Whitney numbers for poset cones

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- Zaslavsky's Theorems
  counting chambers in
  hyperplane arrangements
  and cones

- Braid arrangements,
  poset cones, and
  linear extensions

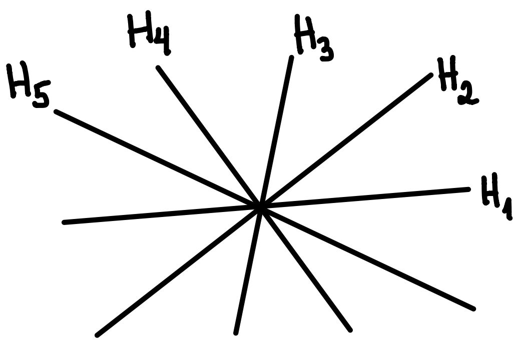
- Two formulas for
  any poset

- Foata's thesis and
  disjoint unions of chains

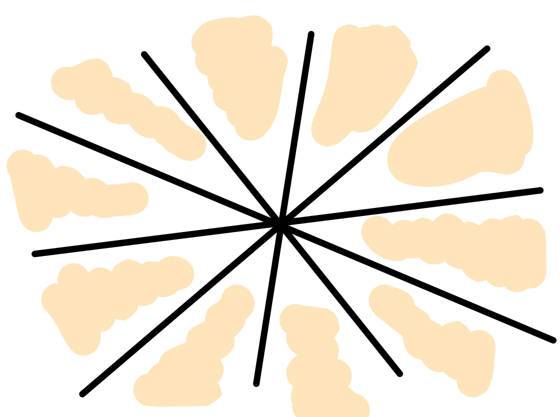
Zaslavsky's Theorems

$\mathcal{A} = \{H_1, H_2, \dots, H_N\}$

an arrangement of hyperplanes in $V = \mathbb{R}^n$
(= codimension one linear subspaces)



dissects the complement $V \setminus \mathcal{A}$ into connected components called chambers

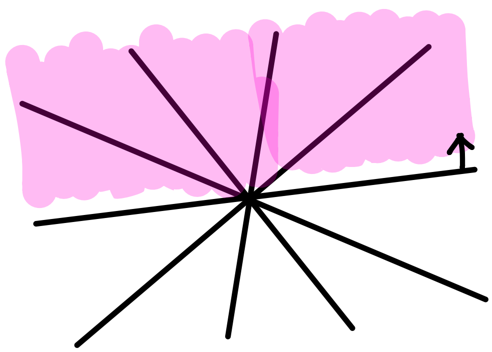


10 chambers

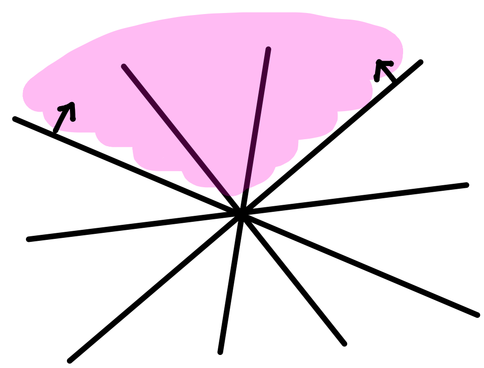
How to count them?

More generally, a cone $\mathcal{K}$ in $\mathcal{A}$ is any intersection of its (open) halfspaces, containing a subset of the chambers of $\mathcal{A}$

How to count them?



$\mathcal{K}_1$



$\mathcal{K}_2$

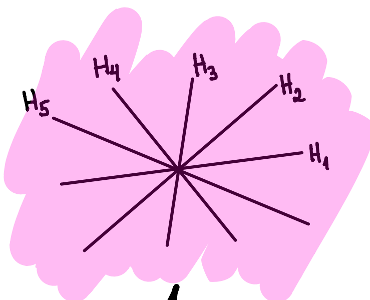
5 chambers inside cone $\mathcal{K}_1$

3 chambers inside cone $\mathcal{K}_2$

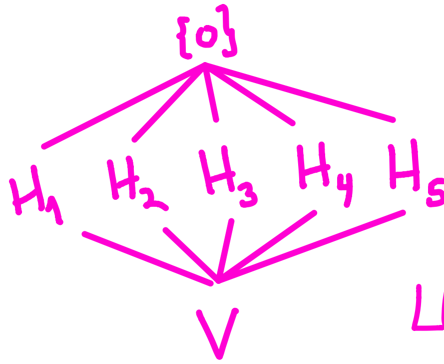
Introduce the poset of intersections

$$L(\mathcal{A}) := \left\{ \begin{array}{l} \text{intersection subspaces} \\ X = H_{i_1} \cap H_{i_2} \cap \dots \cap H_{i_k} \end{array} \right\}$$

ordered via reverse inclusion



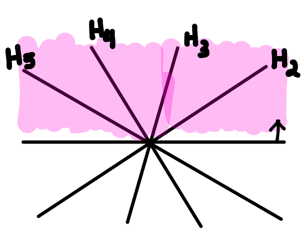
$\mathcal{A}$



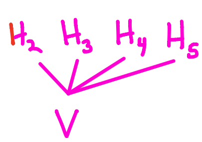
$L(\mathcal{A})$

and for the cones K, the subposet (even order ideal) of interior intersections

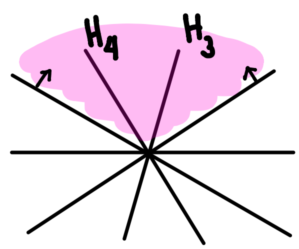
$$L^{int}(K) = \{ X \in L(\mathcal{A}) : X \cap K \neq \emptyset \}$$



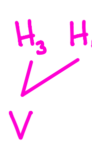
K_1



$L^{int}(K_1)$

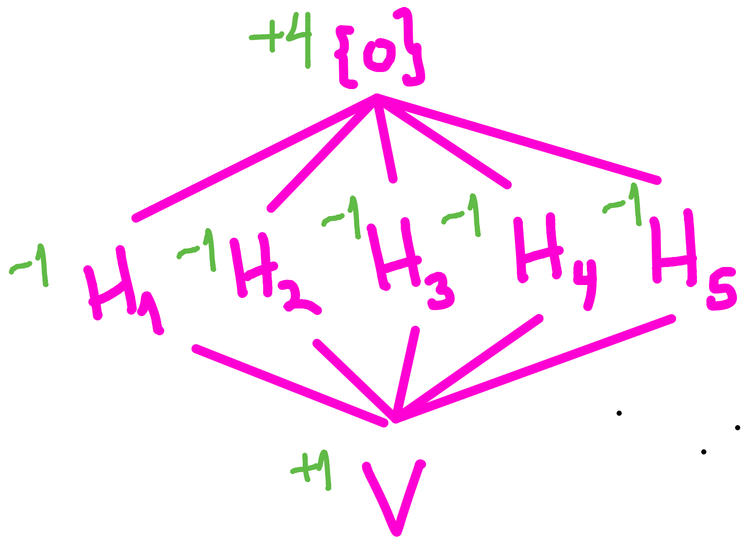


K_2



$L^{int}(K_1)$

To count chambers, label $X \in L(\mathcal{A})$ by Möbius function values $\mu(V,X)$



$c_2 = 4$

$c_1 = 5$

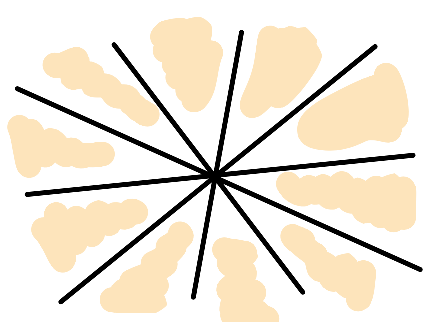
$c_0 = 1$

THEOREM (Zaslavsky 1974)

$$\#\text{chambers of } \mathcal{A} = \sum_{X \in L(\mathcal{A})} |\mu(V,X)| = c_0 + c_1 + \dots + c_n$$

where $c_k = \sum_{\substack{X \in L(\mathcal{A}): \\ \operatorname{codim}(X) = k}} |\mu(V,X)|$

k^{th} Whitney number of $\mathcal{A}$



#chambers

$10 = 1 + 5 + 4$

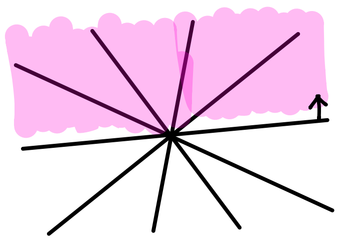
$= c_0 + c_1 + c_2$

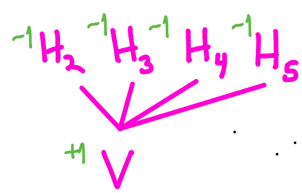
More generally ...

THEOREM (Zaslavsky 1977) For any cone K in $\mathcal{A}$,

$$\#\text{chambers of } \mathcal{A} \text{ inside } K = \sum_{X \in L^{int}(K)} |\mu(V,X)| = c_0(K) + \dots + c_n(K)$$

where $c_k(K) = \sum_{\substack{X \in L^{int}(\mathcal{A}): \\ \operatorname{codim}(X) = k}} |\mu(V,X)|$



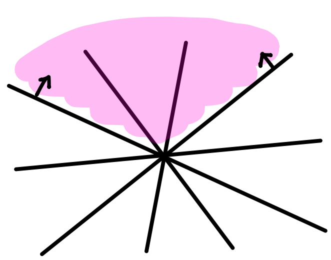


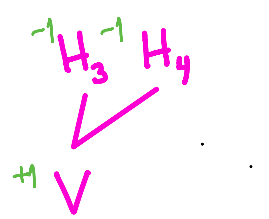
$c_2 = 0$

$c_1 = 4$

$c_0 = 1$

$c_0 + c_1 + c_2 = 1 + 4 + 0$
$= 5$ chambers





$c_2 = 0$

$c_1 = 2$

$c_0 = 1$

$c_0 + c_1 + c_2 = 1 + 2 + 0$
$= 3$ chambers

Define the generating function

$$\mathrm{Poin}(\mathcal{A},t)=c_0+c_1t+c_2t^2+\dots+c_nt^n$$

Poincaré polynomial of $\mathcal{A}$

It gets its name from the interpretation

$$c_k=\mathrm{rank}_{\mathbb{Z}}H_k(\underbrace{\mathbb{C}^n\setminus\mathcal{A}_{\mathbb{C}}}_{\text{complexified complement of }\mathcal{A}},\mathbb{Z})$$

For any cone $\mathcal{K}$ in $\mathcal{A}$, we'll similarly call

$$\mathrm{Poin}(\mathcal{K},t)=c_0(\mathcal{K})+c_1(\mathcal{K})t+c_2(\mathcal{K})t^2+\dots+c_n(\mathcal{K})t^n$$

the Poincaré polynomial of $\mathcal{K}$.

GOAL:

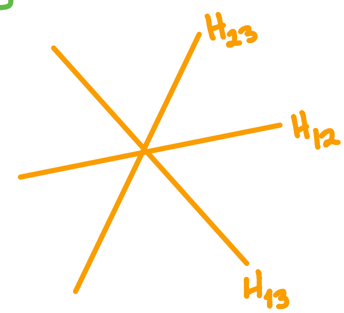
Interpret $\mathrm{Poin}(\mathcal{K},t)$ combinatorially, whenever possible.

Braid arrangements
(the motivating example)

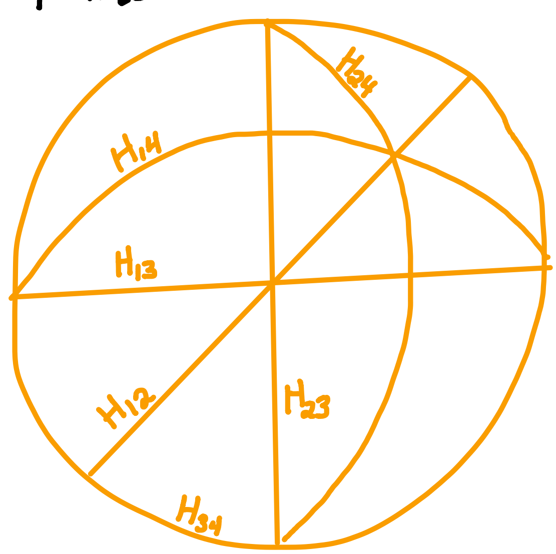
The braid arrangement in $\mathbb{R}^n$ has hyperplanes $H_{ij} = \{x_i = x_j\}$ for $1 \le i < j \le n$

in $\mathbb{R}^3$

$\rightsquigarrow$ intersect with $\begin{bmatrix}1\\1\\1\end{bmatrix}^{\perp}$



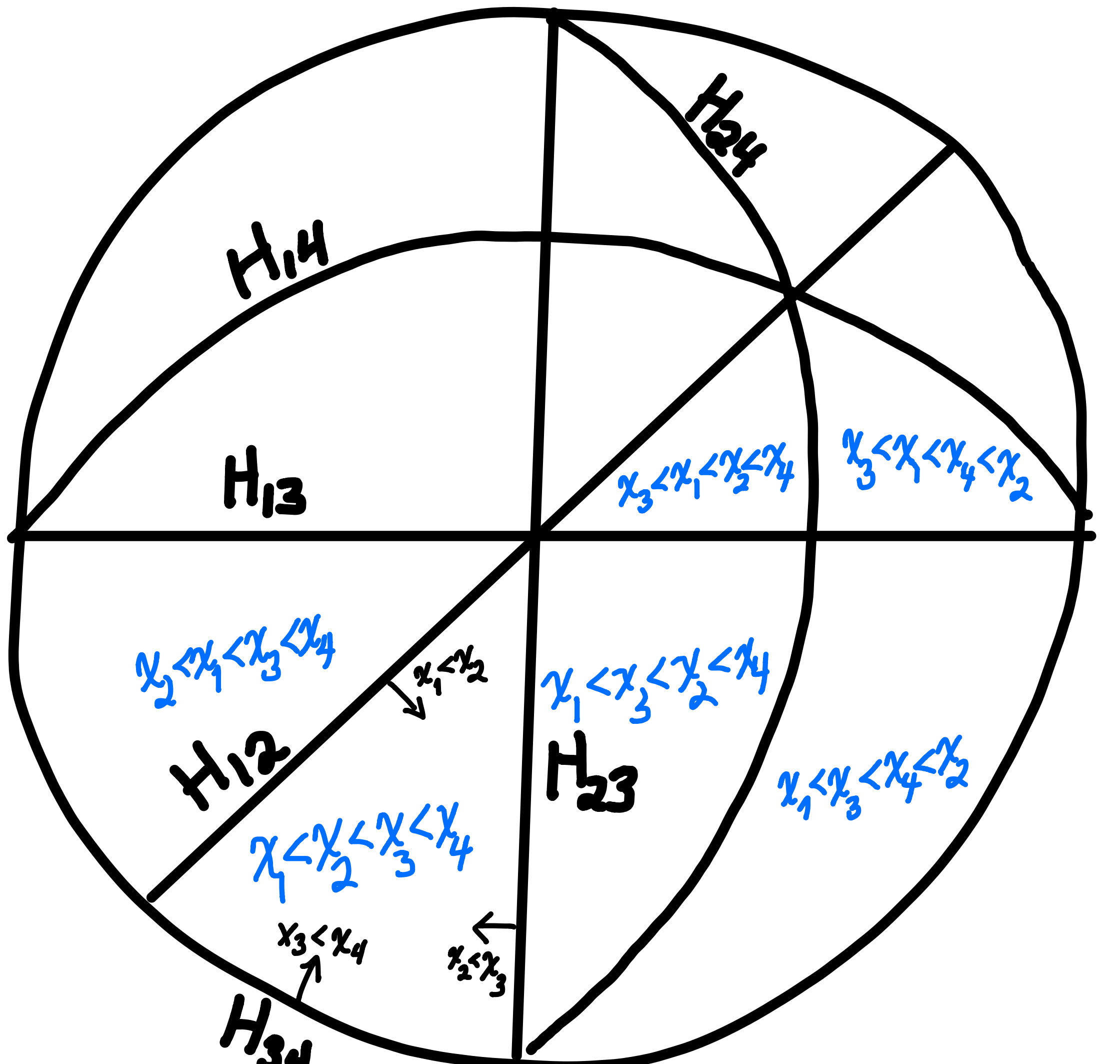
Picture for $n=4$ intersected with unit sphere in $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}^{\perp}$



Inside the braid arrangement,

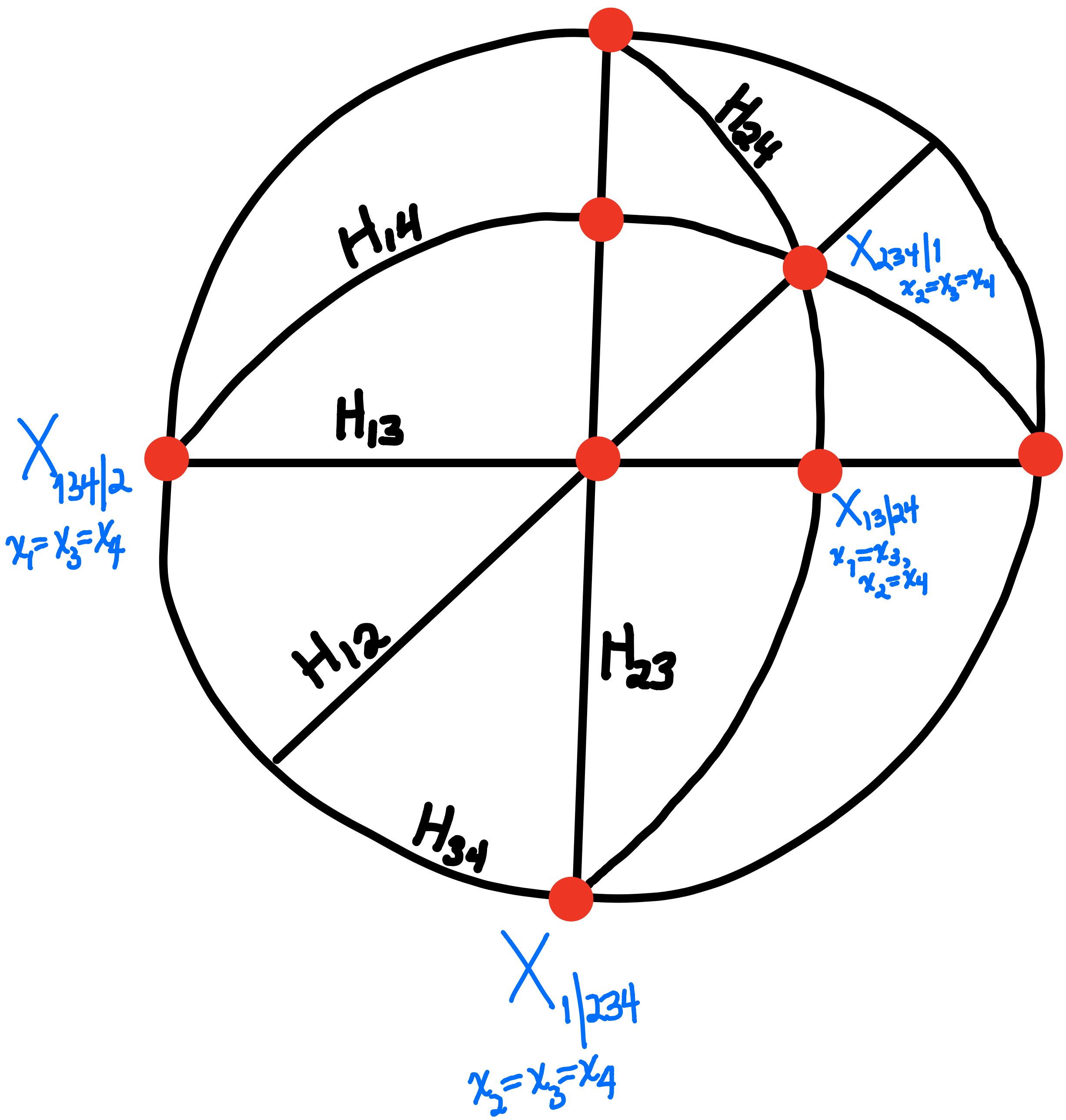
chambers $\longleftrightarrow$ permutations $\sigma = (\sigma_1, \ldots, \sigma_n)$ in S_n

$x_{\sigma_1} < x_{\sigma_2} < \ldots < x_{\sigma_n}$



so $\#\text{chambers} = n!$

intersection subspaces X_π $\longleftrightarrow$ set partitions $\pi = \{B_1, B_2, \ldots, B_k\}$ of $\{1,2,\ldots,n\} = \bigsqcup_{i=1}^{k} B_i$



$\mathcal{A}$ = braid arrangement in $\mathbb{R}^n$ has many expressions for its Poincaré polynomial:

$$\mathrm{Poin}(\mathcal{A},t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

$$= \sum_{\sigma \in S_n} t^{n-\#\mathrm{cycles}(\sigma)}$$

$$= \sum_{\sigma \in S_n} t^{n-\#\mathrm{LRmax}(\sigma)}$$

where $\mathrm{LRmax}(\sigma)$ = left-to-right maxima of σ

e.g. $\sigma = 418253697$ has

$\#\mathrm{LRmax}(\sigma) = 3$

Inside the braid arrangement,

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cones $\mathcal{K}_P$ $\longleftrightarrow$ posets P on $\{1,2,\dots,n\}$

$$= \bigcap_{i <_P j} \{x_i < x_j\}$$

e.g. $\mathcal{K}_P = \{x_1 < x_2\} \cap \{x_3 < x_4\}$ $\longleftrightarrow$ $P = \begin{array}{cc} 2 & 4 \\ | & | \\ 1 & 3 \end{array}$

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chambers inside $\mathcal{K}_P$ $\longleftrightarrow$ linear extensions $\sigma = (\sigma_1 < \sigma_2 < \dots < \sigma_n)$ of P $=: \mathrm{LinExt}(P)$

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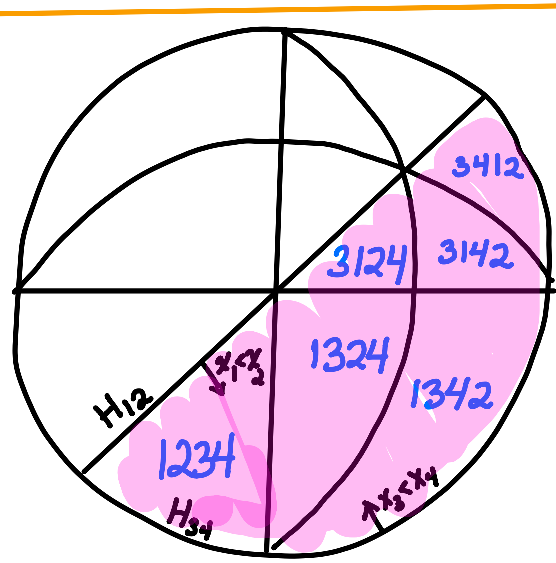
e.g.



$P = \begin{array}{cc} 2 & 4 \\ | & | \\ 1 & 3 \end{array}$

$\mathrm{LinExt}(P) = \{1234, 1324, 1342, 3124, 3142, 3412\}$

Thus Zaslavsky's Theorem for cones gives

COROLLARY

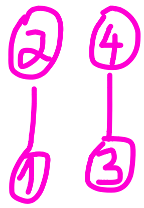
$$\#\mathrm{LinExt}(P) = c_0(K_P) + c_1(K_P) + \cdots + c_n(K_P)$$
$$= \left[\mathrm{Poin}(K_P, t)\right]_{t=1}$$

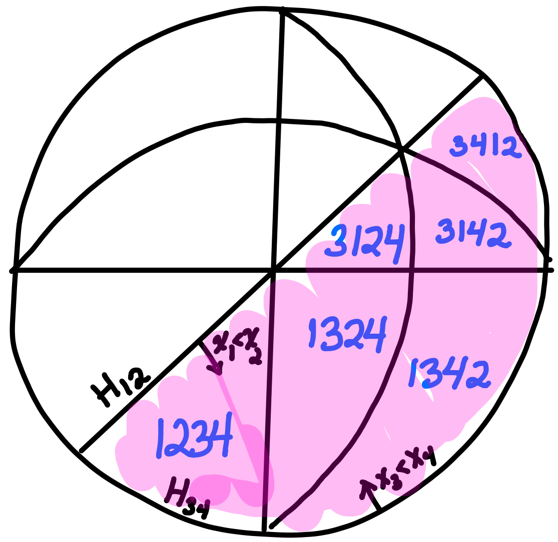
(#P-)hard to compute for arbitrary posets P [Brightwell-Winkler] 1991

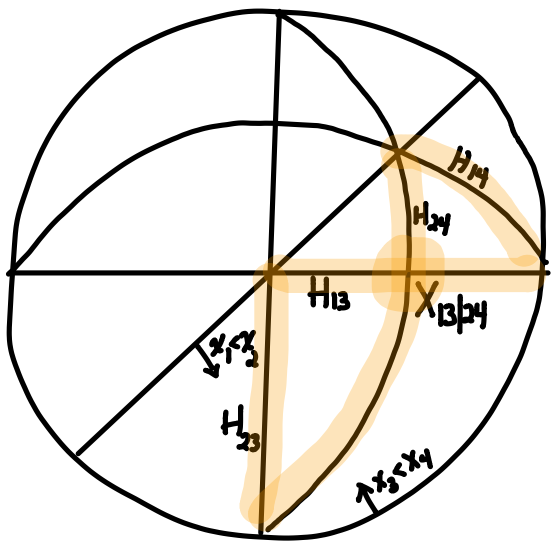
PROBLEM:

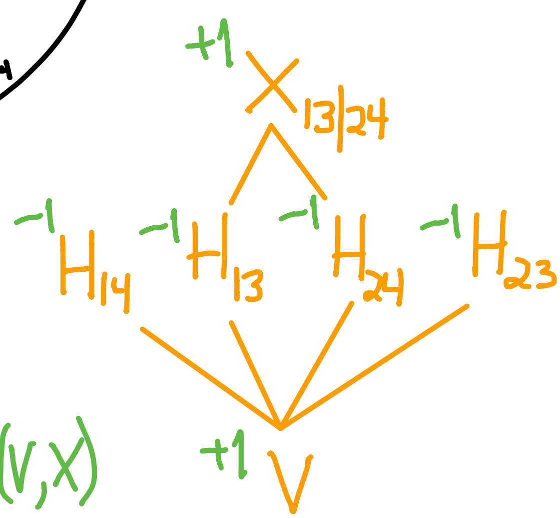
Interpret $\mathrm{Poin}(K_P, t)$ for posets P, by refining the count $\#\mathrm{LinExt}(P)$.

EXAMPLE

$P =$ 







$\mu(V,X)$

$c_2 = 1$

$c_1 = 4$

$c_0 = 1$

$\#\mathrm{LinExt}(P) = \#\text{chambers} = 1+4+1 = 6$

PROBLEM: Interpret $\mathrm{Poin}(K_P, t)$ for posets P, by refining the count $\#\mathrm{LinExt}(P)$.

We had three solutions for $P = ①\ ②\ \cdots\ ⓝ$ where

$$\mathrm{LinExt}(P) = S_n$$

$$\#\mathrm{LinExt}(P) = n!$$

$$\mathrm{Poin}(K_P, t) = (1+t)(1+2t)(1+3t)\cdots(1+(n-1)t)$$

$$= \sum_{\sigma \in S_n} t^{n - \#\mathrm{cycles}(\sigma)}$$

$$= \sum_{\sigma \in S_n} t^{n - \#\mathrm{LRmax}(\sigma)}$$

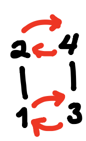
Two formulas for any poset

THEOREM (Dorpalen-Barry – Kim – R. 2019)

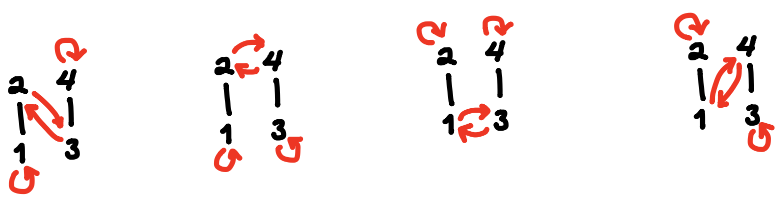
$$\mathrm{Poin}(K_P, t) = \sum_{\substack{P\text{-transverse} \\ \text{permutations} \\ \sigma \in S_n}} t^{\,n-\#\mathrm{cycles}(\sigma)}$$

DEFINITION:

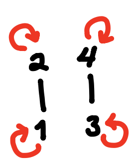
- cycles of σ are antichains in P
- the quotient pre-poset P/σ collapses no strict order relations $i <_P j$



$C_2 = 1$

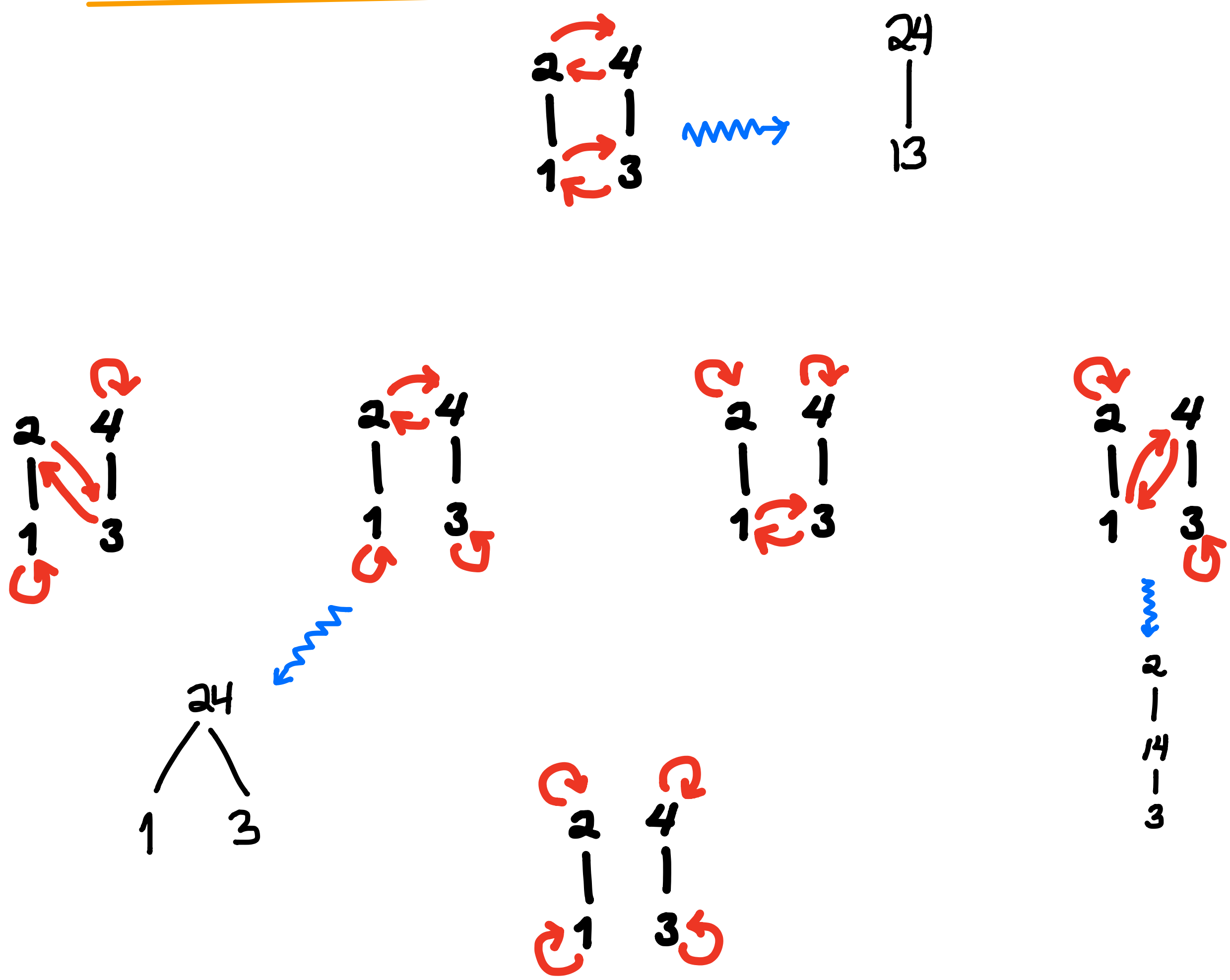


$C_1 = 4$



$C_0 = 1$

What some of those quotient pre-posets look like:



P-transverse $\overset{\text{DEF}}{=}$ $\left\{ \right.$

- cycles of σ are antichains in P
- the quotient pre-poset P/σ collapses no strict order relations $i <_P j$

Note the summation in

$$\mathrm{Poin}(K_P, t) = \sum_{\substack{P\text{-transverse} \\ \text{permutations} \\ \sigma \in S_n}} t^{n - \#\mathrm{cycles}(\sigma)}$$

is not over LinExt(P), unlike here:

THEOREM (Dorpalen-Barry – Kim – R. 2019)

$$\mathrm{Poin}(K_P, t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{n - \#P\text{-LRmax}(\sigma)}$$

where P-LRmax(σ) generalizes LRmax(σ) (in an interesting way, not described here)

The proof is a bijection

$$\begin{array}{ccc} \mathrm{LinExt}(P) & \longrightarrow & P\text{-transverse permutations} \\ \tau & \longmapsto & \sigma \end{array}$$

with $\#P\text{-LRmax}(\tau) = \#\mathrm{cycles}(\sigma)$

Foata's thesis and disjoint unions of chains

Can we have the best of both worlds,

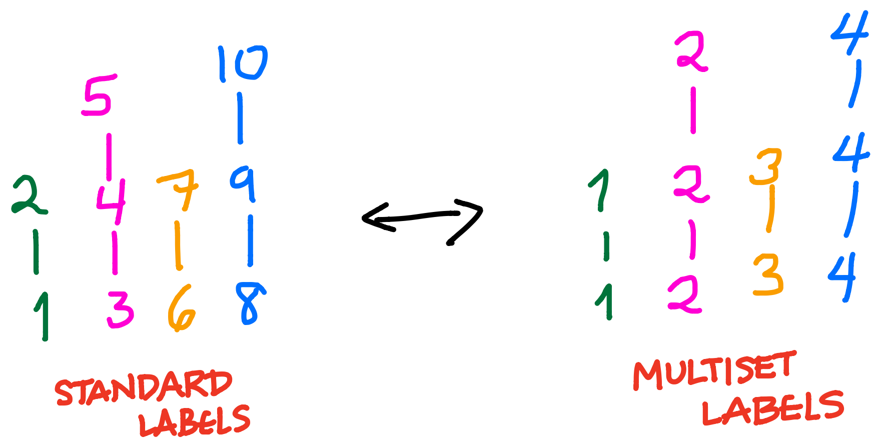
i.e.
$$\mathrm{Poin}(K_P, t) = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{n - \#\text{"cycles}(\sigma)\text{"}}$$

for some natural notion of "cycles" when $\sigma \in \mathrm{LinExt}(P)$?

Amazingly, the answer is YES when P is a disjoint union of chains, where Foata's 1965 thesis can be re-interpreted as giving a natural factorization into cycles for elements of LinExt(P).

Labeling disjoint unions of chains two ways

$P_{(2,3,2,3)} =$



gives an easy bijection

$\mathrm{LinExt}(P_{\underline{a}}) \longleftrightarrow$ permutations of the multiset $1^{a_1}2^{a_2}3^{a_3}\cdots$

$3\,8\,9\,6\,1\,4\,2\,7\,10\,5 \longleftrightarrow 2\,4\,4\,3\,1\,2\,1\,3\,4\,2$

Foata defined intercalation product on multiset permutations in 2-line notation:

$$\begin{pmatrix}2&3&4\\4&2&3\end{pmatrix} \top \begin{pmatrix}1&1&2&2&3&4&4\\2&4&3&1&1&4&2\end{pmatrix} = \begin{pmatrix}1&1&2&2&2&3&3&4&4&4\\2&4&4&3&1&2&1&3&4&2\end{pmatrix}$$

And then he showed they have an essentially unique factorization into (ordinary) cycles, e.g.

$$\begin{pmatrix}1&1&2&2&2&3&3&4&4&4\\2&4&4&3&1&2&1&3&4&2\end{pmatrix}$$

$$= \begin{pmatrix}2&3&4\\4&2&3\end{pmatrix} \top \begin{pmatrix}1&1&2&2&3&4&4\\2&4&3&1&1&4&2\end{pmatrix}$$

$$= \begin{pmatrix}2&3&4\\4&2&3\end{pmatrix} \top \begin{pmatrix}4\\4\end{pmatrix} \top \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} \top \begin{pmatrix}1&2&4\\4&1&2\end{pmatrix}$$

allowed to swap these two, because they commute

THEOREM (Dorpalen-Barry - Kim - R. 2019)

When $P_{\underline{a}}$ is a disjoint union of chains of sizes $\underline{a}=(a_1, a_2, \ldots, a_n)$, then

$$\mathrm{Poin}(K_{P_{\underline{a}}}, t) = \sum_{\sigma \in \mathrm{LinExt}(P_{\underline{a}})} t^{n - \#\text{Foata-cycles}(\sigma)}$$

where $\#$Foata-cycles(σ) means the number of cycles in Foata's unique decomposition for the permutation of the multiset $1^{a_1} 2^{a_2} 3^{a_3} \ldots$ corresponding to σ.

EXAMPLE $\underline{a}=(2,2)$

$$P = \begin{array}{cc} 2 & 4 \\ | & | \\ 1 & 3 \end{array} \quad \longleftrightarrow \quad \begin{array}{cc} 1 & 2 \\ | & | \\ 1 & 2 \end{array}$$

$\sigma \in \mathrm{LinExt}(P)$	permutation of $1^2 2^2$	# Foata-cycles(σ)	
1234	$\begin{pmatrix}1&1&2&2\\1&1&2&2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$	4	$c_0=1$
1324	$\begin{pmatrix}1&1&2&2\\1&2&1&2\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$	3	
1342	$\begin{pmatrix}1&1&2&2\\1&2&2&1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix}$	3	
3124	$\begin{pmatrix}1&1&2&2\\2&1&1&2\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix} \top \begin{pmatrix}2\\2\end{pmatrix}$	3	$c_1=4$
3142	$\begin{pmatrix}1&1&2&2\\2&1&2&1\end{pmatrix} = \begin{pmatrix}2\\2\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1\\1\end{pmatrix}$	3	
3412	$\begin{pmatrix}1&1&2&2\\2&2&1&1\end{pmatrix} = \begin{pmatrix}1&2\\2&1\end{pmatrix} \top \begin{pmatrix}1&2\\2&1\end{pmatrix}$	2	$c_2=1$

Foata used his theory for generatingfunctionology including a new proof of MacMahon's Master Theorem.

We used it to get this generating function for $\mathrm{Poin}(P_{\underline{a}}, t)$'s:

THEOREM (Dorpalen-Barry - Kim - R. 2019)

$$\sum_{\underline{a}} \mathrm{Poin}(P_{\underline{a}}, t)\, x_1^{a_1} x_2^{a_2} \cdots = \frac{1}{1 - \sum_{j \geq 1} e_j(\underline{x})\,(t-1)(2t-1)\cdots((j-1)t-1)}$$

where $e_j(\underline{x})$ = j^{th} elementary symmetric function in $x_1, x_2, \ldots$

QUESTION:

Is there a Foata-style factorization theory for $\mathrm{LinExt}(P)$ of all posets P, not just disjoint unions of chains?

Thanks for
your
attention!