Thrall's problem and coarsenings

Banff workshop on Positivity in Algebraic Combinatorics

August 14-16, 2015

- Thrall's problem
- Known cases
- Coarsening

REFERENCES:

- Gessel & Reutenauer '93
  Counting permutations with given cycle structure and descent set
- Schocker '03
  Multiplicities of higher Lie characters
- Stanley
  Enumerative Combinatorics, Vol. 2 Exer. 7.89
- Thrall '42
  On symmetrized Kronecker powers and the structure of the free Lie ring
- Sundaram '94
  The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice
- R.–Saliola–Welker '14
  Spectra of symmetrized shuffling operators, §IV.7
- Hersh–R. '15
  Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$

Thrall's problem: For $\lambda \vdash n$, combinatorially interpret the coefficients a_μ^λ in the Schur function expansion

$$L_\lambda = \sum_{\mu \vdash n} a_\mu^\lambda s_\mu$$

if

$$L_\lambda := \sum_{\substack{\text{permutations} \\ w \text{ in } \mathfrak{S}_n \\ \text{of cycle type } \lambda}} F_{\mathrm{Des}(w)},$$

$$F_{\mathrm{Des}(w)} := \sum_{\substack{1 \le i_1 \le i_2 \le \cdots \le i_n: \\ i_j < i_{j+1} \text{ if } w_j > w_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

Gessel's fundamental quasisymmetric function

$n=3$

$$L_{(1,1,1)} = F_{\emptyset} = s_{(3)}$$

$$\emptyset = \mathrm{Des}(123), \quad 123 = (1)(2)(3)$$

$$L_{(2,1)} = F_{\{1\}} + F_{\{2\}} + F_{\{1,2\}}$$

$$\{1\} = \mathrm{Des}(2\cdot 13), \quad 2\cdot13 = (12)(3)$$

$$\{2\} = \mathrm{Des}(13\cdot 2), \quad 13\cdot2 = (23)(1)$$

$$\{1,2\} = \mathrm{Des}(3\cdot 2\cdot 1), \quad 3\cdot2\cdot1 = (13)(2)$$

$$= s_{(2,1)} + s_{(1,1,1)}$$

$$L_{(3)} = F_{\{2\}} + F_{\{1\}} = s_{(2,1)}$$

$$\{2\} = \mathrm{Des}(23\cdot 1), \quad 23\cdot1 = (123)$$

$$\{1\} = \mathrm{Des}(3\cdot 12), \quad 3\cdot12 = (132)$$

$$\sum_{\lambda \vdash n}{}' L_\lambda = \sum_{w \in \mathfrak{S}_n}{}' F_{Des(w)} = (x_1 + x_2 + \dots)^n = (s_{\square})^n$$

(RSK, pointing to the second "=")

$n=4$	$s_{(4)}$	$s_{(3,1)}$	$s_{(2,2)}$	$s_{(2,1,1)}$	$s_{(1,1,1,1)}$
$L_{(1,1,1,1)}$	1				
$L_{(2,1,1)}$		1		1	
$L_{(2,2)}$			1		1
$L_{(3,1)}$		1	1	1	
$L_{(4)}$		1		1	
TOTALS:	1	3	2	3	1

Why should L_λ be symmetric, and why Schur-positive?

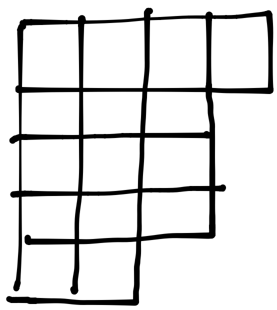
Let's reformulate it.

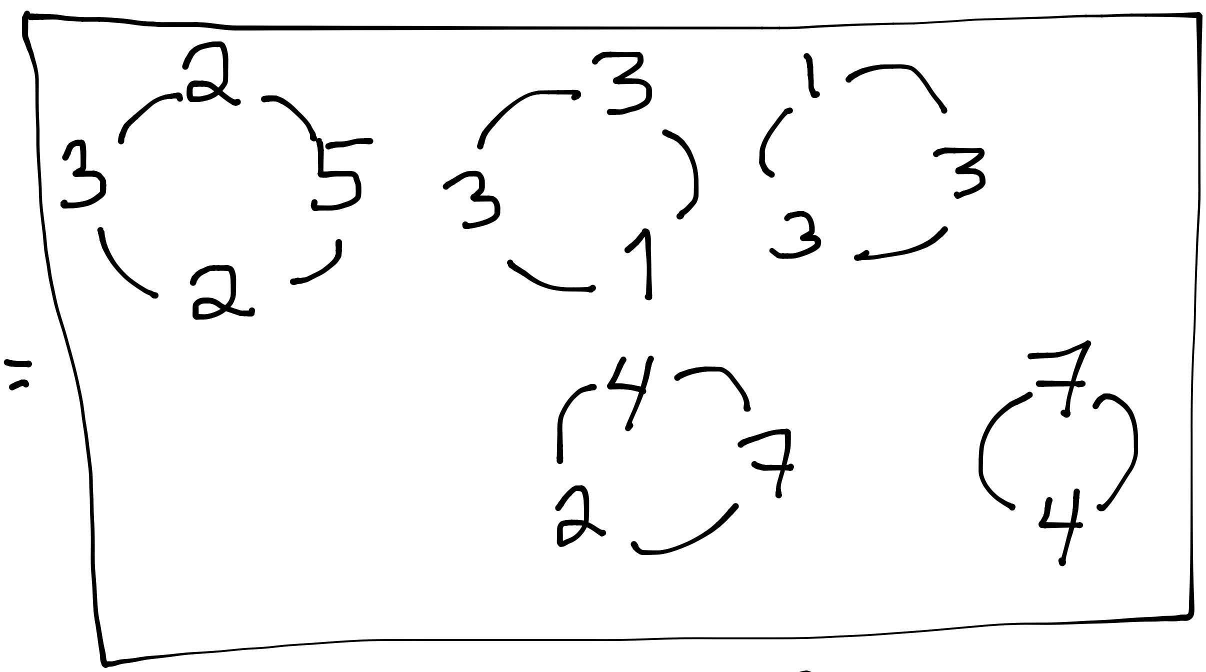
$$L_\lambda = \sum_{w \in \mathfrak{S}_n \text{ of cycle type } \lambda} F_{\operatorname{Des}(w)}$$

$$= \sum_{\text{multisets } \mathcal{O} \text{ of primitive necklaces of sizes } \lambda} x_{\mathcal{O}} \quad \Leftarrow \text{symmetric!}$$

Gessel's necklace bijection

EXAMPLE:

$\lambda =$ 

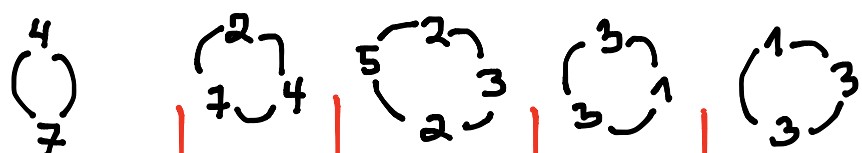
$\theta =$ 

$$\underline{x}_\theta = (x_3 x_2 x_5 x_2)(x_1 x_3 x_3)^2 (x_4 x_7 x_2)(x_4 x_7)$$

$$(s_1)^n = (x_1 + x_2 + \dots)^n = \sum_{\substack{\text{words } u \\ \text{in } \{1,2,\dots\}^n}} x_{u_1} x_{u_2} \cdots x_{u_n}$$

$$= \sum_{\lambda \vdash n} \sum_{\substack{\text{words } u \\ \text{whose Lyndon factorization} \\ \text{has type } \lambda}} x_{u_1} x_{u_2} \cdots x_{u_n}$$

$$= \sum_{\substack{\text{multisets } \omega \text{ of} \\ \text{primitive necklaces} \\ \text{of sizes } \lambda}} x_{\omega} = L_{\lambda}$$



$u = 47 \mid 247 \mid 2325 \mid 133 \mid 133$

lexicographic $\geq \quad \geq \quad \geq \quad \geq$

Hence if $\lambda = 1^{m_1} 2^{m_2} 3^{m_3} \cdots$ then

$$L_\lambda = h_{m_1}[L_{(1)}] \cdot h_{m_2}[L_{(2)}] \cdot h_{m_3}[L_{(3)}] \cdots$$

plethysm $f[g]$

where

$$L_{(n)} = \sum_{\substack{\text{primitive} \\ \text{necklaces } \nu \in \{1,2,\dots\}^n}} \underline{x}_\nu$$

$$= \frac{1}{n} \sum_{d \mid n} (x_1^d + x_2^d + \cdots)^{n/d} \mu(d)$$

Frobenius characteristic

$$= \mathrm{ch}\left(\chi \uparrow_{C_n}^{\mathfrak{S}_n} \right)$$ ← Schur-positive!

with $\chi : C_n \longrightarrow \mathbb{C}^\times$

$(1\,2 \cdots n) \longmapsto e^{2\pi i/n}$

Two other interpretations of $L_{(n)}$:

① $V = \mathbb{C}x_1 \oplus \mathbb{C}x_2 \oplus \cdots$

has tensor algebra $T(V) = \bigoplus_{n \geq 0} V^{\otimes n}$

containing the

free Lie algebra $\mathcal{L}(V) = \bigoplus_{n \geq 0} \mathcal{L}_n(V)$

where

$\mathcal{L}_1(V) = V$

$\mathcal{L}_2(V) = [V, V] = \mathbb{C}\{x \otimes y - y \otimes x\}$

$\mathcal{L}_3(V) = [[V, V], V] = [V, [V, V]]$

$\vdots$

Then $L_{(n)} = GL(V)$-character of $\mathcal{L}_n(V)$

$= \mathfrak{S}_n$-Frob. characteristic of the $x_1 x_2 \cdots x_n$-weight space in $\mathcal{L}_n(V)$

② $L_{(n)} = \mathfrak{S}_n$-Frob. characteristic of

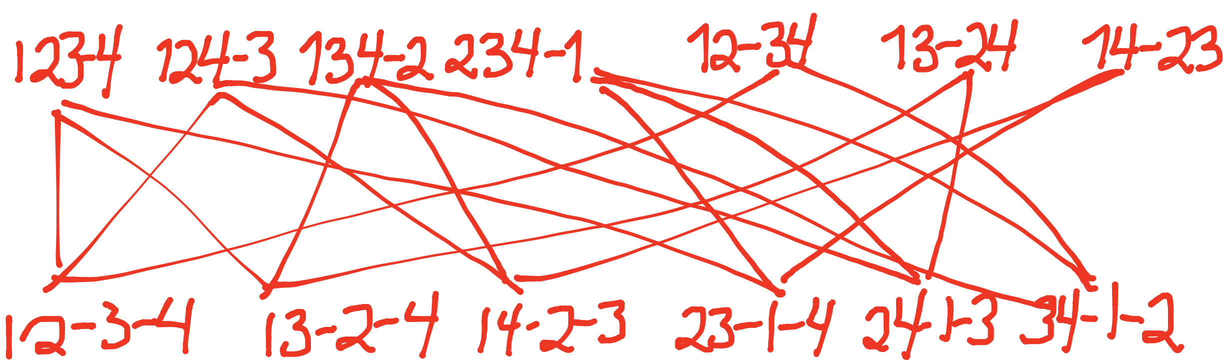
$$\mathrm{sgn} \otimes \tilde{H}_{n-3}(\overline{\Pi}_n)$$

Stanley '82
Hanlon '81

proper part of the lattice of set partitions of $\{1,2,\ldots,n\}$

$n=4$

$$L_{(4)} = s_{(3,1)} + s_{(2,1,1)} = ch(\mathrm{sgn} \otimes \tilde{H}_1(\overline{\Pi}_4))$$



Thrall's motivation, finally:

$(x_1+x_2+\dots)^n = GL(V)$-character of $T(V)$

$$T(V) = \mathcal{U}(\mathcal{L}(V)) \overset{\text{PBW}}{\cong} \mathrm{Sym}(\mathcal{L}(V))$$

universal enveloping algebra (for $\mathcal{U}$); $\cong$ as $GL(V)$-rep

$$= \bigoplus_{\lambda = 1^{m_1}2^{m_2}\cdots} \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \dots$$

Call this $\mathcal{L}_\lambda(V)$

Then $L_\lambda = GL(V)$-character of $\mathcal{L}_\lambda(V)$,

so $L_\lambda = \sum_\mu a_\mu^\lambda s_\mu$

gives its $GL(V)$-irreducible decomposition.

REMARK: There is an important variant of

$$L_\lambda = \prod_{i \geq 1} h_{m_i}[L_{(i)}]:$$

Setting $\pi_n := \omega(L_{(n)}) = ch(H_{n-3}(\Pi_n))$,

define

$$W_\lambda := h_{m_1}[\pi_1]\, e_{m_2}[\pi_2]\, h_{m_3}[\pi_3]\, e_{m_4}[\pi_4] \cdots$$

$$= \prod_{\substack{m \geq 1 \\ \text{odd}}} h_m[\pi_m] \cdot \prod_{\substack{m \geq 2 \\ \text{even}}} e_m[\pi_m].$$

Then there are multiple interpretations for

$$L_n^i := \sum_{\substack{\lambda \vdash n: \\ \ell(\lambda)=i}} L_\lambda \quad , \quad W_n^i := \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=i}} W_\lambda$$

- $W_n^i \cong i^{th}$ piece of type A_{n-1} Orlik-Solomon algebra

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- $W_n^i \cong i^{th}$ Whitney homology of Π_n

$$:= \bigoplus_{\substack{\pi \in \Pi_n: \\ \mathrm{rank}(\pi)=i}} \tilde{H}_{i-2}((\hat{0},\pi))$$

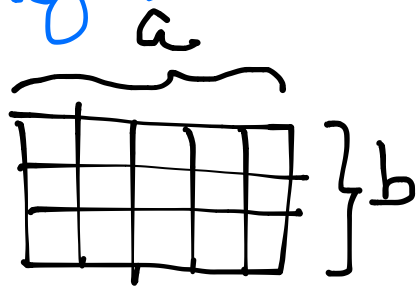
- $\mathrm{ch} H^{i(d-1)}\left(\begin{array}{l}\text{Configuration space of } n \\ \text{ordered distinct points in } \mathbb{R}^d\end{array}\right) \cong \begin{cases} L_n^i & \text{if } d \text{ odd} \\ W_n^i & \text{if } d \text{ even}\end{cases}$

Sundaram-Welker '97

Known cases and reductions for Thrall's problem

① $L_\lambda = h_{m_1}[L_{(1)}]\, h_{m_2}[L_{(2)}] \cdots$

reduces the difficulty, via Littlewood-Richardson rule, to the rectangular case

$\lambda = a^b =$ 

where $L_{(a^b)} = h_b[L_{(a)}]$

② $L_{(1)} = s_{\square}$,

so $L_{(1^m)} = h_m[L_{(1)}] = s_{\underbrace{\square\square\square\cdots\square}_{m}}$

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③ $L_{(2)} = e_2$,

so $L_{(2^m)} = h_m[e_2]$

$$= \sum_{\substack{\mu \vdash 2m \\ \text{with even column sizes}}} s_\mu$$

(Littlewood)

EXAMPLE:

$$L_{\boxplus} = L_{(2^2)} = h_2[e_2] = s_{(1^4)} + s_{\boxplus}$$

④ $$L_{(n)} = \sum_{\substack{\text{standard Young tableaux } Q \text{ of size } n \\ \text{with } \operatorname{maj} Q \equiv 1 \bmod n}} s_{\operatorname{shape}(Q)}$$

Klyachko '74
Kraśkiewicz-Weyman '87

EXAMPLES:

$$L_{(3)} = L_{(3)} = s_{(2,1)}$$

$$\begin{matrix} 1 & 3 \\ 2 & \end{matrix} \quad \operatorname{maj} = 1$$

$$L_{(4)} = L_{(4)} = s_{(2,1,1)} + s_{(3,1)}$$

$$\begin{matrix} 1 & 2 \\ 3 & \\ 4 & \end{matrix} \qquad \begin{matrix} 1 & 3 & 4 \\ 2 & & \end{matrix}$$

$\operatorname{maj} = 2+3 = 5 \equiv 1 \bmod 4$ $\qquad$ $\operatorname{maj} = 1$

(5) C. Ahlbach has a bijective approach to the previous result (aimed toward more cases...)

(6) Schocker generalized the previous result to the case of $L_{(a^b)}$, but his expansion involves

- <u>negative</u> signs
- <u>denominators</u> of $\frac{1}{b!}$

Coarsening Thrall's Problem

Let's consider first

$$\hat{L}_n := \sum_{\substack{\text{derangements} \\ w \text{ in } \mathfrak{S}_n}} F_{\mathrm{Des}(w)} = \sum_{\substack{\lambda \vdash n: \\ \lambda_i \geq 2}} L_\lambda$$

$$= \sum_{\substack{w \text{ in } \mathfrak{S}_n: \\ \text{first ascent of} \\ w \text{ is even}}} F_{\mathrm{Des}(w)}$$

$$= \sum_{\substack{\text{standard Young} \\ \text{tableaux } Q: \\ \text{first ascent of } Q \text{ is even}}} s_{\mathrm{shape}(Q)}$$

Désarménien-Wachs '88

What if we consider

$$\hat{L}_n^k := \sum_{\substack{\text{derangements} \\ w \text{ in } \mathfrak{S}_n \\ \text{with } k \text{ cycles}}} F_{\mathrm{Des}(w)} = \sum_{\substack{\lambda \vdash n: \\ \lambda_i \geq 2, \\ \ell(\lambda) = k}} L_\lambda$$

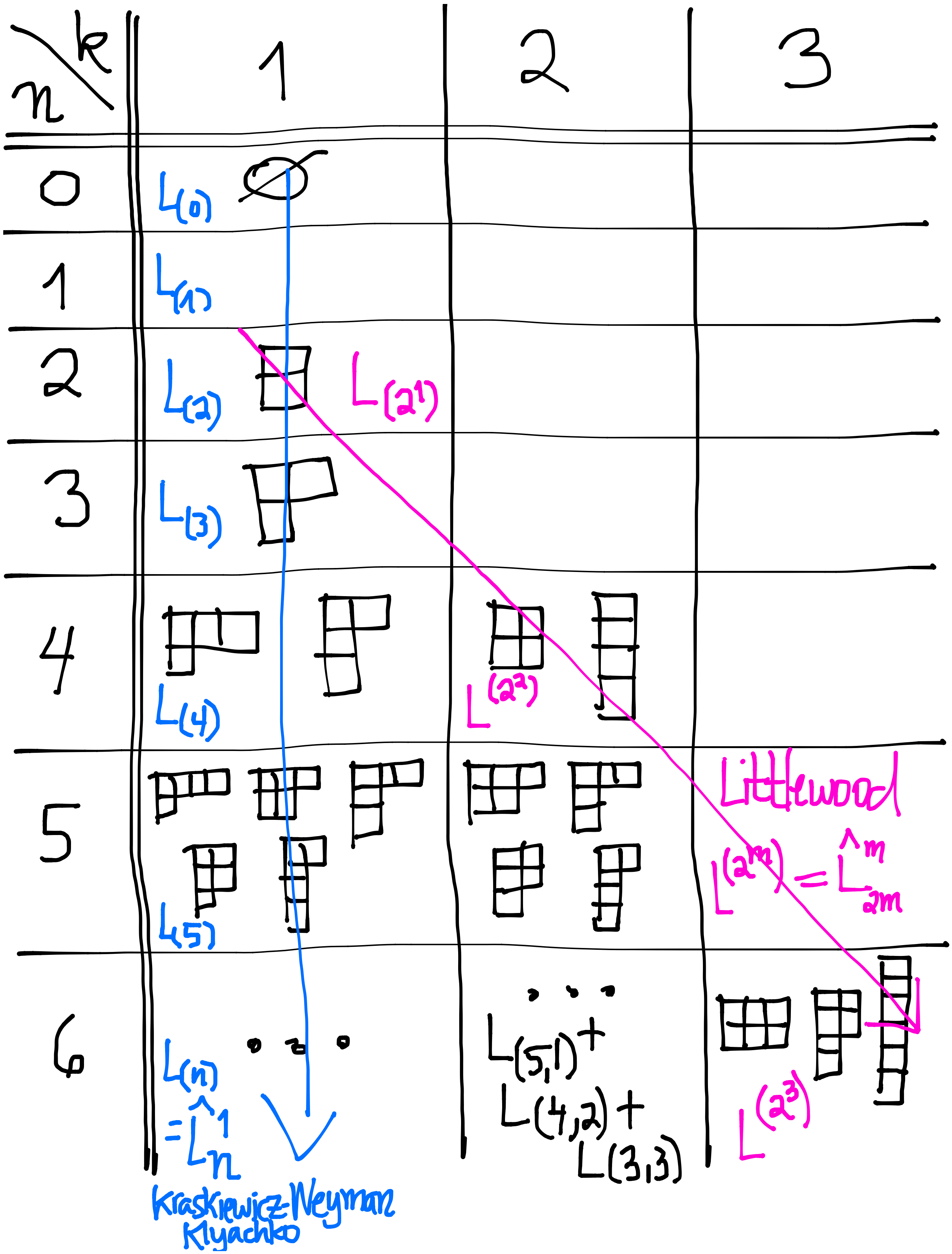
Coarsened Thrall Problem:

Interpret $a_\mu^{k,n}$ in the expansion

$$\hat{L}_n^k = \sum_{\mu \vdash n} a_\mu^{k,n} s_\mu .$$

More tractable? We have as guidance

- Désarménien-Wachs for $\hat{L}_n$
- Branching rules for $\hat{L}_n^k$ (Hersh-R. '15)
- Extreme values of k

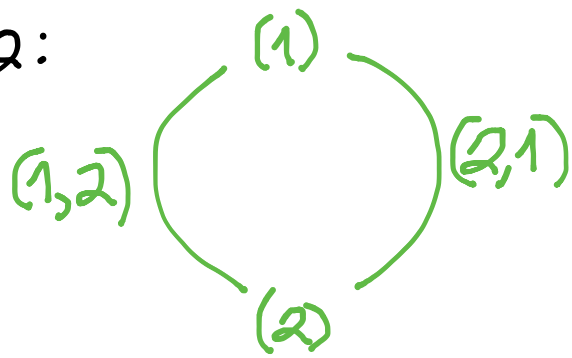

n \ k
1
2
3
0
1
2
3
4
5
6
$L_{(0)}$
$L_{(1)}$
$L_{(2)}$
$L_{(2^1)}$
$L_{(3)}$
$L_{(4)}$
$L^{(2^2)}$
$L_{(5)}$
Littlewood
$L^{(2^m)} = \hat{L}^m_{2m}$
$L_{(n)} = \hat{L}^1_n$
$L_{(5,1)} + L_{(4,2)} + L_{(3,3)}$
$L^{(2^3)}$
Kraskiewicz-Weyman
Klyachko

Why would such coarsenings be important?

$\hat{L}_n = \mathfrak{S}_n$-Frobenius characteristic of the homology of the complex of injective words on $\{1,2,\ldots,n\}$

R.-Webb '04

$n=2$:



$\hat{L}_2 = s_{\square\square}$

$\hat{L}_n^k$ = same for the k^{th} piece in the Eulerian idempotent refinement or Hodge decomposition of the above homology

Hanlon-Hersh '04

Additionally, for d odd

if $X_n := \begin{pmatrix} \text{configuration space of} \\ n \text{ ordered distinct points} \\ \text{in } \mathbb{R}^d \end{pmatrix}$

then

$\hat{L}_n = \mathfrak{S}_n$-Frobenius characteristic on the FI-module generators for all of $H^*(X_n)$

(in sense of Church-Ellenberg-Farb '15)

$\hat{L}_n^{n-i} =$ same thing but more specifically for $H^i(X_n)$

(Similarly for d even, replacing $\hat{L} \rightsquigarrow \hat{W}$)

THANKS FOR COMING!

REFERENCES:

- Gessel & Reutenauer '93
  Counting permutations with given cycle structure and descent set
- Schocker '03
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- Hersh-R. '15
  Representation stability for cohomology of configuration spaces in $\mathbb{R}^d$