# Cyclic sieving: Old and New

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1. What is cyclic sieving?

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(OLD, annoying)

2. EXAMPLE
   - alternating sign matrices

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(OLD, annoying)

3. EXAMPLE
   - polygon triangulations

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4. OLD EXAMPLE with a NEW twist
   - necklaces

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(NEW-ish)

5. EXAMPLE
   - transposition factorizations

# 1. What is cyclic sieving?

## REFERENCES

- B.E. Sagan
  - Surveys in combinatorics 2011
- G. Gaiffi & A. Iraci
  - Rivista dell'Unione Matematica Italiana 2017
- D. Stanton, D. White & V.R.
  - Notices of the AMS 2014
  - J. Comb. Thy. Ser. A 2004

# DEFINITION (RSW 2004)

Given

- $X$ a finite set,
- permuted by a cyclic group $C = \{1, c, c^2, \ldots, c^{n-1}\} \cong \mathbb{Z}/n\mathbb{Z}$,
- a polynomial $X(q) \in \mathbb{Z}[q]$,

Say that $(X, C, X(q))$ exhibits the **cyclic sieving phenomenon (CSP)**

if for every $d$

$$\#\{x \in X : c^d(x) = x\} = \left[X(q)\right]_{q = \zeta^d}$$

where $\zeta := e^{\frac{2\pi i}{n}}$

# SPECIAL CASE

$$C = \{1, c\} \cong \mathbb{Z}/2\mathbb{Z}$$
$$c^2 = 1$$

is Stembridge's 1994

$q = -1$ phenomenon

$$X(1) = \#X$$
$$X(-1) = \#\{x \in X : c(x) = x\}$$

(OLD, annoying)

# 2. EXAMPLE

**DEFINITION** (Mills, Robbins, Rumsey 1982)

An $n \times n$ alternating sign matrix has entries in $\{0,+1,-1\}$ and each row and column has its nonzero entries alternating in sign,

$$+ - + - \dots + - + - +$$

and summing to $+1$

EXAMPLE

$$A = \begin{bmatrix} 0 & +1 & 0 & 0 & 0 & 0 \\ +1 & -1 & 0 & 0 & +1 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & +1 & -1 & +1 & -1 & +1 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 \end{bmatrix} \in ASM_6$$

$= \{6 \times 6$ alternating sign matrices$\}$

They generalize permutation matrices.

$$C = \{1, c, c^2, c^3\} \cong \mathbb{Z}/4\mathbb{Z} \text{ acts on } ASM_n$$

via $0°, 90°, 180°, 270°$ rotations

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EXAMPLE $n=3$

$$ASM_3 =$$

$$\left\{
\begin{tikzcd}
\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \arrow[r, bend left, "c"] & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \arrow[d, bend left, "c"] \\
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \arrow[u, bend left, "c"] & \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \arrow[l, bend left, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \arrow[r, bend left, "c"] & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \arrow[l, bend left, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \arrow[loop right, "c"]
\end{tikzcd}
\right\}$$

For $X = ASM_n$,

what is $|X|$?

**THEOREM**

$$|ASM_n| = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\,(n+2)!\cdots(2n-1)!}$$

conjectured by Mills, Robbins, Rumsey 1982

proven by Zeilberger 1992
Kuperberg 1995

- What is
$$|X^{c^2}| = |\{A \in ASM_n : c^2(A) = A\}| \ ?$$
half-turn symmetric ASMs

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- What is
$$|X^{c}| = |\{A \in ASM_n : c(A) = A\}| \ ?$$
quarter-turn symmetric ASMs

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Formulas by Kuperberg 2002
Razumov–Stroganov 2004

## THEOREM (Stanton 2004)

This triple exhibits the CSP:

$$X = ASM_n$$

$$\circlearrowleft$$

$$C = \underbrace{\{1, c, c^2, c^3\}}_{\text{rotations}} \cong \mathbb{Z}/4\mathbb{Z}$$

$$X(q) = \frac{[1]!_q\,[4]!_q\,[7]!_q \cdots [3n-2]!_q}{[n]!_q\,[n+1]!_q\,[n+2]!_q \cdots [2n-1]!_q}$$

where $[n]!_q = [1]_q[2]_q[3]_q \cdots [n]_q$

$$[n]_q = 1+q+q^2+\ldots+q^{n-1} = \frac{q^n-1}{q-1}$$

**EXAMPLE** $X = ASM_3$ has

$$X(q) = \frac{[1]!_q\,[4]!_q\,[7]!_q}{[3]!_q\,[4]!_q\,[5]!_q} = \frac{[7]_q[6]_q}{[3]_q[2]_q}$$

$$= 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8$$

$q = 1 = \zeta^0$

$$\frac{7\cdot 6}{3\cdot 2} = 7 = |X| = |X^{c^0}|$$

$$\begin{tikzcd}
\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \arrow[r, bend left, "c"] & \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \arrow[d, bend left, "c"] \\
\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \arrow[u, bend left, "c"] & \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \arrow[l, bend left, "c"]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \arrow[r, bend left, "c"] & \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \arrow[l, bend left, "c"]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \arrow[loop right, "c"]
\end{tikzcd}$$

$q = -1 = \zeta^2$

$$\frac{6}{2} = 3 = |X^{c^2}|$$

$$\begin{tikzcd}
\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \arrow[r, bend left, "c"] & \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} \arrow[l, bend left, "c"]
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \arrow[loop right, "c"]
\end{tikzcd}$$

$q = i = \zeta^1$

$$1 = |X^{c^1}|$$

$$\begin{tikzcd}
\begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \arrow[loop right, "c"]
\end{tikzcd}$$

So what's annoying here?
Stanton's proof is **brute force:**

- Do the easy evaluations of $X(q)$ at $q = 1, -1, \pm i$
- Compare the answers with the known formulas by Kuperberg, Razumov-Stroganov!

# What would we like **better**?

## Trace comparison proof

Find a $\mathbb{C}$-vector space $V$ with 2 bases $\{v_x\}_{x \in X}$, $\{w_x\}_{x \in X}$ both indexed by $x \in X$,

- $C$ permuting the $\{v_x\}$ basis: $c(v_x) = v_{c(x)}$
- $C$ scaling the $\{w_x\}$ basis: $c(w_x) = \zeta^{\mathrm{stat}(x)} w_x$

  where $X(q) = \sum_{x \in X} q^{\mathrm{stat}(x)}$

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Then ... $\mathrm{Trace}_V(c^d)$

use $\{v_x\}$: $\mathrm{Trace}_V(c^d) = \left|\{x \in X : c^d(x) = x\}\right|$

use $\{w_x\}$: $\mathrm{Trace}_V(c^d) = \sum_{x \in X} (\zeta^d)^{\mathrm{stat}(x)} = \left[X(q)\right]_{q=\zeta^d}$

For $X = ASM_n$, we don't even have such a stat(x)! Instead…

**THEOREM** (Andrews 1979)

$$\frac{[1]!_q [4]!_q [7]!_q \cdots [3n-2]!_q}{[n]!_q [n+1]!_q [n+2]!_q \cdots [2n-1]!_q} = \sum_{\substack{\text{descending} \\ \text{plane partitions } \pi \\ \text{with parts } \leq n}} q^{|\pi|}$$

(OLD, annoying)

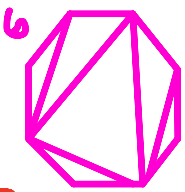
# 3. EXAMPLE

**THEOREM** (RSW 2004)

This triple $(X, C, X(q))$ exhibits the CSP:

$X = \{$triangulations of an $(n+2)$-sided polygon$\}$

n=6



$\circlearrowleft$

$C = \langle c \rangle \cong \mathbb{Z}/(n+2)\mathbb{Z}$ via rotations

$X(q) = \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$ = MacMahon's q-Catalan polynomial

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Here $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$

**EXAMPLE** $n=4$ $\quad C \cong \mathbb{Z}/6\mathbb{Z}$ $\quad \zeta = e^{\frac{2\pi i}{6}}$

$$X(q) = \frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q = \frac{[8]_q[7]_q[6]_q}{[4]_q[3]_q[2]_q}$$

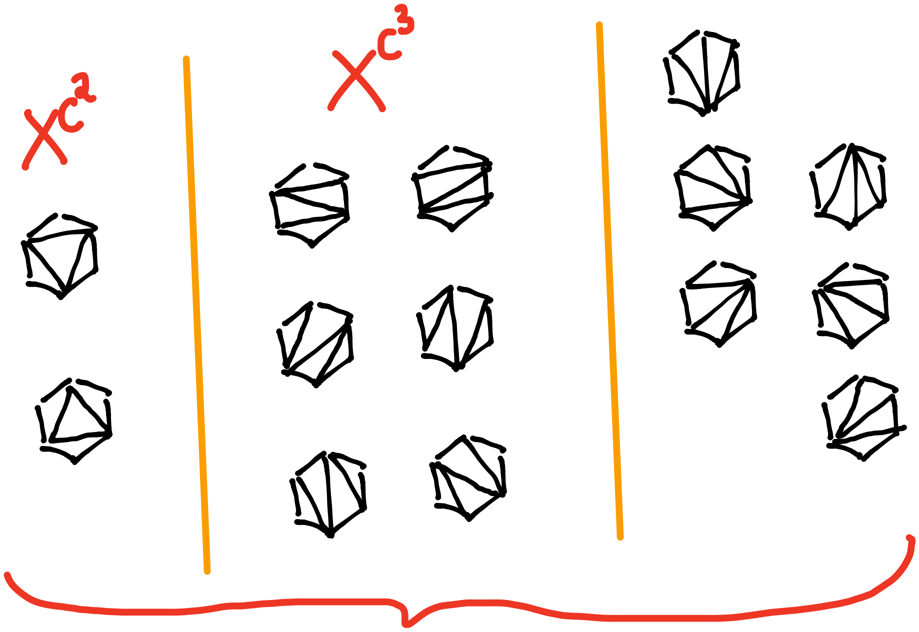
$$= 1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

$q=\zeta^1$: $0$

$q=\zeta^2$: $2$

$q=\zeta^3=-1$: $6$

$q=\zeta^0=1$: $14$




$X = X^{C^0}$

We only know a **brute force** proof!
And it generalizes (still via **brute force** proof):

**THEOREM** (S.-P. Eu & T.-S. Fu 2006)

For finite real reflection group $W$
with degrees $d_1 \leq d_2 \leq \cdots \leq d_n = h$,
one has a CSP triple $(X, C, X(q))$

$X = \{W\text{-clusters}\}$
$\circlearrowleft$

Fomin & Zelevinsky's deformed Coxeter element

$C = \langle \tau \rangle \cong \mathbb{Z}/(h+2)\mathbb{Z}$

$$X(q) = \frac{[h+d_1]_q [h+d_2]_q \cdots [h+d_n]_q}{[d_1]_q [d_2]_q \cdots [d_n]_q} = W\text{-}q\text{-Catalan polynomial}$$

A tantalizingly close result with a good proof...

**THEOREM** (Rhoades 2010)

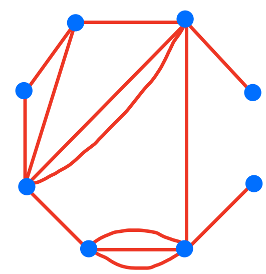
This triple $(X, C, X(q))$ exhibits the CSP:

$X$ = multidissections of a regular $n$-gon using $k$ edges

$\circlearrowright$

$C \cong \mathbb{Z}/n\mathbb{Z}$ via rotations

$n=8$
$k=13$



$X(q) = q^{-k} s_{\underbrace{\square\square\square\square\square}_{k \text{ columns}}}(1, q, q^2, \ldots, q^{n-1})$ $q$-Narayana polynomials

Proof uses $V = k^{th}$ graded component of coordinate ring of $Gr(2, \mathbb{C}^n)$

$\{v_x\}$ = cluster monomial basis

$\{w_x\}$ = semistandard tableaux basis

# 4. OLD EXAMPLE with a NEW twist

Rewriting MacMahon's q-Catalan

$$\frac{1}{[n+1]_q}\begin{bmatrix}2n\\n\end{bmatrix}_q = \frac{1}{[2n+1]_q}\begin{bmatrix}2n+1\\n\end{bmatrix}_q \in \mathbb{Z}[q]$$

one sees it is the special case $(a,b)=(n,n+1)$ of the

rational q-Catalan for $\gcd(a,b)=1$

$$\frac{1}{[a+b]_q}\begin{bmatrix}a+b\\a\end{bmatrix}_q \in \mathbb{Z}[q]$$

considered by Armstrong, Rhoades, Williams 2013
Armstrong, Loehr, Warrington 2013
Bodnar & Rhoades 2015 and others.

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More generally it was observed in RSW 2004 that for any composition $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$ of $n$ with $\gcd(\alpha)=1$

$$C_\alpha(q) := \frac{1}{[n]_q}\begin{bmatrix}n\\\alpha\end{bmatrix}_q = \frac{1}{[n]_q}\begin{bmatrix}n\\\alpha_1,\alpha_2,\ldots,\alpha_\ell\end{bmatrix}_q \in \mathbb{Z}[q].$$

# Why is $\frac{1}{[n]_q}\begin{bmatrix} n \\ \alpha_1,\alpha_2,\dots,\alpha_\ell\end{bmatrix}_q \in \mathbb{Z}[q]$?

**proof:** There is a CSP triple $(X, C, X(q))$

$X = \{$ $\alpha$-words, that is, those with $\alpha_1$ 1's, $\alpha_2$ 2's, $\alpha_3$ 3's, etc. $\}$

$\circlearrowleft$

$C \cong \mathbb{Z}/n\mathbb{Z}$ cycling positions

$$(w_1, w_2, \dots, w_{n-1}, w_n) \xrightarrow{c} (w_n, w_1, w_2, \dots, w_{n-1})$$

$$X(q) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$$

When $\gcd(\alpha)=1$, the $C$-action is free

so $[X(q)]_{q=\zeta^d} = 0$ for all $n^{th}$ roots $\zeta^d \neq 1$.

Hence $[n]_q = 1+q+q^2+\dots+q^{n-1} = \prod_{d=1}^{n-1}(q-\zeta^d)$

divides $X(q) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$ $\blacksquare$

Since $C = \mathbb{Z}/n\mathbb{Z}$ acts freely,

$$\left[C_\alpha(q)\right]_{q=1} = \frac{1}{n}\binom{n}{\alpha}$$

counts $C$-orbits $= \{\alpha\text{-necklaces}\}$

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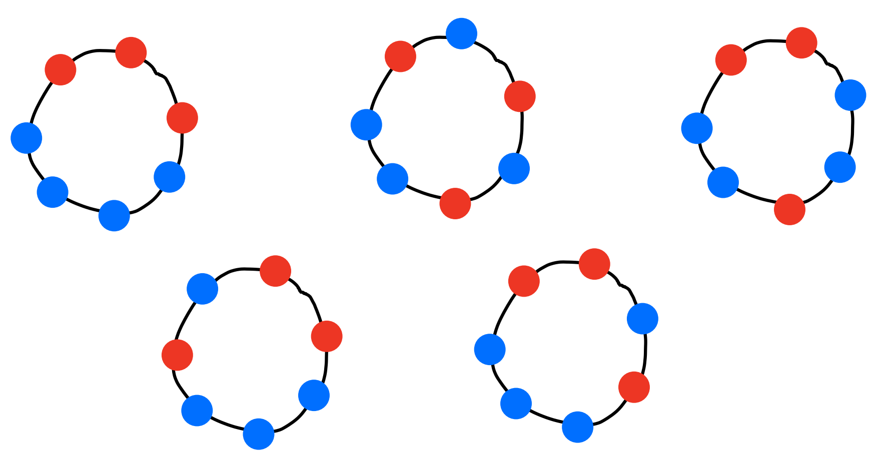
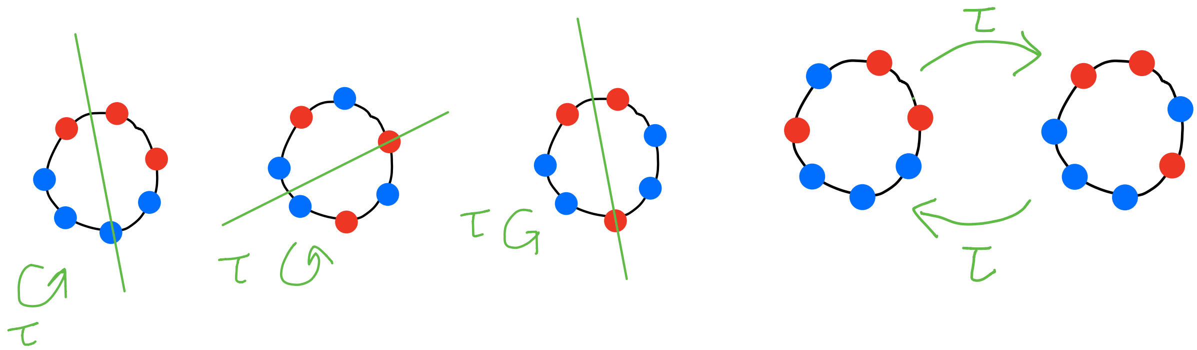
EXAMPLE $\alpha = (3,4)$

$$C_\alpha(q) = \frac{1}{[7]_q}\begin{bmatrix}7\\3\end{bmatrix}_q = \frac{\cancel{[7]_q}[6]_q[5]_q}{\cancel{[7]_q}[3]_q[2]_q} = 1+q^2+q^3+q^4+q^6$$

$\underset{q=1}{\leadsto}$ $\frac{1}{7}\binom{7}{3} = 5$ counts $(3,4)$-necklaces



Some $\alpha$-necklaces are fixed by $\mathbb{Z}/2\mathbb{Z}$-action of reflection $\tau$




**THEOREM** (Stucky 2018) If $\gcd(\alpha)=1$ one has a $q=-1$ phenomenon for

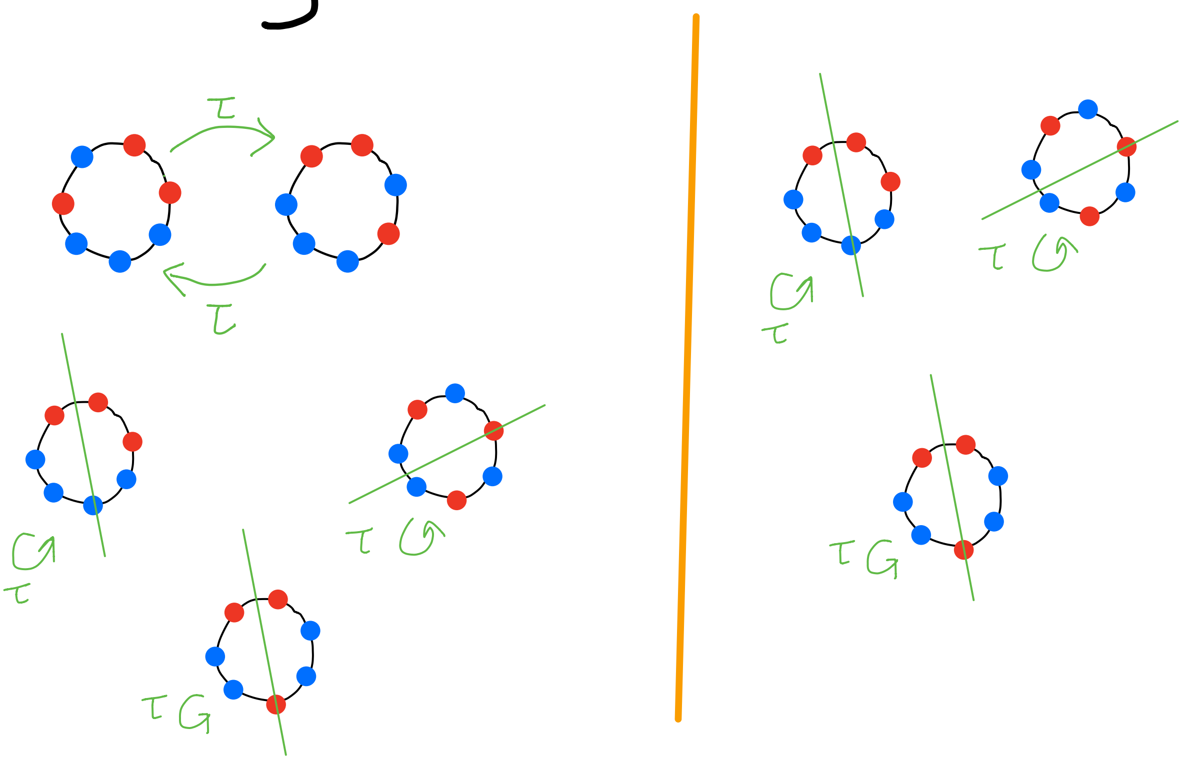
$$Y = \{\alpha\text{-necklaces}\} \circlearrowleft \tau = \text{reflection}$$

$$Y(q) = C_\alpha(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ \alpha \end{bmatrix}_q$$

# EXAMPLE $\alpha = (3,4)$

$$C_\alpha(q) = \frac{1}{[7]_q}\begin{bmatrix}7\\3\end{bmatrix}_q = 1 + q^2 + q^3 + q^4 + q^6$$

$$\xrightarrow{q=1} 5 \qquad \xrightarrow{q=-1} 3$$



Stucky's proof is interesting,
using **Molien's Theorem**
and allows generalization . . .

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cosets

$$X = \left\{ \underset{\text{with } \gcd(\alpha)=1}{\alpha\text{-words}} \right\} \rightsquigarrow \mathfrak{S}_n / H$$

$$\circlearrowright \qquad\qquad\qquad\qquad \circlearrowright \text{ with free action}$$

$$C = \mathbb{Z}/n\mathbb{Z} \qquad\qquad C = \langle (1\,2\,3 \ldots n) \rangle$$

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double cosets

$$Y = \left\{ \alpha\text{-necklaces} \right\} \rightsquigarrow C \backslash \mathfrak{S}_n / H$$

$$\circlearrowright \qquad\qquad\qquad\qquad \circlearrowright$$

$$\tau = \text{reflection} \qquad\qquad \langle \tau \rangle < N_{\mathfrak{S}_n}(C)$$

(NEW-ish)

## 5. EXAMPLE

**THEOREM** (Hurwitz 1891)

$$X = \Big\{ \text{shortest factorizations of } \begin{aligned} c = (1\,2\,3 \cdots n) &= (i_1 j_1)(i_2 j_2) \cdots (i_{n-1} j_{n-1}) \\ &= t_1 \; t_2 \; \cdots \; t_{n-1} \end{aligned} \text{ into transpositions } t = (ij) \text{ in } \mathfrak{S}_n \Big\}$$

has $|X| = n^{n-2}$

**EXAMPLE** $n=3$

$$\left.\begin{aligned} (123) &= (12)(23) \\ &= (13)(12) \\ &= (23)(13) \end{aligned}\right\} |X| = 3^1$$

There is a natural cyclic action $C = \langle \psi \rangle \cong \mathbb{Z}/n(n-1)\mathbb{Z}$ on $X$ considered by Armstrong 2009

$$(123\cdots n) = t_1 t_2 \cdots t_{n-2}\, t_{n-1}$$

$$\left.\begin{array}{l} \big\downarrow \psi \\ = c\, t_{n-1} c^{-1} \cdot t_1 t_2 \cdots t_{n-2} \\ \big\downarrow \psi \\ \vdots \\ \big\downarrow \psi \end{array}\right\} n-1 \text{ times}$$

$$= c t_1 c^{-1} \cdot c t_2 c^{-1} \cdots c t_{n-2} c^{-1} \cdot c t_{n-1} c^{-1}$$

# EXAMPLE

$n=4$

$$|X| = n^{n-2} = 4^2 = 16$$

$$(12)(23)(34) \xrightarrow{\psi} (14)(12)(23) \xrightarrow{\psi} (34)(14)(12) \xrightarrow{\psi} (23)(34)(14) \xrightarrow{\psi} (12)(23)(34)$$

$$(13)(34)(12) \xrightarrow{\psi} (23)(13)(34) \xrightarrow{\psi} (14)(23)(13) \xrightarrow{\psi} (24)(14)(23) \xrightarrow{\psi} (34)(24)(14) \xrightarrow{\psi} (12)(34)(24) \xrightarrow{\psi} (13)(12)(34)$$

$$\xrightarrow{\psi} (14)(13)(12) \xrightarrow{\psi} (23)(14)(13) \xrightarrow{\psi} (24)(23)(14) \xrightarrow{\psi} (12)(24)(23) \xrightarrow{\psi} (34)(12)(24) \xrightarrow{\psi} (13)(34)(12)$$

**THEOREM** (Douvropoulos 2017
Conj. by N. Williams 2013)

One has a CSP for

$$X = \left\{ \begin{array}{c} \text{shortest factorizations} \\ c = (12 \cdots n) = t_1 t_2 \cdots t_{n-1} \end{array} \right\}$$

$\circlearrowleft$

$$C = \langle \psi \rangle \cong \mathbb{Z}/n(n-1)\mathbb{Z}$$

$$X(q) = [n]_{q^2} [n]_{q^3} \cdots [n]_{q^{n-1}}$$

# EXAMPLE $n=4$

$$X(q) = [4]_{q^2}[4]_{q^3}$$

$$= (1+q^2+q^4+q^6)(1+q^3+q^6+q^9)$$

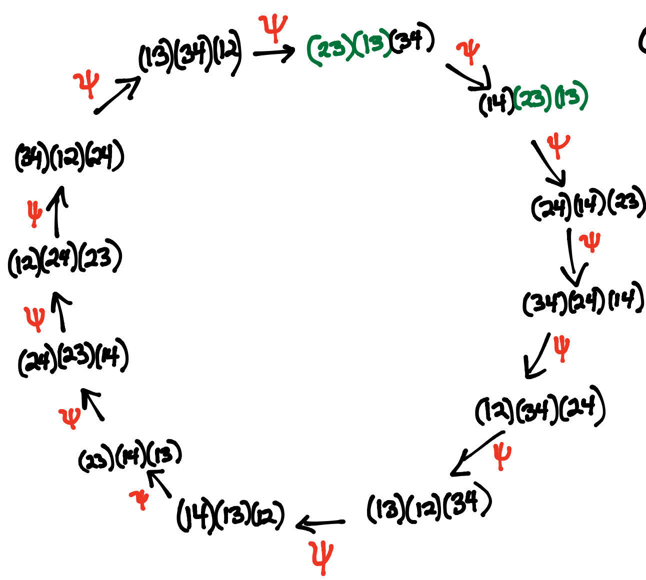
$$\xrightarrow{q=\zeta^0=1} 16 \qquad \xrightarrow{q=\zeta^4=e^{\frac{2\pi i}{3}}} 4$$

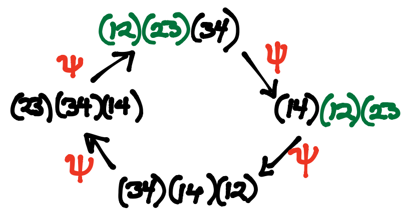
$$\xrightarrow{q=\zeta^1} 0$$

$$\xrightarrow{q=\zeta^2} 0$$

$$\xrightarrow{q=\zeta^3} 0$$

$$\xrightarrow{q=\zeta^6} 0$$



Generalizes to CSP for $W$ a real reflection group with degrees $d_1 \leq d_2 \leq \cdots \leq d_\ell = h$

$$X = \left\{ \begin{array}{l} \text{shortest factorizations} \\ \text{of Coxeter element} \\ c = t_1 t_2 \cdots t_\ell \text{ into} \\ \text{reflections} \end{array} \right\}$$

$$\circlearrowleft$$

$$C = \langle \psi \rangle \cong \mathbb{Z}/h\mathbb{Z}$$

$$X(q) = \frac{[h]_q [2h]_q [3h]_q \cdots [\ell h]_q}{[d_1]_q [d_2]_q [d_3]_q \cdots [d_\ell]_q}$$

= Arnold-Deligne-Bessis formula

Douvropoulos's proof is via trace comparison using Bessis's 2006 work on the geometry of the Lyashko-Looijenga covering.

Thanks for your attention, and thanks again to the Northeast Combinatorics Network!