Stirling numbers and Koszul algebras with symmetry

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1. Stirling numbers $\underset{\text{1st kind}}{c(n,k)}$, $\underset{\text{2nd kind}}{S(n,k)}$

2. Algebras, Hilbert functions/series

3. Koszul algebras & duality

4. Representation theory results

1. Stirling numbers

\# k cycle permutations in $\mathfrak{S}_n$ $=: c(n,k)$ (signless) Stirling # of 1st kind

$c(4,4) = 1$

(1)(2)(3)(4)

$c(4,3) = 6$

(12)(3)(4)
(13)(2)(4)
(14)(2)(3)
(23)(1)(4)
(24)(1)(3)
(34)(1)(2)

$c(4,2) = 11$

(123)(4)
(132)(4)
(124)(3)
(142)(3)
(134)(2)
(143)(2)
(234)(1)
(243)(1)
(12)(34)
(13)(24)
(14)(23)

$c(4,1) = 6$

(1234)
(1243)
(1324)
(1342)
(1423)
(1432)

\# k block set partitions of $\{1,2,\dots,n\}$ $=: S(n,k)$ Stirling # of 2nd kind

$S(4,4) = 1$

1|2|3|4

$S(4,3) = 6$

12|3|4
13|2|4
14|2|3
23|1|4
24|1|3
34|1|2

$S(4,2) = 7$

123|4
124|3
134|2
234|1
12|34
13|24
14|23

$S(4,1) = 1$

1234

Triangle recurrences

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$

- $c(n,k)$: k cycle permutations of $\{1,2,\ldots,n-1,n\}$
- $c(n-1,k-1)$: n is a singleton cycle
- $(n-1)\cdot c(n-1,k)$: n is not a singleton cycle

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
$\vdots$	$\vdots$					$\ddots$

$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

- $S(n,k)$: k block partitions of $\{1,2,\ldots,n-1,n\}$
- $S(n-1,k-1)$: n is a singleton block
- $k\cdot S(n-1,k)$: n is not a singleton block

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
$\vdots$	$\vdots$					$\ddots$

Generating functions

$$1+6t+11t^2+6t^3=(1+t)(1+2t)(1+3t)$$

$$\sum_{i=1}^{n} c(n,n-i)\,t^i = (1+t)(1+2t)\cdots(1+(n-1)t)$$

$c(n,k)$

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮						⋱

$$1+6t+25t^2+\dots = \frac{1}{(1-t)(1-2t)(1-3t)}$$

$$\sum_{i=0}^{\infty} S((n-1)+i,\,n-1)\,t^i = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

$S(n,k)$

	k = 0	1	2	3	4	5
n = 0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮				⋮		⋱

2. Algebras, Hilbert functions/series

$$A = \bigoplus_{i=0}^{\infty} A_i = A_0 \oplus A_1 \oplus A_2 \oplus \dots$$

with $A_i \cdot A_j \subset A_{i+j}$

a graded associative $\mathbb{k}$-algebra

$\mathbb{k}$ a field

has **Hilbert series**

$$\mathrm{Hilb}(A,t) := \sum_{i=0}^{\infty} \underbrace{\dim_{\mathbb{k}}(A_i)}_{\text{called Hilbert function } h(A,i)} \cdot t^i$$

EXAMPLES:

$$\mathrm{Hilb}\left(\Lambda_{\Bbbk}\{x_1,\dots,x_n\},t\right) = \sum_{i=0}^{n}\binom{n}{i}t^i = (1+t)^n$$

$= \Lambda^\bullet V$ where $V = \mathrm{span}_{\Bbbk}\{x_1,\dots,x_n\}$

exterior algebra

$x_i x_j = -x_j x_i$

$x_i^2 = 0$

$$\mathrm{Hilb}\left(\Bbbk[y_1,\dots,y_n],t\right) = \sum_{i=0}^{\infty}\binom{n+i-1}{i}t^i = \frac{1}{(1-t)^n}$$

$= \mathrm{Sym}(V^*)$ where $V^* = \mathrm{span}_{\Bbbk}\{y_1,\dots,y_n\}$

polynomial algebra
(commutative)

$y_i y_j = y_j y_i$

$c(n,k)$ are also a Hilbert function ...

... for two related cohomology algebras A, both with

$$\mathrm{Hilb}(A,t) = \sum_{i=1}^{n} c(n,n-i)\,t^i = (1+t)(1+2t)\cdots(1+(n-1)t)$$

THEOREM: $A := H^\bullet(\mathrm{Conf}_n(\mathbb{C}), \Bbbk)$
V.I. Arnold 1968

$\mathrm{Conf}_n(\mathbb{C}) = \{(z_1,\dots,z_n)\in\mathbb{C}^n : z_i\neq z_j \text{ for } i\neq j\}$

configuration space of n labeled points in $\mathbb{C}$

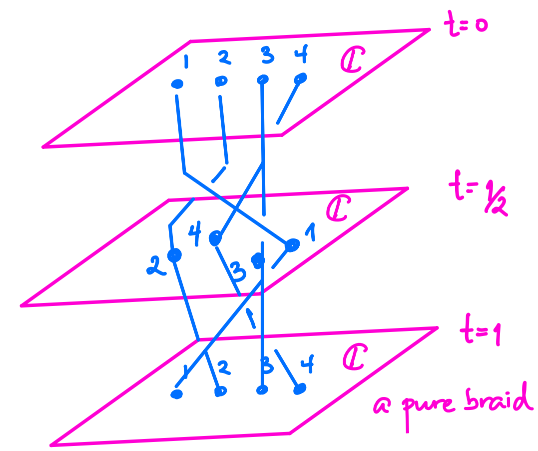
exterior algebra

$$\cong \Lambda_\Bbbk\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}\right)_{1\le i<j<k\le n}$$

$\cong$ group cohomology of pure braid group PB_n

$PB_n = \ker(B_n \to \mathfrak{S}_n)$

B_n: braid group; $\mathfrak{S}_n$: symmetric group



$\cong$ Orlik-Solomon algebra of type A_{n-1} reflection arrangement

THEOREM: Same presentation works for $\mathrm{Conf}(\mathbb{R}^d)$, $d=2,4,6,\ldots$ even (not just $\mathbb{C}=\mathbb{R}^2$)

F. Cohen 1972

and similarly, for $d=3,5,7,\ldots$ odd, one has

$$A := H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

(commutative) polynomial algebra

$$\cong \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \Big(x_{ij}^2,\ \underbrace{x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk}}_{\text{same!}}\Big)_{1\le i<j\le k\le n}$$

$\cong$ graded Varchenko-Gelfand ring of type A_{n-1} reflection arrangement

{ Varchenko-Gelfand 1987, deLongueville-Schultz 2001, Moseley 2017 }

NOTE: We will rescale the grading on both algebras A to divide by $d-1$, making $\deg(x_{ij})=1$ rather than $x_{ij} \in H^{d-1}(\mathrm{Conf}_n(\mathbb{R}^d))$

Why do both A have $\mathrm{Hilb}(A,t) = (1+t)(1+2t)\cdots(1+(n-1)t) = \sum_{i=1}^{n} c(n,n-i)t^i$?

F. Cohen's proof shows both presentations

$$A = \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \Big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \Big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} \end{cases}$$

are Gröbner basis presentations
(exterior, commutative)

with initial terms underlined in green, giving

standard monomial $\Bbbk$-bases for A

= squarefree products of at most one variable from these sets:

$$\begin{matrix} \{x_{12}\}, & \{x_{13}, x_{23}\}, & \{x_{14}, x_{24}, x_{34}\}, & \ldots, & \{x_{1n}, x_{2n}, \ldots, x_{n-1,n}\} \\ (1+t) & \cdot\ (1+2t) & \cdot\ (1+3t) & \cdots & (1+(n-1)t) \end{matrix}$$

Are the $S(n,k)$ also a Hilbert function?

Yes, $$\frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)} = \sum_{i=0}^{\infty} S((n-1)+i,\, n-1)\, t^i = \mathrm{Hilb}(A^!, t)$$

where $A^!$ is the Koszul dual algebra

for either of the quadratic algebras

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \big/ (x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \big/ (x_{ij}^2,\ x_{ij}x_{ik} - x_{ij}x_{jk} + x_{ik}x_{jk})_{1\le i<j\le k\le n} & d = 3,5,7,\ldots \end{cases}$$

3. Koszul algebras & their Koszul duals

Let $A = \bigoplus_{i=0}^{\infty} A_i = \underbrace{A_0}_{=\Bbbk} \oplus A_1 \oplus A_2 \oplus \cdots$

be a standard graded connected associative $\Bbbk$-algebra, meaning

$A = \underbrace{k\langle x_1, \ldots, x_n \rangle}_{\text{tensor algebra } T^\bullet(V)} / I$ for a homogeneous (2-sided) ideal I

on $V = \mathrm{span}_{\Bbbk}\{x_1, \ldots, x_n\} = A_1$

DEFINITION: (Priddy 1970) A is a Koszul algebra if $\Bbbk = A / \underbrace{A_+}_{A_1 \oplus A_2 \oplus A_3 \oplus \cdots}$ has a linear free A-resolution

$$\cdots \to A(-3)^{\beta_3} \xrightarrow{[\cdots]} A(-2)^{\beta_2} \xrightarrow{[\cdots]} A(-1)^{\beta_1} \xrightarrow{[x_1 \cdots x_n]} A \to \Bbbk \to 0$$

$$e_1 \longmapsto x_1$$
$$\vdots \qquad \vdots$$
$$e_n \longmapsto x_n$$

Koszulity of A

- is stronger than being a quadratic algebra: $A = \Bbbk\langle x_1,\dots,x_n\rangle/(I_2)$,

---

- allowed Priddy to write down an elegant and explicit linear A-resolution of $\Bbbk$, based on $A \otimes_{\Bbbk} \underbrace{(A^!)^*}_{\text{graded dual of } A^!}$

where $\underbrace{A^!}_{\substack{\text{quadratic}\\ \text{dual}\\ \text{of } A}} := \Bbbk\underbrace{\langle y_1,\dots,y_n\rangle}_{\substack{\text{tensor algebra}\\ T(V^*) \text{ on}\\ \text{dual } \Bbbk\text{-basis } y_1,\dots,y_n \text{ of } V^*\\ \text{with } \langle y_i, x_j\rangle = \delta_{ij}}}/(J_2)$ where $J_2 = I_2^{\perp} \subset V^* \otimes V^* = (V \otimes V)^*$

- exactness of Priddy's resolution $\Rightarrow$ $\boxed{\mathrm{Hilb}(A,t) \cdot \mathrm{Hilb}(A^!,-t) = 1}$

i.e. $\mathrm{Hilb}(A^!,t) = \dfrac{1}{\mathrm{Hilb}(A,-t)}$

More generally, any group G of graded symmetries of A also acts on $A^!$, and has virtual (equivariant) G-character identities, recurrences:

$$\boxed{\mathrm{Hilb}_{eq}(A,t)\cdot\mathrm{Hilb}_{eq}((A^!)^*,-t)=1} \quad \text{in} \quad \underbrace{R(G)}_{\text{ring of complex } G\text{-characters}}[[t]]$$

or equivalently

$$\boxed{(A_i^!)^* = A_1\otimes(A_{i-1}^!)^* - A_2\otimes(A_{i-2}^!)^* + A_3\otimes(A_{i-3}^!)^* - \cdots \pm A_i} \quad \text{in} \quad R(G)$$

Koszul recurrence for $\{A_i^!\}$ (as G-reps) in terms of $\{A_i\}$ (as G-reps)

EXAMPLE

$$A = \underbrace{\Lambda_{\Bbbk}\{x_1,\dots,x_n\}}_{\Lambda^\bullet V} \cong \Bbbk\langle x_1,\dots,x_n\rangle / \underbrace{(x_i^2,\ x_ix_j + x_jx_i)}_{I=(I_2)}$$ is Koszul

$$A^! = \underbrace{\Bbbk[y_1,\dots,y_n]}_{\mathrm{Sym}^\bullet(V^*)} \cong \Bbbk\langle y_1,\dots,y_n\rangle / \underbrace{(y_iy_j - y_jy_i)}_{J=(J_2)}$$ is its Koszul dual, where $J_2 = I_2^\perp \subset T^2(V^*)$

and Priddy's complex = (usual) Koszul complex resolving $\Bbbk$ over $\Bbbk[\underline{y}]$:

$$0 \to \Lambda^n V \otimes_{\Bbbk} \mathrm{Sym}(V^*) \to \dots \to \Lambda^2 V \otimes_{\Bbbk} \mathrm{Sym}(V^*) \to V \otimes \mathrm{Sym}(V^*) \to \mathrm{Sym}(V^*) \to \Bbbk \to 0$$

$$x_1 \longmapsto y_1 \longmapsto 0$$
$$\vdots$$
$$x_n \longmapsto y_n \longmapsto 0$$

$$x_i \wedge x_j \longmapsto y_ix_j - y_jx_i$$

How to prove an algebra A is Koszul?

THEOREM: (Folklore + Fröberg 1975 for monomial case) When A is commutative $\Bbbk[x_1,\ldots,x_n]/I$ or anticommutative $\Lambda_{\Bbbk}\{x_1,\ldots,x_n\}/I$ and I has some quadratic Gröbner basis then A is Koszul.

e.g. $A = H^{\bullet}(\mathrm{Conf}_n(\mathbb{R}^d),\Bbbk)$ is Koszul

$$\cong \begin{cases} \Lambda_{\Bbbk}\{x_{ij}\}_{1\le i<j\le n} \big/ \left(x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=2,4,6,\ldots \\ \Bbbk[x_{ij}]_{1\le i<j\le n} \big/ \left(\underline{x_{ij}^2},\ x_{ij}x_{ik} - x_{ij}x_{jk} + \underline{x_{ik}x_{jk}}\right)_{1\le i<j\le k\le n} & d=3,5,7,\ldots \end{cases}$$

$$A^! = \Bbbk\langle y_{ij}\rangle_{1\le i<j\le n} \big/ \left([y_{ij},y_{k\ell}]\right)_{\{i,j\}\cap\{k,\ell\}=\emptyset} + \left([y_{ij},y_{ik}+y_{jk}]\right)_{1\le i<j<k\le n}$$

is its Koszul dual where $[a,b] := \begin{cases} ab-ba & \text{if } d \text{ even} \\ ab+ba & \text{if } d \text{ odd} \end{cases}$

REMARK: Supersolvable hyperplane arrangements are lurking here!

COROLLARY: $A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$ (for d even or odd) have

$$\mathrm{Hilb}(A^!, t) = \frac{1}{\mathrm{Hilb}(A,t)} = \frac{1}{(1-t)(1-2t)\cdots(1-(n-1)t)}$$

$$= \sum_{i=0}^{\infty} S((n-1)+i, n-1)\, t^i$$

i.e. $\boxed{\dim_{\Bbbk}(A^!_i) = S((n-1)+i,\, n-1)}$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	2	3	1		
4	0	6	11	6	1	
5	0	24	50	35	10	1
⋮	⋮					⋱

$n=4$: $\mathrm{Hilb}(A,t)$

$c(n,k)$

$n \backslash k$	0	1	2	3	4	5
$n=0$	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
⋮	⋮			⋮		⋱

$n=4$: $\mathrm{Hilb}(A^!,t)$

$S(n,k)$

Topological REMARK:

$$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk) \text{ for } d \geq 3$$

has

$$A^! \cong H_\bullet(\Omega\,\mathrm{Conf}_n(\mathbb{R}^d), \Bbbk)$$

(based pointed) loop space

studied, e.g., by Cohen-Gitler 2002

QUESTION:

Can this help us better understand the $\mathfrak{S}_n$-reps on $A^!$?

4. Representation theory

$A = H^\bullet(\mathrm{Conf}_n(\mathbb{R}^d), \mathbb{k})$ carry actions of the symmetric group $\mathfrak{S}_n$ on $\{1,2,\ldots,n\}$.

QUESTION: What do the $\mathfrak{S}_n$-representations on the graded components of A, $A^!$ look like?

Can one decompose them into the $\mathfrak{S}_n$-irreducible representations $\{S^\lambda\}$, indexed by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$ of n?

$A = H^\cdot \mathrm{Conf}_n(\mathbb{R}^d)$ = Stirling reps of 1st kind have generating function formulas involving plethysms
Sundaram & Welker 1997

$\Longrightarrow$ A_i very computable as $\mathfrak{S}_n$-reps in SAGE/CoCalc

$n=4$ $\qquad 1 + 6t + 11t^2 + 6t^3$

	A_0	A_1	A_2	A_3	total rep'n (ungraded)
$d=3,5,7,\ldots$ odd	$S_{(4)}$	$S_{(3,1)}$, $S_{(2,1,1)}$	$S_{(3,1)}$, $S_{(2,1,1)}$, $S_{(2,2)}$, $S_{(2,2)}$, $S_{(1,1,1,1)}$	$S_{(3,1)}$, $S_{(2,1,1)}$	$\Bbbk[\mathfrak{S}_4]$ = regular rep.
$d=2,4,6,\ldots$ even	$S_{(4)}$	$S_{(3,1)}$, $S_{(2,2)}$	$S_{(3,1)}$, $S_{(2,1,1)}$		2 copies of $\Bbbk[\mathfrak{S}_4/(\mathfrak{S}_2\times\mathfrak{S}_1\times\mathfrak{S}_1)]$
		$S_{(4)}$	$S_{(3,1)}$, $S_{(2,2)}$	$S_{(3,1)}$, $S_{(2,1,1)}$	

QUESTION: What about $A(n)^!$

for $A(n) = H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$?

i.e. Stirling reps of the 2^{nd} kind ?

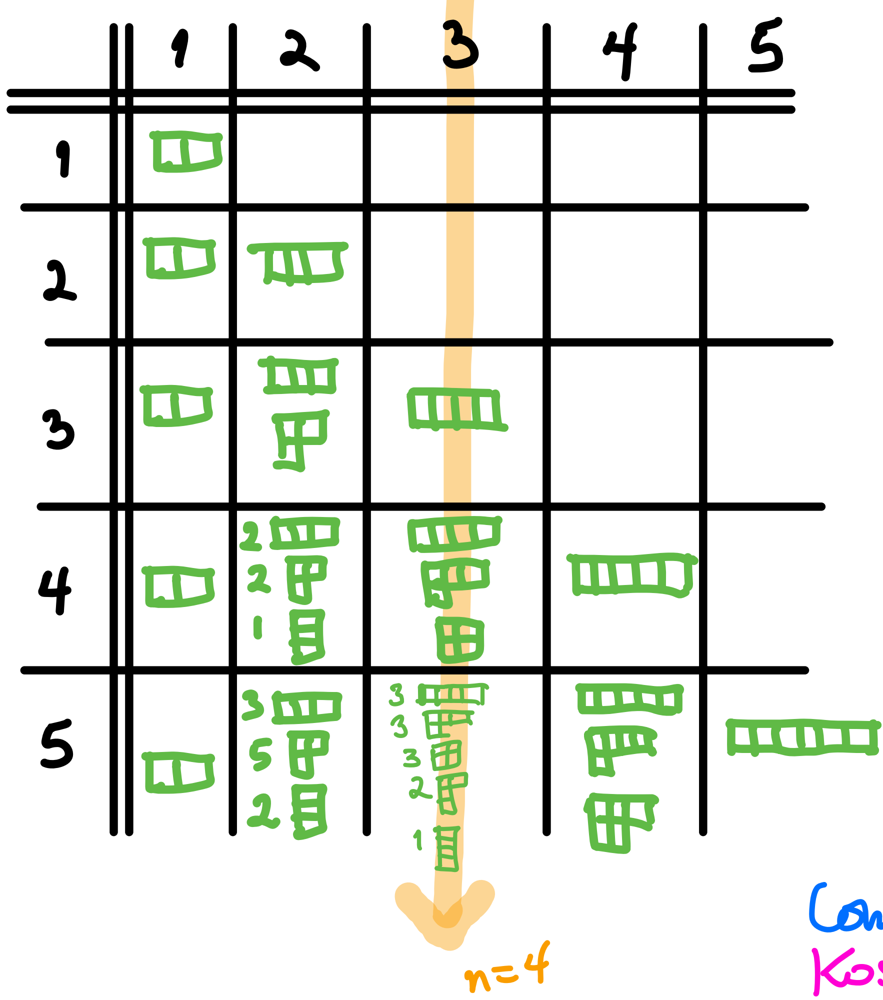
$S(n,k)$

n= \ k=	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

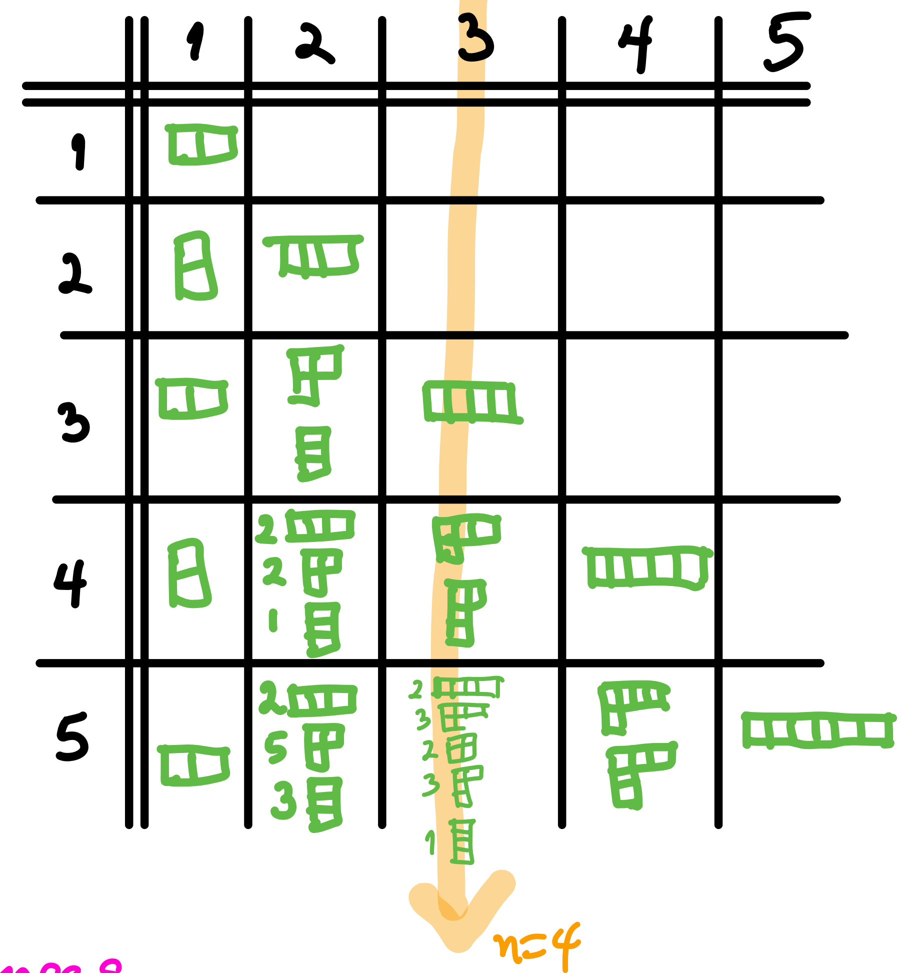
n=4

$$\dim A(n)^!_i = S((n-1)+i,\, n-1)$$

d=2,4,6,--- even



d=3,5,7,--- odd



Computed via Koszul recurrences.

THEOREM: The triangular Stirling recurrences

(Almousa-R.-Sundaram 2023+)

$$c(n,k) = c(n-1,k-1) + (n-1)\cdot c(n-1,k)$$
$$S(n,k) = S(n-1,k-1) + k\cdot S(n-1,k)$$

lift to short exact sequences of graded $\mathfrak{S}_{n-1}$-representations describing how $A(n)_i$ and $A(n)^!_i$ branch/restrict from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$:

---

defining permutation rep of $\mathfrak{S}_{n-1}$

$$0 \to A(n-1) \to A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to \chi^{(n-1)}_{def} \otimes A(n-1)(-1) \to 0$$

$$0 \to \chi^{(n-1)}_{def} \otimes \left(A(n)\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}\right)(-1) \to A(n)^!\big\downarrow^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \to A(n-1)^! \to 0$$

This reflects a general Koszul algebra branching relation ...

PROPOSITION: (ARS 2023+)

Given Koszul algebras $B \subset A$ with symmetries $H < G$, $\left(\text{e.g. } H^\cdot \mathrm{Conf}_{n-1}(\mathbb{R}^d) \subset H^\cdot \mathrm{Conf}_n(\mathbb{R}^d) \text{ with } \mathfrak{S}_{n-1} < \mathfrak{S}_n\right)$

and a $\mathbb{k}H$-module $\mathcal{U}$,

one has a sequence of character identities in $R(H)$

$$\boxed{A_i \downarrow^G_H = B_i + \mathcal{U} \otimes B_{i-1}} \quad \text{for } A$$

$$\Updownarrow$$

$$\boxed{A^!_i \downarrow^G_H = B^!_i + \mathcal{U}^* \otimes A^!_{i-1} \downarrow^G_H} \quad \text{for } A^!$$

REMARK: Supersolvable arrangements lurking here again!

Representation Stability

DEFINITION: (Church & Farb 2013) A sequence of $\mathfrak{S}_n$-representations $\{V_n\}_{n=1,2,3,\ldots}$ are called representation-stable if

$\exists$ some N, and partitions $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(t)}$
and multiplicities $c_1, c_2, \ldots, c_t$

such that $\forall\, n \geq N$, one has

$$V_n \cong \bigoplus_{j=1}^{t} \left(S^{\substack{\overbrace{\hspace{4em}}^{n-|\lambda^{(j)}|} \\ \lambda^{(j)}}} \right)^{\oplus c_j}$$

---

e.g. **THEOREM:** (Church & Farb 2013) Fixing $i \geq 0$, $\left\{ H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d) \right\}_{n=1,2,\ldots}$ is representation-stable.

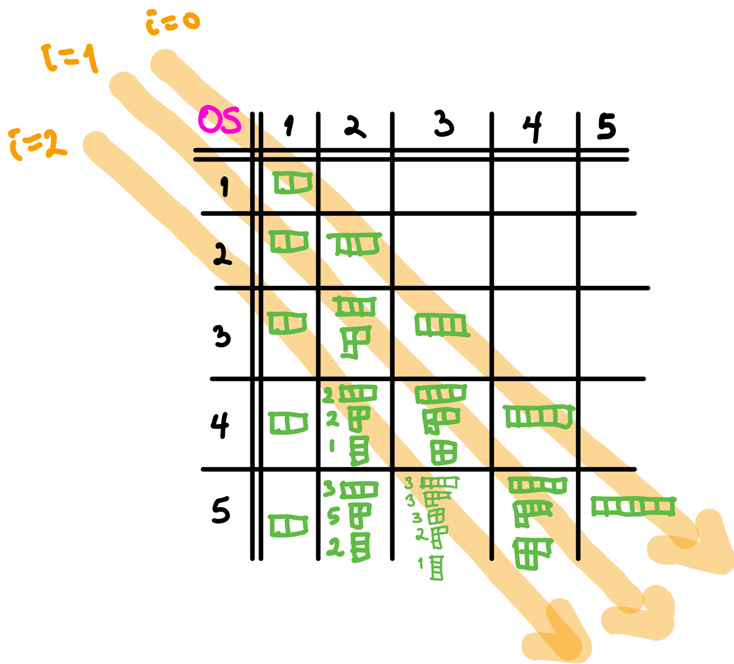
THEOREM: (P. Hersh & R. 2016) The above stability starts at $n = \begin{cases} 3i & \text{for } d=3,5,7,\ldots \text{ odd} \\ 4i & \text{for } d=2,4,6,\ldots \text{ even} \end{cases}$

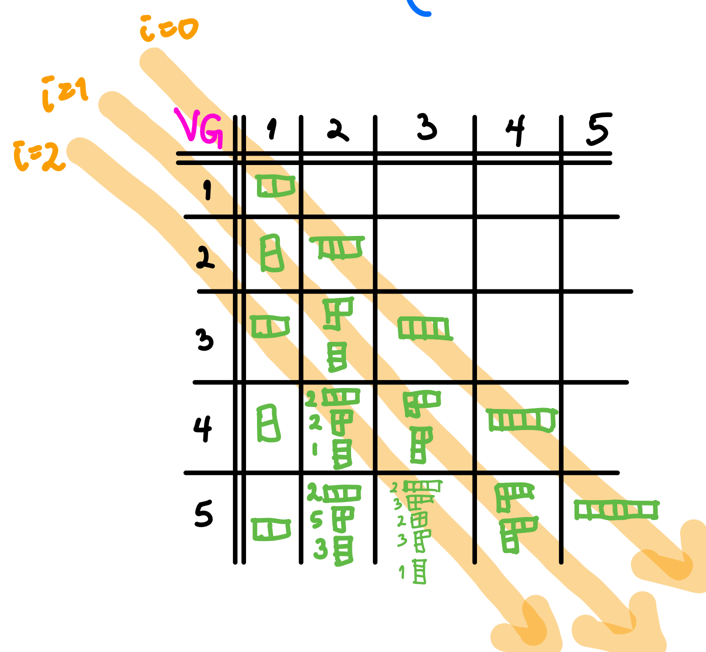
THEOREM: (ARS 2023^+) Assuming $\{A(n)\}_{n=1,2,\ldots}$ are Koszul, then

$\{A(n)_i\}_{n=1,2,\ldots}$ rep-stable past $n = c\cdot i$ $\Rightarrow$ same for $\{A(n)^!_i\}_{n=1,2,\ldots}$

COROLLARY: (ARS 2023^+) For $A(n) := H^\bullet \mathrm{Conf}_n(\mathbb{R}^d)$,

the $\{A(n)^!_i\}_{n=1,2,\ldots}$ are rep-stable past $n = \begin{cases} 3i & \text{for } d=3,5,7,\ldots \text{ odd} \\ 4i & \text{for } d=2,4,6,\ldots \text{ even} \end{cases}$





THEOREM: (ARS 2023+) For $d=2,4,6,\ldots$ even,

- $\mathrm{Hilb}_{eq}(H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d), t)$ is divisible by $1+t$ for $d=2,4,6,\ldots$ even

  because multiplication by $x_1+x_2+\ldots+x_n$ makes $H^{\cdot}\mathrm{Conf}_n(\mathbb{R}^d) =: A$

  a G-equivariant exact cochain complex

  $$0 \to H^0 \to H^1 \to H^2 \to \cdots \to H^{n-1} \to 0$$

  (Yuzvinsky 2001)

- $\mathrm{Hilb}_{eq}(A^!, t)$ is divisible by $1+t+t^2+\ldots = \frac{1}{1-t}$ for $d=2,4,6,\ldots$ even

  because multiplication on the right by $y_1+y_2+\ldots+y_n$

  gives G-equivariant injective maps

  $$A^!_0 \hookrightarrow A^!_1 \hookrightarrow A^!_2 \hookrightarrow \cdots$$

Permutation representations

The $\mathfrak{S}_n$-representations on $A_i := H^{i(d-1)} \mathrm{Conf}_n(\mathbb{R}^d)$ are not permutation representations.

But when $d = 2, 4, 6, \ldots$ even,

A_i ! turned out be permutation representations surprisingly often:

- for $i = 0, 1$ (and $\frac{1}{2}$ a perm rep for $i = 2$!)
- for $n = 1, 2, 3, 4, 5$

(but failed for $n = 9$ with $i = 3$,
$n = 6$ with $i = 5$)

checked with T. Karn's Burnside Solver

QUESTION: Is there a reason why this occurs ?

Thanks for your attention!

$S(n,k)$

$n= \backslash k=$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

$A = H^{\cdot}\,\mathrm{Conf}_n(\mathbb{R}^d)$

$\dim A^{!}_{i} = S((n-1)+i,\, n-1)$