Whitney numbers for poset cones

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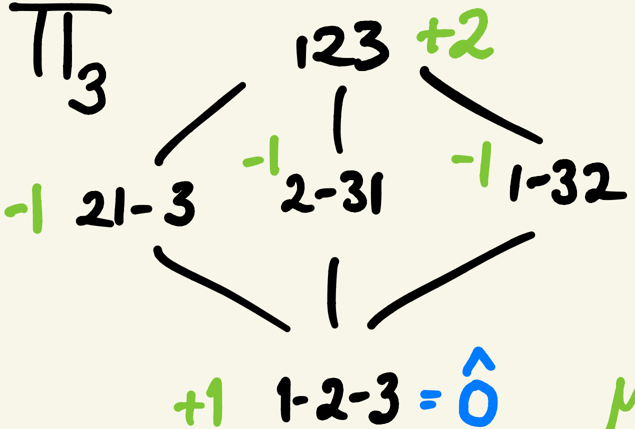
This talk is being recorded

- Stirling numbers of 1st kind
  - 4 formulas
- Whitney numbers for cones
- Poset cones and 4 formulas

- (Signless) Stirling numbers of the 1^{st} kind $c(n,k)$

$$\sum_{k=1}^{n} c(n,k)\, t^k = \sum_{\substack{\text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{\text{blocks}(\pi)}$$

Π_3



$\mu(\hat{0},\pi)$ labeled

$2 = c(3,1)$

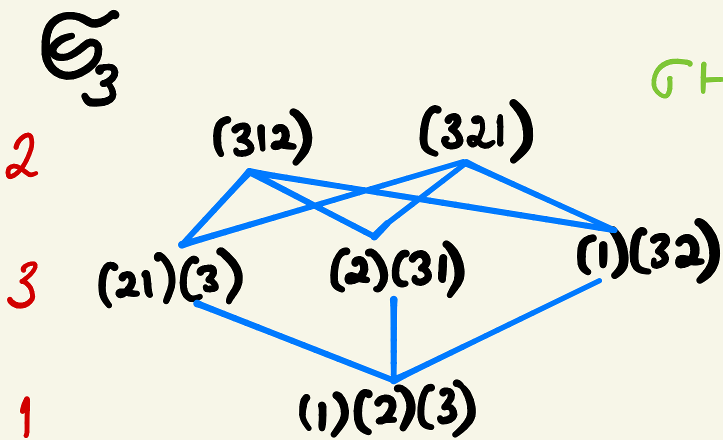
$3 = c(3,2)$

$1 = c(3,3)$

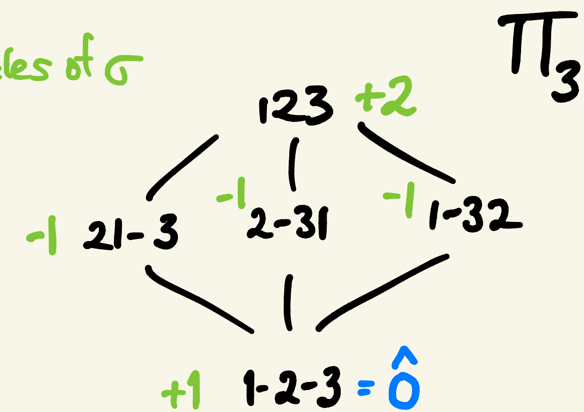
$$\sum_{k=1}^{n} c(n,k)\, t^{k} = \sum_{\substack{\text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{\text{blocks}(\pi)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cyc}(\sigma)}$$

$\text{cyc}(\sigma)$ = number of cycles of σ



$\sigma \longmapsto$ cycles of σ

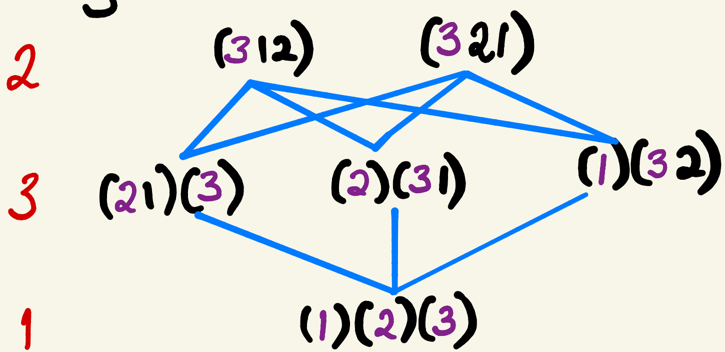


$$\sum_{k=1}^{n} c(n,k)\, t^{k} = \sum_{\substack{\text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{\text{blocks}(\pi)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cyc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{LRmax}(\sigma)}$$

LRmax(σ) = # of left-to-right maxima in $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$

$\mathfrak{S}_3$



$\longmapsto$

Foata's 1st fundamental transformation

(write cycles with biggest element first, list cycles in increasing order)

$\mathfrak{S}_3$

312 321

213 231 132

123

$$\sum_{k=1}^{n} c(n,k)\, t^{k} = \sum_{\substack{\text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{\text{blocks}(\pi)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\text{cyc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{LRmax}(\sigma)}$$

$$= t(t+1)(t+2)\cdots(t+(n-1))$$

$n=3$:

$$\underset{\substack{\| \\ c(3,3)}}{1} \cdot t^3 + \underset{\substack{\| \\ c(3,2)}}{3} t^2 + \underset{\substack{\| \\ c(3,1)}}{2} t = t(t+1)(t+2)$$

Want to generalize $\mathfrak{S}_n$ = linear orders on $\{1,2,\dots,n\}$

$\rightsquigarrow$ $\mathrm{LinExt}(P)$ = linear extensions of a poset

and generalize the 4 formulas

$$\sum_{k=1}^{n} c(n,k)\,t^k = \sum_{\pi \in \Pi_n} |\mu(\hat{0},\pi)|\, t^{\mathrm{blocks}(\pi)}$$ $\rightsquigarrow$ all posets P

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{cyc}(\sigma)}$$ $\rightsquigarrow$ all posets P

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{LRmax}(\sigma)}$$ $\rightsquigarrow$ all posets P

$$= t(t+1)(t+2)\cdots(t+(n-1))$$ $\rightsquigarrow$ disjoint unions of chains

Whitney numbers for cones

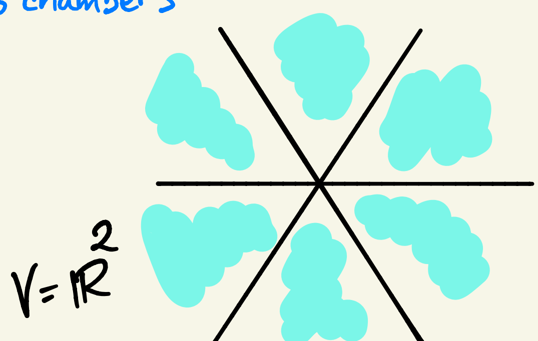
THEOREM (Winder 1966, Zaslavsky 1975)

For an arrangement $\mathcal{A}$ of hyperplanes in $V = \mathbb{R}^d$,

$$\#\text{ chambers of } \mathcal{A} = \sum_{X \in \mathcal{L}(\mathcal{A})}' |\mu(V, X)|$$

lattice of intersection subspaces $X = H_1 \cap \ldots \cap H_m$ ordered via (reverse) inclusion

6 chambers



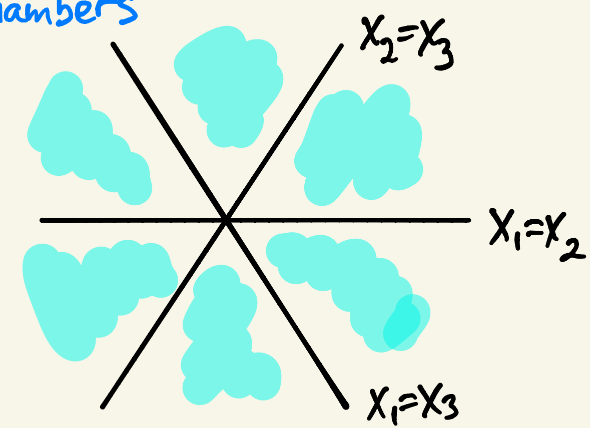
THEOREM (Winder 1966, Zaslavsky 1975)

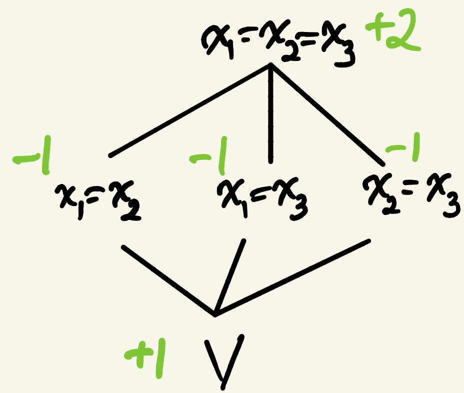
For an arrangement $\mathcal{A}$ of hyperplanes in $V=\mathbb{R}^d$,

$$\#\text{ chambers of } \mathcal{A} = \sum_{X \in \mathcal{L}(\mathcal{A})} |\mu(V,X)| = \sum_{k=0}^{d} c_k \quad \text{where } c_k = \sum_{\substack{X \in \mathcal{L}(\mathcal{A}):\\ \operatorname{codim}(X)=k}} |\mu(V,X)|$$

Whitney numbers of 1st kind

6 chambers



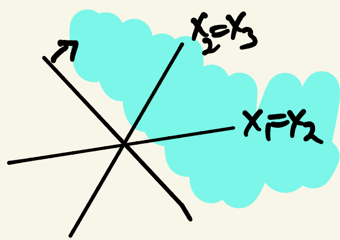


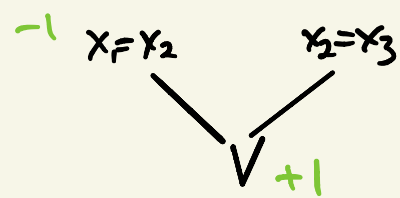
$$\begin{array}{r} 2 \\ +\ 3 \\ +\ 1 \\ \hline 6 \end{array}$$

THEOREM (Zaslavsky 1977) Essentially the same works for counting chambers inside cones K within $\mathcal{A}$:

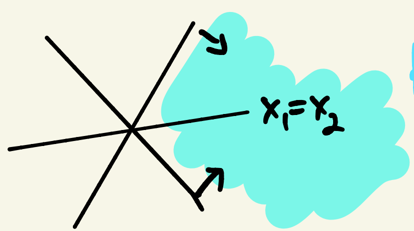
$$\#\text{chambers of } \mathcal{A} \text{ inside cone } K = \sum_{X \in \mathcal{L}^{int}(K)} |\mu(V,X)| = \sum_{k=0}^{d} c_k(K)$$

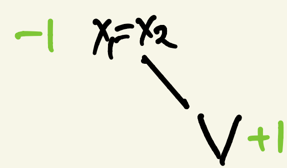
intersection subspaces X with $X \cap K \neq \phi$





$$\begin{array}{r} 2 \\ +1 \\ \hline 3 \end{array}$$ chambers in K





$$\begin{array}{r} 1 \\ +1 \\ \hline 2 \end{array}$$ chambers in K

Poset cones and 4 formulas

When the arrangement $\mathcal{A}$ is the type A_{n-1} reflection arrangement or braid arrangement

Permutations $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$ in $\mathfrak{S}_n$ $\longleftrightarrow$ chambers $x_{\sigma(1)} < x_{\sigma(2)} < \cdots < x_{\sigma(n)}$ in $\mathcal{A}$

Posets P on $\{1, 2, \ldots, n\}$ $\longleftrightarrow$ cones $K_P = \bigcap_{i <_P j} \{x_i < x_j\}$

$\mathrm{LinExt}(P)$ = linear extensions of P $\longleftrightarrow$ chambers of $\mathcal{A}$ inside the cone K_P

Intersection lattice $L(\mathcal{A})$ $\longleftrightarrow$ set partition lattice Π_n

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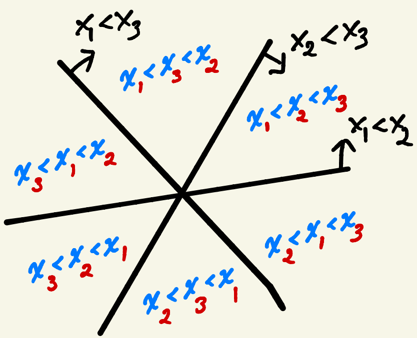
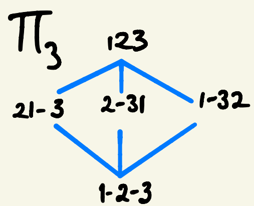
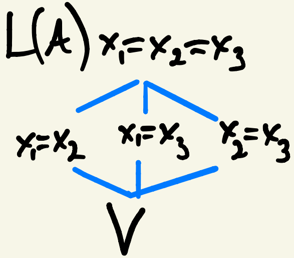
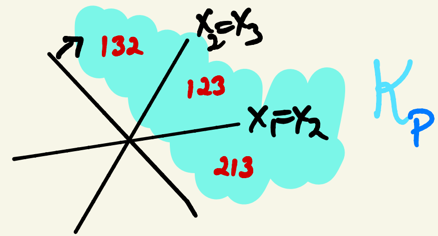
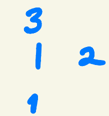


chambers $\leftrightarrow \mathfrak{S}_3$

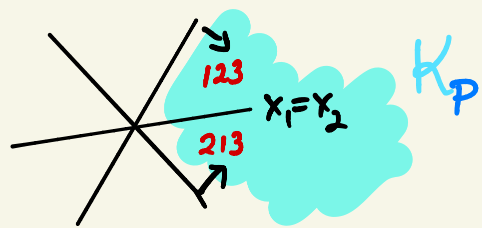


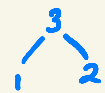




$P =$ 

$\mathrm{LinExt}(P) = \{123, 213, 132\}$



$P =$ 

$\mathrm{LinExt}(P) = \{123, 213\}$

UPSHOT: For a poset P on $\{1,2,\ldots,n\}$, if we define $$c_k(P) := \sum_{\substack{X \in \mathcal{L}^{int}(K_P):\\ \operatorname{codim} X = k}} |\mu(V,X)|$$ for $k=0,1,\ldots,n-1$

Whitney numbers of the 1st kind for P

then Zaslavsky's Theorem implies

$$\#\mathrm{LinExt}(P) = c_0(P) + c_1(P) + \cdots + c_{n-1}(P)$$

(= $e(P)$ in Stanley's talk)

depend on P only up to isomorphism, not labeling

So what about the 4 formulas, corresponding to the antichain poset $P = \underset{1}{\bullet} \quad \underset{2}{\bullet} \quad \cdots \quad \underset{n}{\bullet}$?

$$\sum_{k=1}^{n} c(n,k)\, t^{k} = \sum_{\pi \in \Pi_n} |\mu(\hat{0},\pi)|\, t^{\mathrm{blocks}(\pi)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{cyc}(\sigma)}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{LRmax}(\sigma)}$$

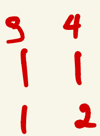
$$= t(t+1)(t+2)\cdots(t+(n-1))$$

$$\sum_{k=1}^{n} c(n,k)\, t^k = \sum_{\pi \in \Pi_n} |\mu(\hat{0},\pi)|\, t^{\mathrm{blocks}(\pi)} \rightsquigarrow$$

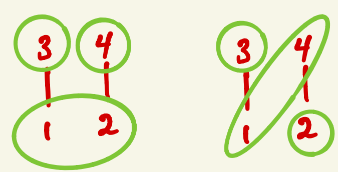
PROPOSITION: (Dorpalen-Barry, Kim, R. 2019) For any poset P on $\{1,2,\ldots,n\}$,

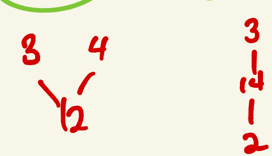
$$\sum_{k=0}^{n-1} c_k(P)\, t^k = \sum_{\substack{P\text{-transverse} \\ \text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{n-\mathrm{blocks}(\pi)}$$

The quotient preposet P/π never has $i \equiv j$ for $i \lneq_P j$

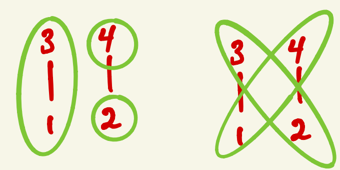
$P=$ 

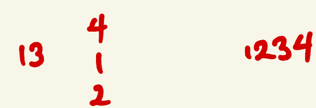
P-transverse π



P/π 

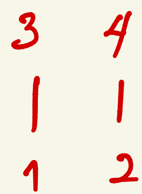
not P-transverse π





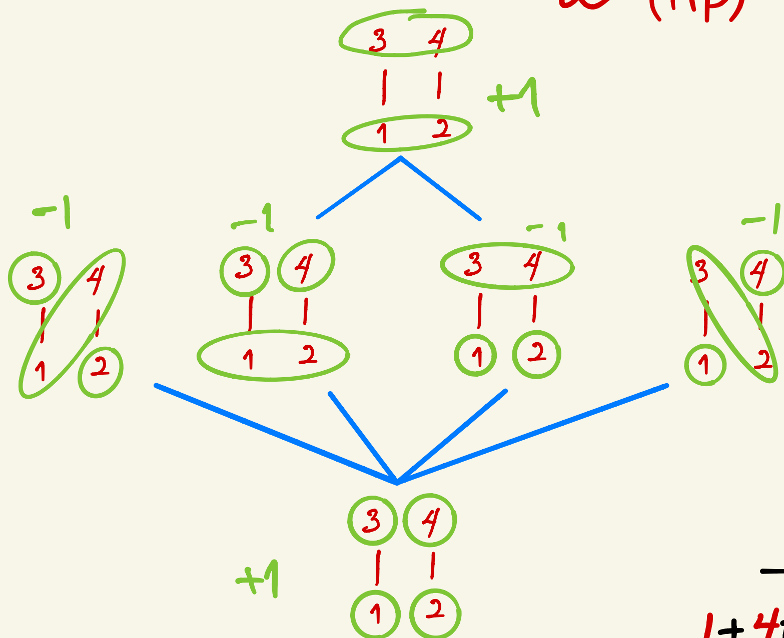
$$\sum_{k=0}^{n-1} c_k(P)\, t^k = \sum_{\substack{P\text{-transverse} \\ \text{set partitions} \\ \pi \in \Pi_n}} |\mu(\hat{0},\pi)|\, t^{n-\text{blocks}(\pi)}$$

$$\mathcal{L}^{int}(K_P) = \{P\text{-transverse partitions } \pi\}$$

$P=$ 

$\mathrm{LinExt}(P) = \{1234, 1243, 1324, 2134, 2143, 2413\}$

$\#\mathrm{LinExt}(P) = 6$



$1\cdot t^2$

$+\, 4\cdot t^1$

$+\, 1\cdot t^0$

$$1 + 4t + 1\cdot t^2 \xrightarrow{t=1} 6$$

$$\sum_{k=1}^{n} c(n,k)\, t^{k} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{cyc}(\sigma)} \rightsquigarrow$$

PROPOSITION (Dorpalen-Barry, Kim, R. 2019) For any poset P on $\{1,2,\ldots,n\}$,

$$\sum_{k=0}^{n-1} c_k(P)\, t^{k} = \sum_{\substack{P\text{-transverse}\\ \text{permutations}\\ \sigma \in \mathfrak{S}_n}} t^{\,n-\mathrm{cyc}(\pi)}$$

its cycle partition is a P-transverse set partition

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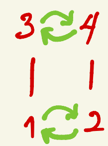
proof is immediate from $\left|\mu(\hat{0},\pi)\right| = \prod_i (\#B_i - 1)! = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has cycle partition } \pi\}$

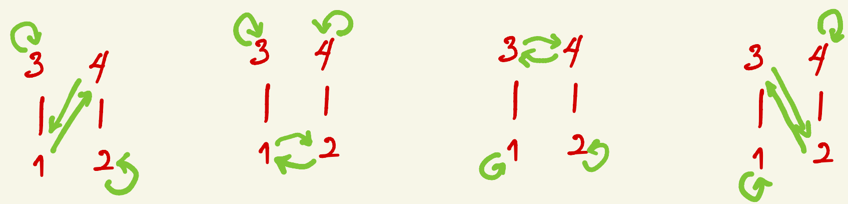
if $\pi = \{B_1, B_2, \ldots\}$

$$\sum_{k=0}^{n-1} c_k(P)\, t^k = \sum_{\substack{P\text{-transverse} \\ \text{permutations} \\ \sigma \in \mathfrak{S}_n}} t^{\,n-\mathrm{cyc}(\pi)}$$

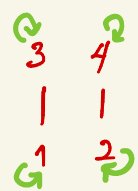
sadly, not LinExt(P)



$1 \cdot t^2$



$+\, 4 \cdot t^1$



$+\, 1 \cdot t^0$

$$\sum_{k=1}^{n} c(n,k)\, t^k = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{LRmax}(\sigma)}$$

THEOREM (Dorpalen-Barry, Kim, R. 2019) There is a bijection generalizing Foata's first fundamental transformation

$$\left\{ \begin{array}{c} P\text{-transverse} \\ \text{permutations} \\ \sigma \in \mathfrak{S}_n \end{array} \right\} \longleftrightarrow \mathrm{LinExt}(P)$$

$$\sigma \longmapsto \hat{\sigma}$$

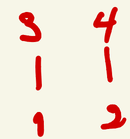
with $\mathrm{cyc}(\sigma) = \mathrm{LRmax}_P(\hat{\sigma})$

generalizes $\mathrm{LRmax}(-)$

and hence $\displaystyle\sum_{k=0}^{n-1} c_k(P)\, t^k = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{\mathrm{LRmax}_P(\sigma)}$

$$\sum_{k=1}^{n} c(n,k)\,t^k = t(t+1)(t+2)\cdots(t+(n-1))$$ generalizes so far only to

$P_{\underline{a}} = P_{(a_1,a_2,\ldots,a_\ell)}$ = disjoint unions of chains, of sizes $a_1, a_2, \ldots, a_\ell$

e.g. $P_{(2,2)} =$ 

THEOREM
(Dorpalen-Barry, Kim, R. 2019)

$$\sum_{\underline{a} \in \mathbb{N}^\ell} x_1^{a_1} \cdots x_\ell^{a_\ell} \sum_k c_k(P_{\underline{a}})\,t^k$$

$$= \frac{1}{1 - \sum_{j=1}^{\ell} e_j(x_1, x_2, \ldots, x_\ell)\,(t-1)(2t-1)\cdots((j-1)t-1)}$$

elementary symmetric function

Extracting coefficient of $x_1^1 x_2^1 \cdots x_n^1$ gives the factored formula for $P_{(1,1,\ldots,1)}$ = antichain

The proof uses Foata's 1965 thesis, where he defined the **interpolation product** of **multiset permutations**, and proved they have a **unique factorization** into prime cycles (up to commutation)

---

e.g.

$$\sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \top \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{pmatrix}$$

a multiset permutation of 1122233444

$= 1^{a_1} 2^{a_2} 3^{a_3} 4^{a_4}$

where $\underline{a} = (2,3,2,3)$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 4 & 4 \\ 4 & 1 & 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \top \begin{pmatrix} 4 \\ 4 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix}$$

commute

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \top \begin{pmatrix} 4 \\ 4 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \top \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix}$$

Say **pcyc(σ) = 4** because it has 4 prime cycles

One can identify a multiset permutation σ of $1^{a_1}2^{a_2}\cdots \ell^{a_\ell}$ with a linear extension σ of $P_{(a_1,a_2,\ldots,a_\ell)}$

e.g. $P_{(2,3,2,3)} = \begin{array}{cccc} & 5 & & 10 \\ & | & & | \\ 2 & 4 & 7 & 9 \\ | & | & | & | \\ 1 & 3 & 6 & 8 \end{array} \quad \overset{\text{relabel}}{\longleftrightarrow} \quad \begin{array}{cccc} & 2 & & 4 \\ & | & & | \\ 1 & 2 & 3 & 4 \\ | & | & | & | \\ 1 & 2 & 3 & 4 \end{array}$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 8 & 9 & 6 & 1 & 4 & 2 & 7 & 10 & 5 \end{pmatrix} \quad \underset{\text{relabel}}{\longleftrightarrow} \quad \sigma = \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}$$

THEOREM (Dorpalen-Barry, Kim, R. 2019) For any disjoint union of chains $P_{\underline{a}}$, Foata's prime cycle decomposition gives a bijection

$$\begin{array}{ccc} \mathrm{LinExt}(P_{\underline{a}}) & \longleftrightarrow & \left\{ \begin{array}{c} P_{\underline{a}}\text{-transverse} \\ \text{permutations} \end{array} \right\} \\ \| & & \\ \text{multiset permutations} & & \\ \text{of } 1^{a_1}2^{a_2}\cdots \ell^{a_\ell} & & \end{array}$$

$$\sigma \longmapsto \hat{\sigma}$$

with $\mathrm{pcyc}(\sigma) = \mathrm{cyc}(\hat{\sigma})$

Consequently,

$$\sum_k c_k(P_{\underline{a}})\, t^k = \sum_{\substack{\text{multiset permutations} \\ \sigma \text{ of } 1^{a_1}2^{a_2}\cdots \ell^{a_\ell}}} t^{\,n-\mathrm{pcyc}(\sigma)}$$

where $n = a_1 + a_2 + \cdots + a_\ell$

$$P_{(2,2)} = \begin{matrix} 2 & 4 \\ | & | \\ 1 & 3 \end{matrix} \longleftrightarrow \begin{matrix} 1 & 2 \\ | & | \\ 1 & 2 \end{matrix}$$

pcyc(σ)

$$\begin{pmatrix} 1&2&3&4 \\ 1&2&3&4 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 1&1&2&2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad 4 \qquad \Big\} \; c_0 = 1$$

$$\begin{pmatrix} 1&2&3&4 \\ 1&3&2&4 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 1&2&1&2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad 3$$

$$\begin{pmatrix} 1&2&3&4 \\ 1&3&4&2 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 1&2&2&1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \qquad 3$$

$$\begin{pmatrix} 1&2&3&4 \\ 3&1&2&4 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 2&1&1&2 \end{pmatrix} = \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \top \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad 3$$

$$\begin{pmatrix} 1&2&3&4 \\ 3&1&4&2 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 2&1&2&1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \top \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \top \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad 3$$

$$c_1 = 4$$

$$\begin{pmatrix} 1&2&3&4 \\ 3&4&1&2 \end{pmatrix} \quad \begin{pmatrix} 1&1&2&2 \\ 2&2&1&1 \end{pmatrix} = \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \top \begin{pmatrix} 1&2 \\ 2&1 \end{pmatrix} \qquad 2 \qquad \Big\} \; c_2 = 1$$

QUESTION: Is there an intercalation product and factorization theory for all posets P giving a statistic $\mathrm{pcyc}(-)$ on $\mathrm{LinExt}(P)$ with

$$\sum_k c_k(P)\, t^k = \sum_{\sigma \in \mathrm{LinExt}(P)} t^{\,n-\mathrm{pcyc}(\sigma)} \quad ?$$

Thanks for your attention;
and...