimes (A_{i-3}^!)^* - \cdots \pm A_i \quad \text{in } R(G)$$

EXAMPLE

$$A = \underbrace{\Lambda_{\Bbbk}\{x_1,\dots,x_n\}}_{\Lambda^\bullet V} \cong \Bbbk\langle x_1,\dots,x_n\rangle \big/ \underbrace{(x_i^2\,,\ x_ix_j + x_jx_i)}_{I=(I_2)} \quad \text{is Koszul}$$

$$A^! = \underbrace{\Bbbk[y_1,\dots,y_n]}_{\mathrm{Sym}^\bullet(V^*)} \cong \Bbbk\langle y_1,\dots,y_n\rangle \big/ \underbrace{(y_iy_j - y_jy_i)}_{J=(J_2)} \quad \text{is its Koszul dual}$$

where $J_2 = I_2^\perp \subset T^2(V^*)$

and Priddy's complex = (usual) Koszul complex resolving $\Bbbk$ over $\Bbbk[\underline{y}]$:

$$0 \to \Lambda^n V \otimes_{\Bbbk} \mathrm{Sym}(V^*) \to \cdots \to \Lambda^2 V \otimes_{\Bbbk} \mathrm{Sym}(V^*) \to V \otimes \mathrm{Sym}(V^*) \to \mathrm{Sym}(V^*) \to \Bbbk \to 0$$

$$\begin{aligned} x_1 &\longmapsto y_1 \longmapsto 0 \\ &\ \ \vdots \\ x_n &\longmapsto y_n \longmapsto 0 \end{aligned}$$

$$x_i \wedge x_j \longmapsto y_i x_j - y_j x_i$$

How to prove some A is Koszul?

THEOREM: when A is commutative $\Bbbk[x_1,\dots,x_n]/I$ or anticommutative $\Lambda_{\Bbbk}\{x_1,\dots,x_n\}/I$

(Folklore + Fröberg 1975 for monomial case)

and I has a quadratic Gröbner basis for some monomial order on $\Bbbk[x_1,\dots,x_n]$ or $\Lambda_{\Bbbk}\{x_1,\dots,x_n\}$, then A is Koszul.

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e.g. $A = H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d),\Bbbk)$ is Koszul

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=2,4,6,\dots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=3,5,7,\dots \end{cases}$$

$$A^! = \Bbbk\langle y_{ij}\rangle_{1\le i<j\le n} \Big/ \left([y_{ij},y_{k\ell}]\right)_{\{i,j\}\cap\{k,\ell\}=\emptyset} + \left([y_{ij}, y_{ik}+y_{jk}]\right)_{1\le i<j<k\le n}$$

is its Koszul dual where $[a,b] := \begin{cases} ab-ba & \text{if } d \text{ even} \\ ab+ba & \text{if } d \text{ odd} \end{cases}$

"infinitesimal braid" relations
Drinfeld-Kohno

COROLLARY: $A = H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ (for d even or odd) have

$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A,t)} = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

$$= \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i$$

i.e. $\dim_{\Bbbk}(A^!_i) = S((n-1)+i,\, n-1)$

	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮	⋮					⋱

$n=4$ $\mathrm{Hilb}(A,t)$

$c(n,k)$

	$k=0$	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮	⋮			⋮		⋱

$n=4$ $\mathrm{Hilb}(A^!,t)$

$S(n,k)$

4. Supersolvable hyperplane arrangements

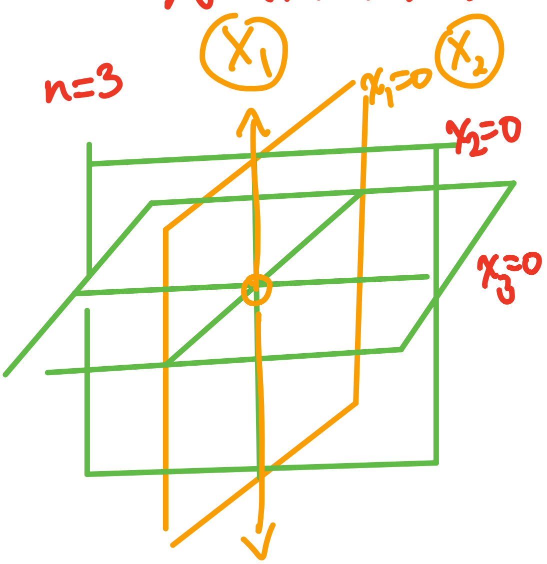
DEFINITION: An arrangement $\mathcal{H} = \{H_1, H_2, \ldots, H_m\} \subset \mathbb{R}^n$ of linear hyperplanes is **supersolvable** if its poset of flats (= intersections $X = H_{i_1} \cap \ldots \cap H_{i_k}$) contains a maximal flag $\{0\} \subset X_1 \subset X_2 \subset \ldots \subset X_{n-1} \subset \mathbb{R}^n$ of modular flats X_i

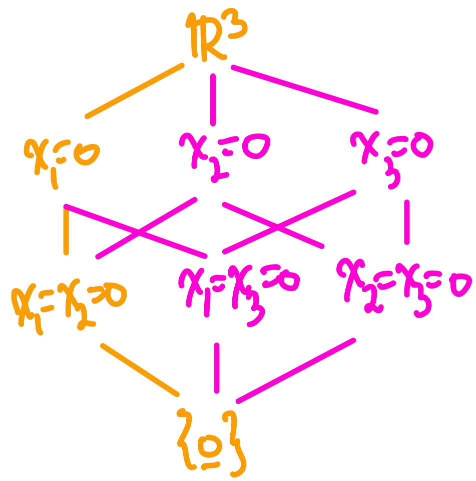
modular flats: $\forall$ flats X, one has

$$\dim X + \dim X_i = \dim X \cap X_i + \dim \underbrace{X \vee X_i}_{\substack{\text{smallest flat}\\ \text{containing } X + X_i}}$$

e.g. Boolean/coordinate arrangement

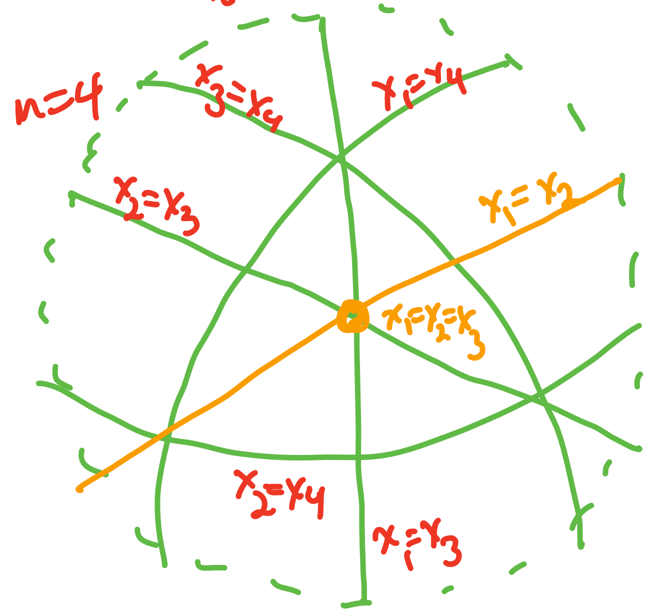
$\mathcal{H} = \{x_1 = 0, x_2 = 0, \ldots, x_n = 0\}$

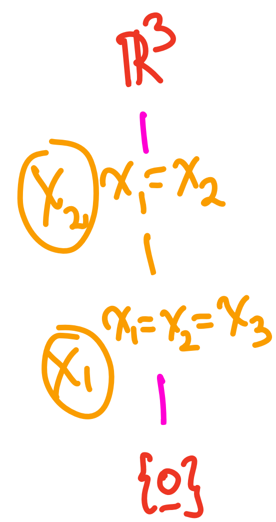




e.g. braid / Type A reflection arrangement

$\mathcal{H} = \mathcal{B}r_n = \{x_i = x_j\}_{1 \le i < j \le n}$





For any hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\} \subset \mathbb{R}^n$,
the cohomology algebras for their "$\mathbb{R}^d$-thickened complements"

$$A = H^\bullet\Big(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{R}^d - \bigcup_{H_i \in \mathcal{H}} H_i \otimes_{\mathbb{R}} \mathbb{R}^d, \Bbbk\Big)$$

$\cong \begin{cases} OS(\mathcal{H}) = \text{Orlik-Solomon algebra (1980)} & \text{for } d = 2, 4, 6, \ldots \\ VG(\mathcal{H}) = \text{graded Varchenko-Gelfand algebra (1987) (Moseley 2017)} & \text{for } d = 3, 5, 7, \ldots \end{cases}$

have

- simple combinatorial presentations,
- simple Gröbner bases,
- standard monomial $\Bbbk$-bases (called NBC bases)
  no broken circuit

... but supersolvable $\mathcal{H}$ are even better:

THEOREM (Björner-Ziegler 1991, Peeva 2003, Dorpalen-Barry 2023)
For supersolvable $\mathcal{H}$ with modular flats $\{0\} \subset X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset \mathbb{R}^n$,

- the Gröbner basis presentations for both $A = \begin{cases} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{cases}$ are quadratic, so they are Koszul algebras
- both have $\mathrm{Hilb}(A,t) = (1+e_1 t)(1+e_2 t)\cdots(1+e_n t)$ with the exponents $(e_1, \ldots, e_n)$ defined by $e_p := \#\{H_i \in \mathcal{H} : H_i \supset X_p \text{ but } H_i \not\supset X_{p+1}\}$

⇝ our starting point ...

COROLLARY (Almousa-R.-Sundaram 2023)
For supersolvable hyperplane arrangements $\mathcal{H} \subset \mathbb{R}^n$, both $A = \begin{cases} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{cases}$ have Koszul duals $A^! = \mathbb{k}\langle y_1, \ldots, y_n \rangle / J$ with

- simple (noncommutative) quadratic Gröbner bases for J
- simple standard monomial $\mathbb{k}$-bases
- $\mathrm{Hilb}(A^!, t) = \dfrac{1}{(1-e_1 t)(1-e_2 t)\cdots(1-e_n t)}$

EXAMPLES

$\mathcal{H}$	A	$A^!$	$(e_1, e_2, \dots, e_n)$ $\mathrm{Hilb}(A,t)$, $\mathrm{Hilb}(A^!,t)$
~~Boolean~~ coordinate arrangement $\{x_i=0\}_{i=1,\dots,n}$	$OS(\mathcal{H}) = \Lambda_{\Bbbk} V = \Lambda_{\Bbbk}\{x_1,\dots,x_n\}$ $VG(\mathcal{H}) = \Bbbk[x_1,\dots,x_n]/(x_i^2)$	$OS(\mathcal{H})^! = \mathrm{Sym}(V^*) = \Bbbk[y_1,\dots,y_n]$ $VG(\mathcal{H})^! = \Bbbk\langle y_1,\dots,y_n\rangle/(y_iy_j+y_jy_i)$	$(1,1,\dots,1)$ $(1+t)^n$, $\frac{1}{(1-t)^n}$
Braid arrangement $Br_n = \{x_i=x_j\}_{1\le i<j\le n}$	$\left.\begin{matrix} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{matrix}\right\} = H^\bullet(\mathrm{Conf}_n \mathbb{R}^d)$ for d even, d odd	loop space $\left.\begin{matrix} OS(\mathcal{H})^! \\ VG(\mathcal{H})^! \end{matrix}\right\} = H^\bullet(\Omega \mathrm{Conf}_n \mathbb{R}^d)$ for d even, d odd $(d \geq 3)$	$(1,2,\dots,n-1)$ $(1+t)(1+2t)\cdots(1+(n-1)t)$, $\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$
Type B_n braid arrangement $\{x_i=0\}_{i=1,\dots,n} \cup \{x_i = \pm x_j\}_{1\le i<j\le n}$	$\left.\begin{matrix} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{matrix}\right\} = H^\bullet(\mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}} \mathbb{R}^d)$	$\left.\begin{matrix} OS(\mathcal{H})^! \\ VG(\mathcal{H})^! \end{matrix}\right\} = H^\bullet(\Omega \mathrm{Conf}_n^{\mathbb{Z}/2\mathbb{Z}} \mathbb{R}^d)$	$(1,3,5,\dots,2n-1)$ $(1+t)(1+3t)(1+5t)\cdots(1+(2n-1)t)$, $\frac{1}{(1-t)(1-3t)(1-5t)\cdots(1-(2n-1)t)}$

5. Representation theory results

$\mathcal{H} = \begin{cases} \text{coordinate/Boolean arrangements} \\ \text{braid arrangements } Br_n \end{cases}$

both carry actions of the symmetric group $\mathfrak{S}_n$ on $\{1,2,\ldots,n\}$.

Q: What do the $\mathfrak{S}_n$-representations on the graded components of

$$A = OS(\mathcal{H}),\ VG(\mathcal{H})$$

$$A^! = OS(\mathcal{H})^!,\ VG(\mathcal{H})^!$$

look like?

Can one decompose them into the

$\mathfrak{S}_n$-irreducible representations $\{S^\lambda\}$,

indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of n?

For $\mathcal{H}$ = coordinate/Boolean arrangement,

both $\begin{Bmatrix} A = OS(\mathcal{H}) \\ A^! = OS(\mathcal{H})^! \end{Bmatrix}$ are already well-understood classically:

$$OS(\mathcal{H}) = \Lambda^{\bullet} V = \Lambda_{\Bbbk}\{x_1, \dots, x_n\} = \bigoplus_{i=0}^{n} \underbrace{\Lambda^i V}_{\cong\, S^{(n-i+1,\,1^{i-1})} \,\oplus\, S^{(n-i,\,1^{i})}}$$

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e.g. $n=4$

$$OS(\mathcal{H}) = \Lambda^{4} V = \Lambda^0 V \oplus \Lambda^1 V \oplus \Lambda^2 V \oplus \Lambda^3 V \oplus \Lambda^4 V$$

$$\begin{array}{ccccccccc}
S^{(4)} & & S^{(3,1)} & & S^{(2,1,1)} & & S^{(1,1,1,1)} & & \\
 & & \oplus & & \oplus & & \oplus & & \\
 & & S^{(4)} & & S^{(3,1)} & & S^{(2,1,1)} & & S^{(1,1,1,1)}
\end{array}$$

$$\Rightarrow \mathrm{Hilb}_{eq}(\Lambda^{\bullet} V, t) = \underbrace{(1+t)}_{\text{note this factor of } 1+t} \otimes \left(S^{(4)} + S^{(3,1)}\, t + S^{(2,1,1)}\, t^2 + S^{(1,1,1,1)}\, t^3 \right)$$

On the other hand, classical invariant theory results show

$$\mathcal{O}S(\mathcal{H})^! = \mathrm{Sym}(V^*) = \Bbbk[y_1,\dots,y_n]$$

$$\underset{\substack{\text{as graded}\\ \text{representations}\\ \text{of } \mathfrak{S}_n}}{\cong} \underbrace{\Bbbk[y_1,\dots,y_n]^{\mathfrak{S}_n}}_{\substack{\text{symmetric}\\ \text{polynomials}\\ \cong \Bbbk[e_1,\dots,e_n]}} \otimes \underbrace{\Bbbk[y_1,\dots,y_n]\big/\left(\Bbbk[y_1,\dots,y_n]^{\mathfrak{S}_n}_+\right)}_{\substack{\text{coinvariant algebra}\\ \Bbbk[\underline{y}]/(e_1,\dots,e_n)}}$$

$e_1 = y_1+y_2+\dots+y_n$, $e_n = y_1y_2\cdots y_n$

$$\Rightarrow \mathrm{Hilb}_{eq}\left(\mathrm{Sym}(V^*), t\right) = \mathrm{Hilb}\left(\Bbbk[e_1,\dots,e_n], t\right) \cdot \mathrm{Hilb}_{eq}\left(\Bbbk[\underline{y}]/(e_1,\dots,e_n), t\right)$$

$$= \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \cdot \sum_{\substack{\text{standard}\\ \text{Young tableaux}\\ Q \text{ with size } n}} \mathbb{S}^{\mathrm{shape}(Q)} \cdot t^{\mathrm{maj}(Q)}$$

note the factor of $\frac{1}{1-t}$

Lusztig-Stanley formula (1979)

$\mathfrak{S}_n$-action on $A(n) = \begin{Bmatrix} OS(\mathrm{Br}_n) \\ VG(\mathrm{Br}_n) \end{Bmatrix}$ = Stirling reps of 1st kind are well-studied, but not completely understood.

$n=4$	1 +	$6t$ +	$11t^2$ +	$6t^3$	total rep'n (ungraded)
	A_0	A_1	A_2	A_3	
$VG(\mathrm{Br}_4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$ $S_{(2,2)}$ $S_{(2,2)}$ $S_{(1,1,1,1)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	$\mathbb{k}[\mathfrak{S}_4]$ = regular rep.
$OS(\mathrm{Br}_4)$	$S_{(4)}$	$S_{(3,1)}$ $S_{(2,2)}$ $S_{(4)}$	$S_{(3,1)}$ $S_{(2,1,1)}$ $S_{(3,1)}$ $S_{(2,2)}$	$S_{(3,1)}$ $S_{(2,1,1)}$	2 copies of $\mathbb{k}[\mathfrak{S}_4 / \mathfrak{S}_2 \times \mathfrak{S}_1 \times \mathfrak{S}_1]$

THEOREM (Sundaram-Welker 1997) One has symmetric function formulas for $A(n) = \begin{cases} VG(Br_n) \\ OS(Br_n) \end{cases}$

$$\sum_{n=0}^{\infty} \sum_{k=1}^{n} ch\, A(n)_{n-k}\, t^k =$$

$$\begin{cases} \displaystyle\sum_{\lambda = 1^{m_1} 2^{m_2} \cdots} t^{\ell(\lambda)} \cdot \prod_{j=1}^{\infty} h_{m_j}[Lie_j] = \prod_{m=1}^{\infty} (1-p_m)^{-a_m(t)} & \text{for VG} \\ \displaystyle\sum_{\lambda = 1^{m_1} 2^{m_2} \cdots} t^{\ell(\lambda)} \prod_{j \text{ odd}} h_{m_j}[\pi_j] \cdot \prod_{j \text{ even}} e_{m_j}[\pi_j] = \prod_{m=1}^{\infty} (1+(-1)^m p_m)^{a_m(-t)} & \text{for OS} \end{cases}$$

plethysm formulas vs. product generating functions

where $a_m(t) := \frac{1}{m} \sum_{d|m} \mu(d)\, t^{m/d}$

Q: Can we understand

Stirling reps of 2^{nd} kind

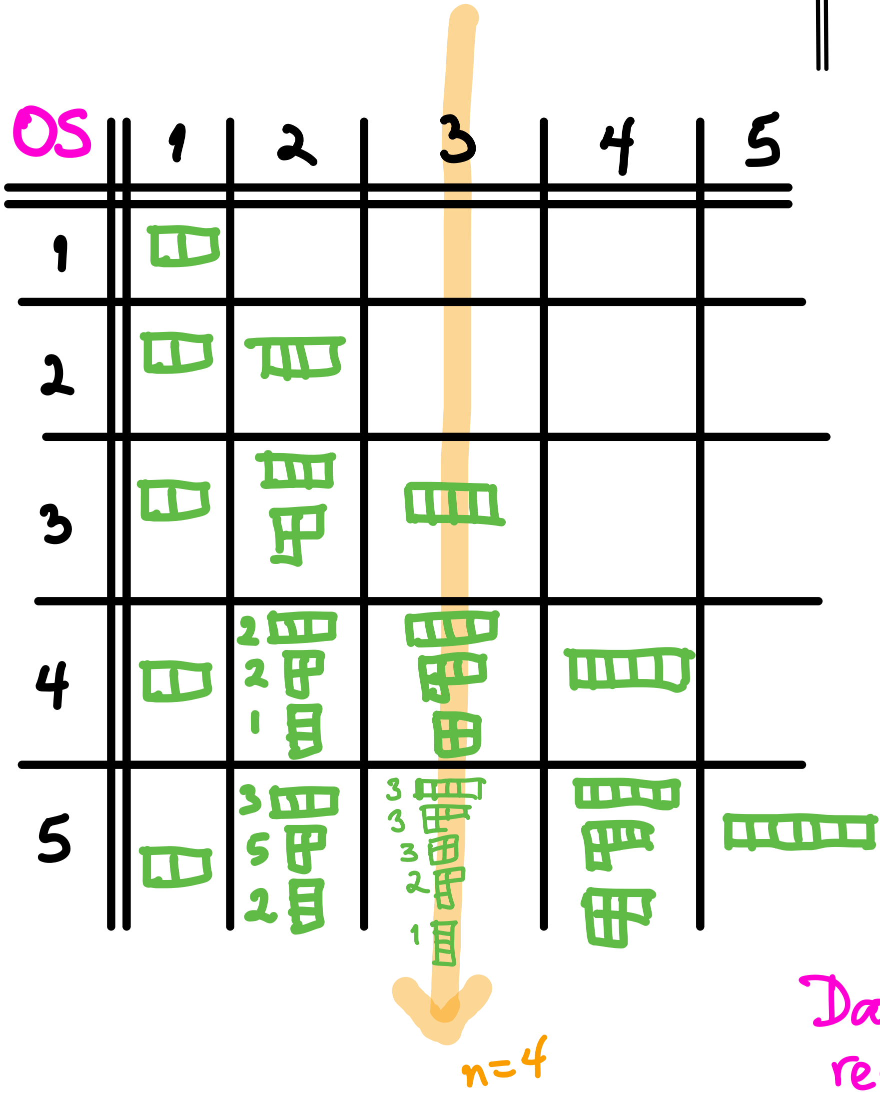
$$A(n)^!_i := \begin{cases} OS(Br_n)^!_i \\ VG(Br_n)^!_i \end{cases} \; ?$$

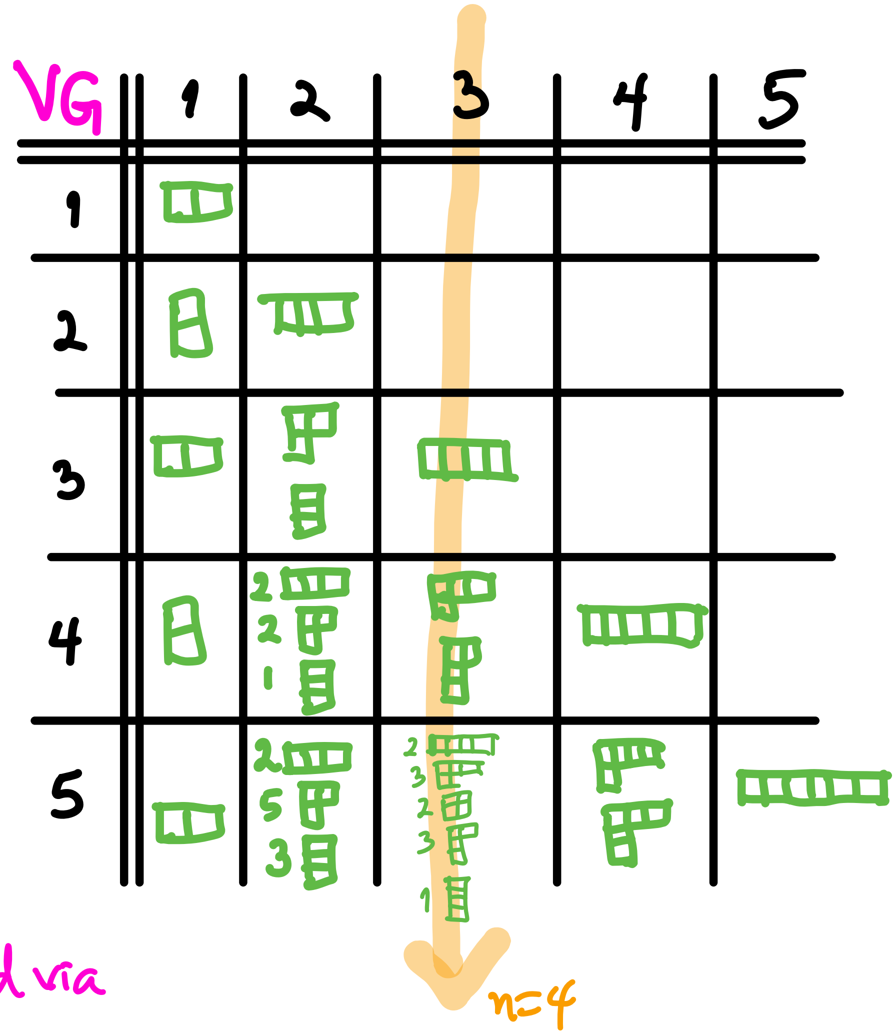
$S(n,k)$

n= \ k=	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

n=4

$$\dim A(n)^!_i = S((n-1)+i,\, n-1)$$





Data computed via recurrences.

THEOREM (ARS 2023): For any supersolvable $\mathcal{H}$ with symmetry group G,

- $\mathrm{Hilb}_{eq}(OS(\mathcal{H}), t)$ is divisible by $1+t$

  because multiplication $OS(\mathcal{H}) \xrightarrow{\cdot(x_1+\dots+x_n)} OS(\mathcal{H})$

  gives a G-equivariant exact cochain complex

  $$0 \to OS_0 \to OS_1 \to OS_2 \to \cdots \to OS_{n-1} \to 0$$

  (Yuzvinsky 2001)

- $\mathrm{Hilb}_{eq}(OS(\mathcal{H})^!, t)$ is divisible by $1+t+t^2+\dots = \frac{1}{1-t}$

  because multiplication $OS(\mathcal{H})^! \xrightarrow{\cdot(y_1+\dots+y_n)} OS(\mathcal{H})^!$

  gives G-equivariant injective maps

  $$OS_0^! \hookrightarrow OS_1^! \hookrightarrow OS_2^! \hookrightarrow OS_3^! \hookrightarrow \cdots$$

THEOREM: (ARS 2023) The triangular Stirling recurrences

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$
$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

lift to short exact sequences of graded $\mathfrak{S}_{n-1}$-representations describing how $A(n)_i = \begin{cases} OS(Br_n)_i \\ VG(Br_n)_i \end{cases}$ and $A(n)^!_i$ restrict from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

defining permutation rep of $\mathfrak{S}_{n-1}$

$$0 \to A(n-1) \to A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to \chi^{(n-1)}_{def} \otimes A(n-1)(-1) \to 0$$

$$0 \to \chi^{(n-1)}_{def} \otimes \left(A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right)(-1) \to A(n)^!\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to A(n-1)^! \to 0$$

REMARK: This generalizes to supersolvable $\mathcal{H}$ with modular flats $\{0\} \subset X_1 \subset X_2 \subset \cdots \subset X_{n-1} \subset \mathbb{R}^n$

relating $A := \begin{cases} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{cases}$ to $B := \begin{cases} OS(\mathcal{H}_{X_1}) \\ VG(\mathcal{H}_{X_1}) \end{cases}$ where $\mathcal{H}_{X_1} = \left\{ \begin{array}{c} \text{hyperplanes } H_i \in \mathcal{H} \\ \text{with } H_i \supset X_1 \end{array} \right\}$

$\circlearrowright$ symmetries G $\qquad$ $\circlearrowright$ symmetries $H := \{g \in G : g(X_1) = X_1\}$

again giving short exact sequences of graded H-representations:

$$0 \to B \longrightarrow A\big\downarrow^G_H \longrightarrow \mathbb{C}[\mathcal{H} - \mathcal{H}_{X_1}] \otimes B(-1) \to 0$$

$$0 \to \mathbb{C}[\mathcal{H} - \mathcal{H}_{X_1}] \otimes \left(A^!\big\downarrow^G_H\right)(-1) \to A^!\big\downarrow^G_H \longrightarrow B^! \to 0$$

DEFINITION: (Church & Farb 2013) A sequence of $\mathfrak{S}_n$-representations $\{V_n\}_{n=1,2,3,\ldots}$ are called representation-stable if

$\exists$ some N, and partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t)}$
and multiplicities $c_1, c_2, \ldots, c_t$

such that $\forall\, n \geq N$, one has
$$V_n \cong \bigoplus_{j=1}^{t} \left(S^{(\overbrace{n-|\lambda^{(j)}|}^{n-|\lambda^{(j)}|},\ \lambda^{(j)})} \right)^{\oplus c_j}$$

---

e.g.

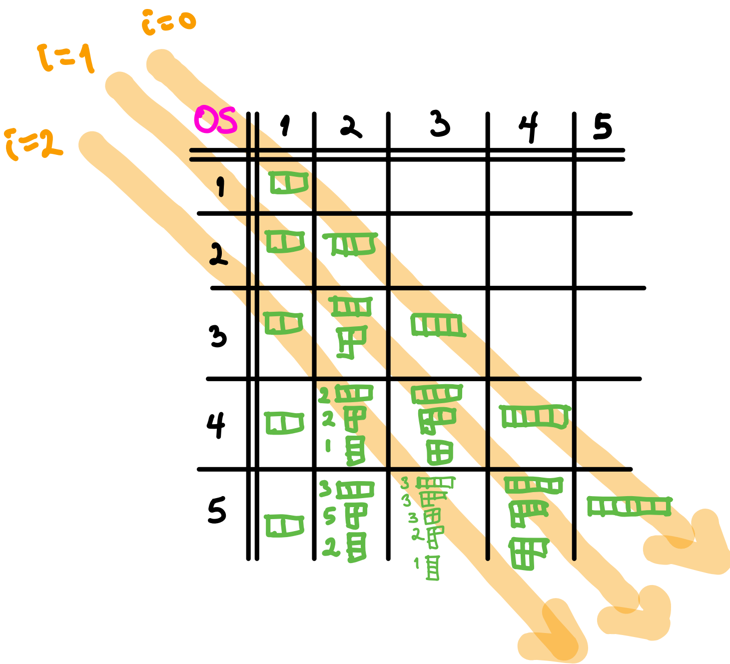
THEOREM: (Church & Farb) Fixing $i \geq 0$, $\left\{ H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) \right\}_{n=1,2,\ldots}$ is representation-stable.

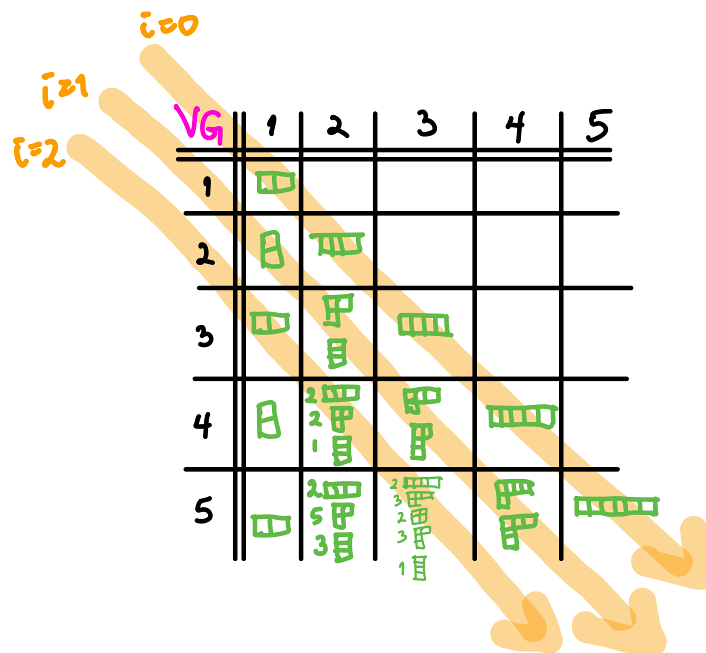
$$= \begin{cases} \{OS(Br_n)_i\}_{n=1,2,\ldots} & d \text{ even} \\ \{VG(Br_n)_i\}_{n=1,2,\ldots} & d \text{ odd} \end{cases}$$

THEOREM (ARS 2023): If $\{A(n)\}_{n=1,2,\ldots}$ are Koszul algebras, and for fixed $i \geq 0$ one has $\{A(n)_i\}_{n=1,2,\ldots}$ representation-stable then $\{A(n)^!_i\}_{n=1,2,\ldots}$ are also representation-stable.

COROLLARY (ARS 2023)

$\{OS(n)^!_i\}_{n=1,2,\ldots}$, $\{VG(n)^!_i\}_{n=1,2,\ldots}$ are reprentation-stable for each fixed $i \geq 0$





REMARK: For supersolvable $\mathcal{H}$ and $A = \begin{cases} OS(\mathcal{H}) \\ VG(\mathcal{H}) \end{cases}$

the Koszul dual $A^!$ is always the universal enveloping algebra

$$A^! \cong U(\mathcal{L}) \text{ for } \mathcal{L} = \bigoplus_{d=0}^{\infty} \mathcal{L}_d \text{ which is either a}$$

graded $\begin{cases} \text{Lie algebra} & \text{for } OS(\mathcal{H})^! \\ \text{super-Lie algebra} & \text{for } VG(\mathcal{H})^! \end{cases}$

$\Longrightarrow$ Poincaré-Birkhoff-Witt Thms

$$A^! \underset{\substack{\uparrow \\ \text{as graded} \\ G\text{-representations}}}{\cong} \begin{cases} \mathrm{Sym}(\mathcal{L}) \\ \mathrm{Sym}^{\pm}(\mathcal{L}) = \Lambda^{\cdot}(\mathcal{L}_{odd}) \otimes \mathrm{Sym}^{\cdot}(\mathcal{L}_{even}) \end{cases}$$

THEOREM (ARS 2023): If $\{A(n)\}$ are Koszul, and $\{A(n)_i\}_{n=1,2,\ldots}$ representation-stable,

$\{A(n)\} = \begin{cases} OS(\mathcal{H}_n) \\ VG(\mathcal{H}_n) \end{cases}$

then $\{\mathcal{L}(n)_i\}_{n=1,2,\ldots}$ are also representation-stable.

REMARK: Very mysteriously, we find that for supersolvable $\mathcal{H}$ with symmetries $G = \mathrm{Aut}(\mathcal{H})$, some of these G-representations are permutation representations:

- For $OS(\mathcal{H})_i$, $VG(\mathcal{H})_i$ it happens rarely.
- For $VG(\mathcal{H})_i^!$ it happens somewhat more often.
- For $OS(\mathcal{H})_i^!$ it happens a lot, but not always.

We really do not understand why!

Thanks for your attention!

$S(n,k)$

$n=$ \ $k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$$\dim A(n)^!_i = S((n-1)+i,\, n-1)$$