Cyclic symmetry? You can count on it!

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Fisk Lecture Series
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1. Some famous counts
   - binomial
   - Catalan

2. (Less?) famous q-counts
   - q-binomial
   - q-Catalan

3. A cyclic symmetry phenomenon
   (joint w/ Dennis Stanton, Dennis White)
   with examples

4. Proof philosophies

1. Well-known counts

Binomial coefficients

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

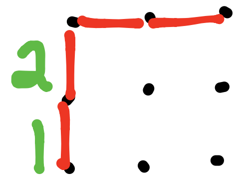
$$= \frac{n(n-1)\cdots(n-k+2)(n-k+1)}{k(k-1)\cdots 2 \cdot 1}$$

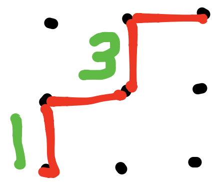
count

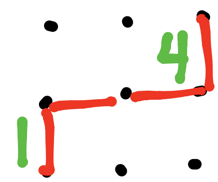
k-element subsets of $\{1,2,\ldots,n\}$

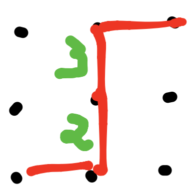
or

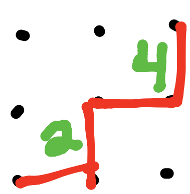
paths from $(0,0) \to (n-k,k)$ taking unit north or east steps e.g. $\binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3}{2 \cdot 1} = 6$

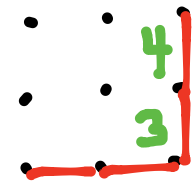
$\{1,2\}$ 

$\{1,3\}$ 

$\{1,4\}$ 

$\{2,3\}$ 

$\{2,4\}$ 

$\{3,4\}$ 

The Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{1}{2n+1}\binom{2n+1}{n}$$

$$= \frac{(2n)(2n-1)\cdots(n+3)(n+2)}{n\,(n-1)\cdots\ 3\cdot 2}$$

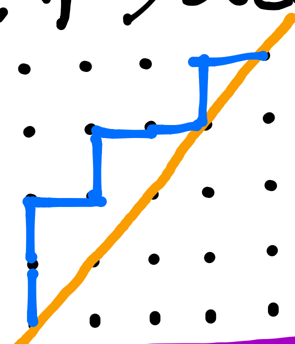
is not even obviously an integer!

But it is, and counts many things

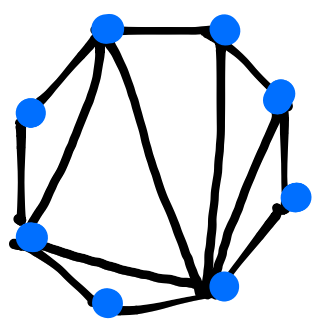
– see recent book by Stanley

Two classic interpretations:
(Euler, Segner, Goldbach 1750's)

- C_n counts lattice paths $(0,0) \to (n,n)$ weakly above $y=x$

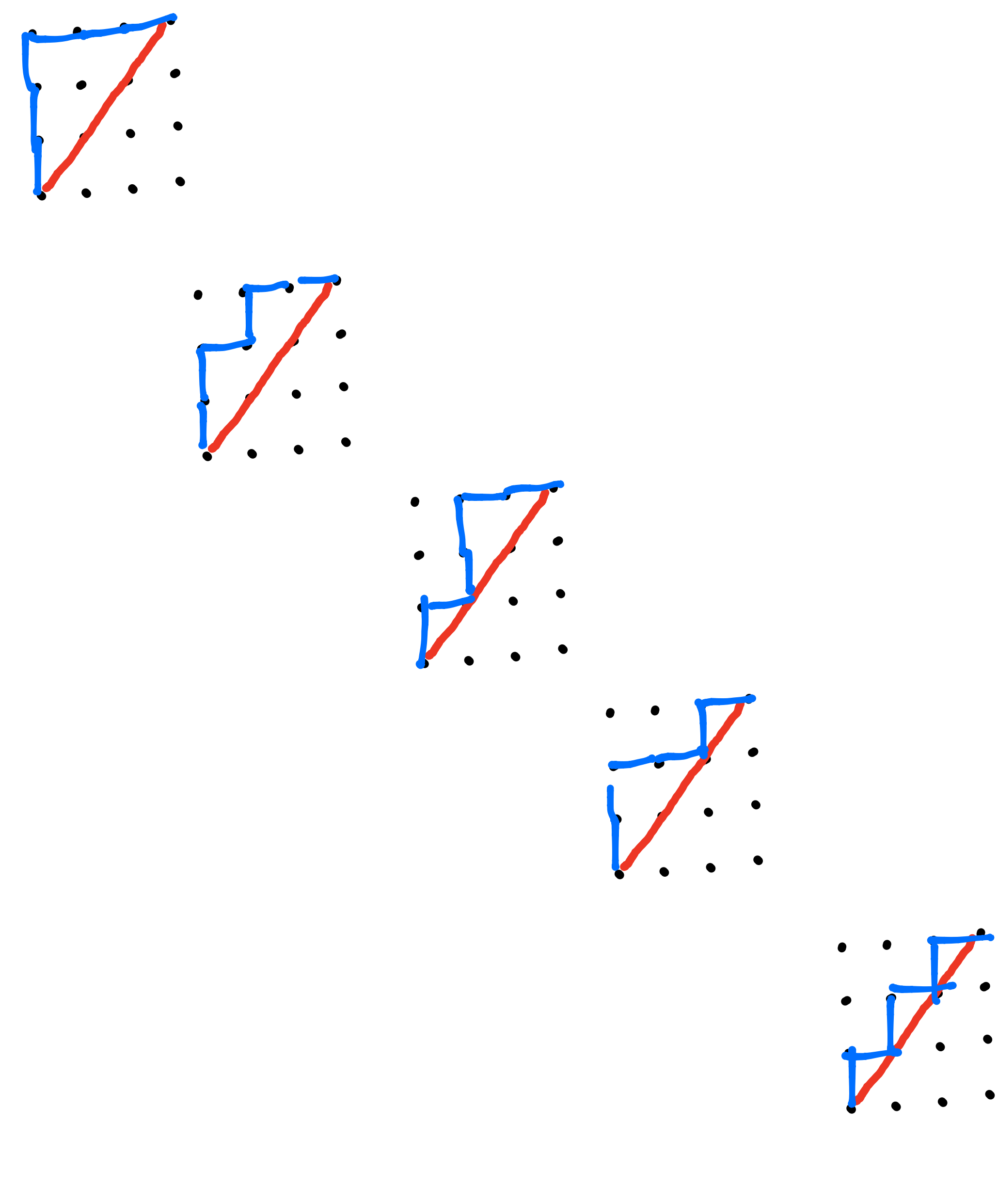


- C_n counts triangulations of an $(n+2)$-sided polygon



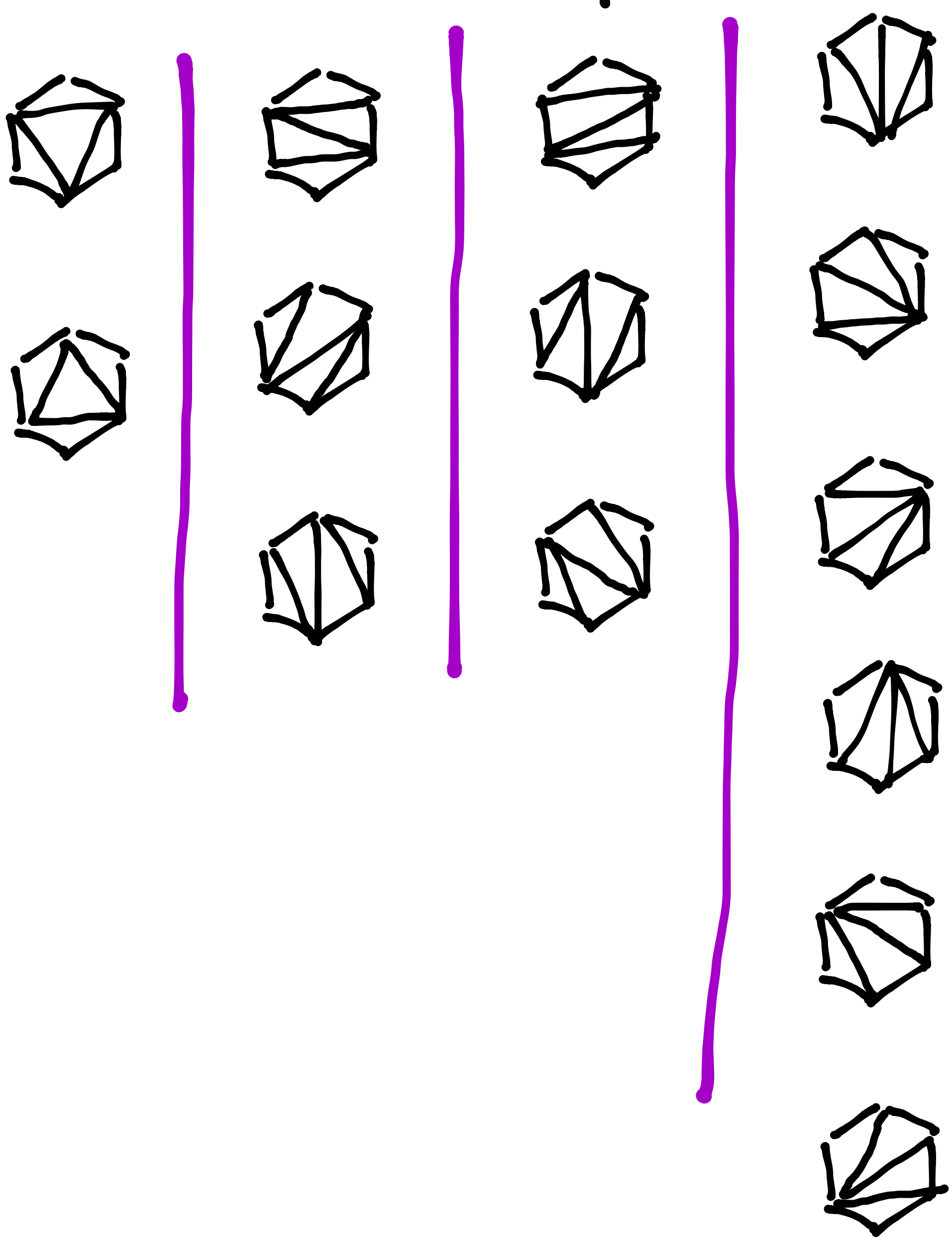
e.g. $n=3$

$$C_3 = \frac{1}{4}\binom{6}{3} = \frac{1}{5}\binom{5}{2} = \frac{6\cdot 5}{3\cdot 2} = 5$$



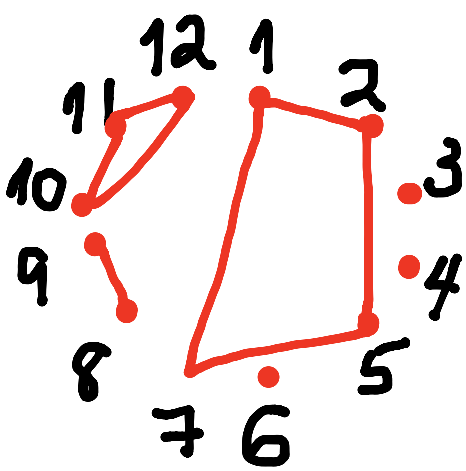
e.g. $n=4$

$$C_4 = \frac{1}{5}\binom{8}{4} = \frac{1}{9}\binom{9}{4} = \frac{8\cdot 7\cdot 6}{4\cdot 3\cdot 2} = 14$$



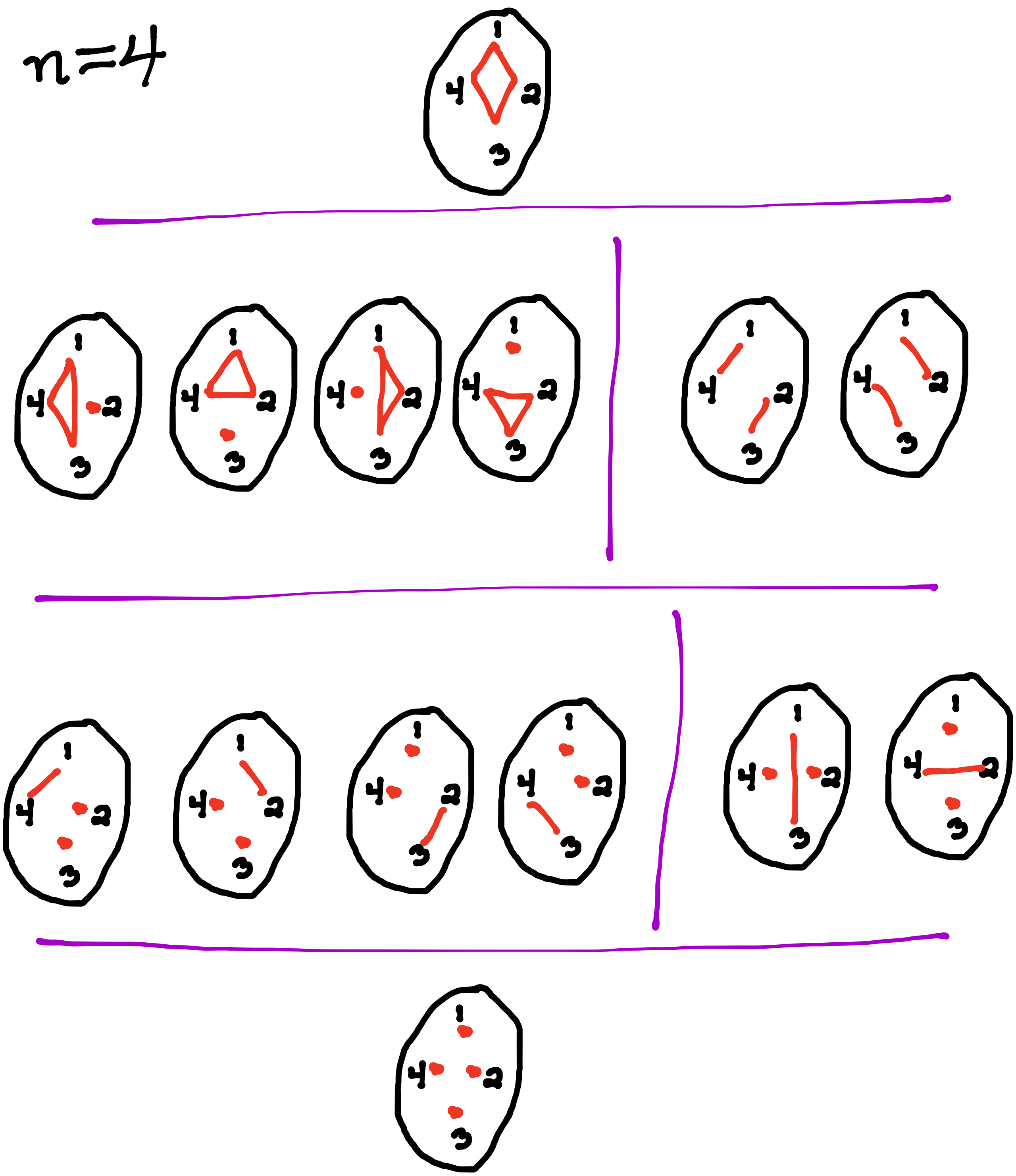
Less classical... (Becker 1948, Motzkin 1948)

THEOREM (Kreweras 1972)
C_n counts partitions of the set $\{1,2,3,\ldots,n\}$ written in cyclic order around a circle into blocks whose convex hulls are noncrossing.



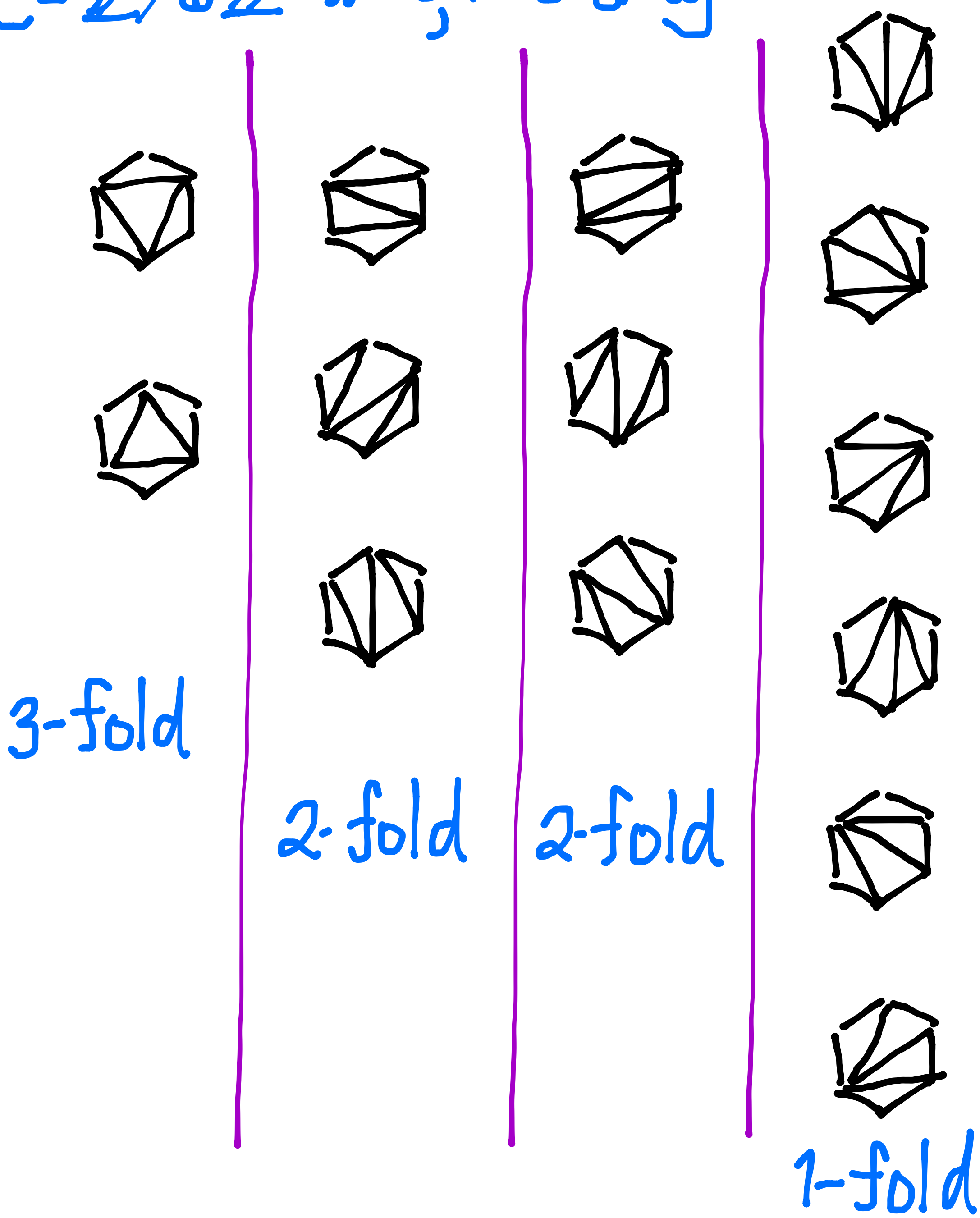
blocks:
$\{1,2,5,7\}$,
$\{3\}$, $\{4\}$, $\{6\}$
$\{8,9\}$, $\{10,11,12\}$

$n=4$

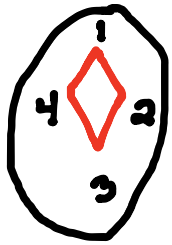


Note cyclic action & symmetry:

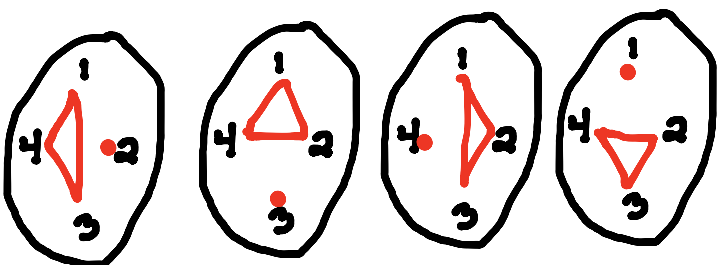
$C = \mathbb{Z}/6\mathbb{Z}$ acts, rotating



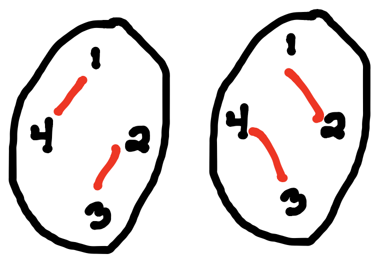
$C = \mathbb{Z}/4\mathbb{Z}$ acts, rotating



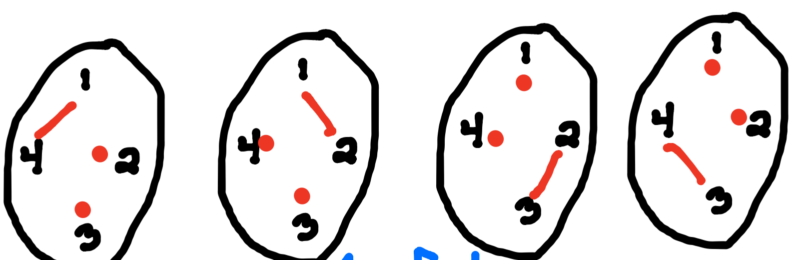
4-fold



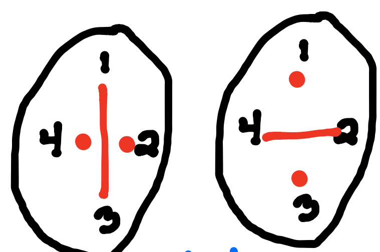
1-fold



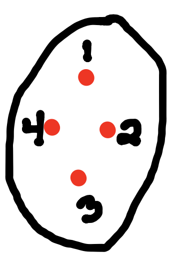
2-fold



1-fold



2-fold



4-fold

2. q-counts

q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q\,[n-k]!_q}$$

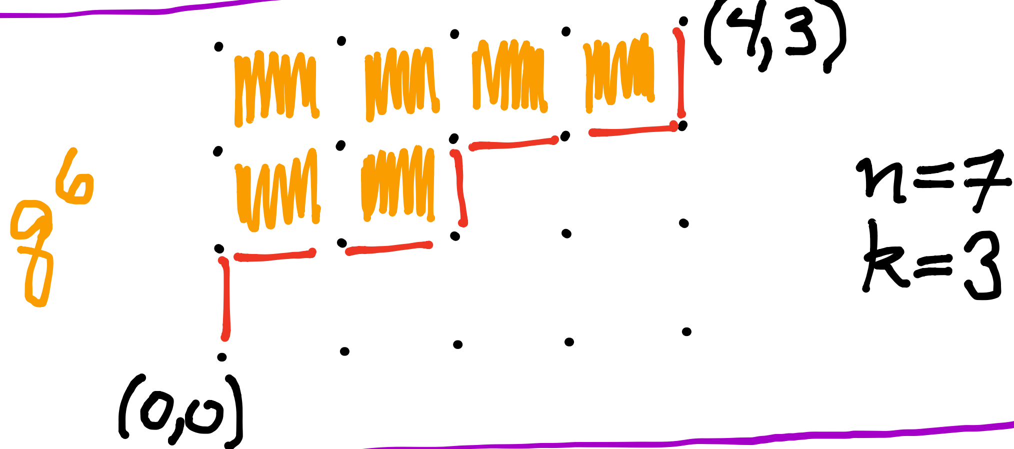
where $[n]_q := 1+q+q^2+\cdots+q^{n-1} = \dfrac{q^n-1}{q-1} \overset{q=1}{\rightsquigarrow} n$

and $[n]!_q := [n]_q [n-1]_q \cdots [2]_q [1]_q \overset{q=1}{\rightsquigarrow} n!$

so $\begin{bmatrix} n \\ k \end{bmatrix}_q \overset{q=1}{\rightsquigarrow} \binom{n}{k}$

THEOREM (Gauss)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\substack{\text{lattice} \\ \text{paths } (0,0)\to(n-k,k)}} q^{\text{area above path}}$$

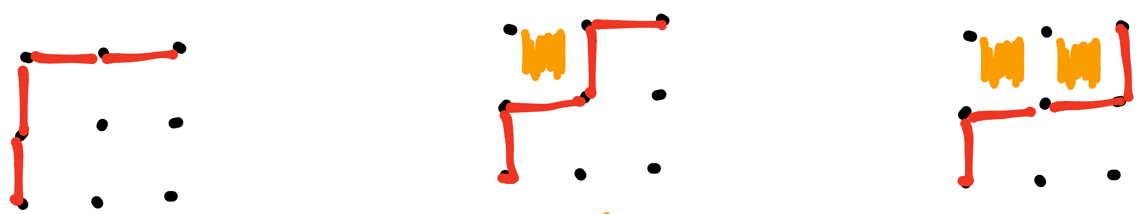


In particular, $\begin{bmatrix} n \\ k \end{bmatrix}_q \in \mathbb{N}[q]$;

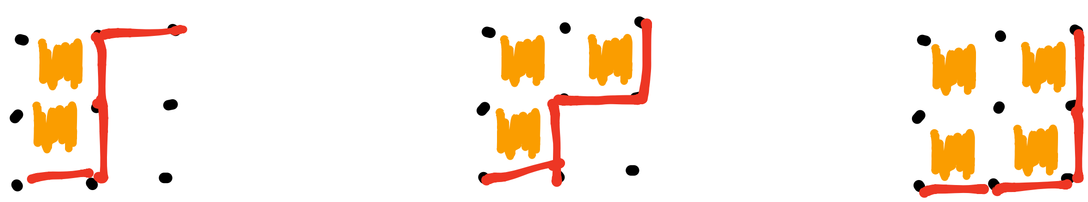
it is polynomial in q, with nonnegative integer coefficients.

$$\begin{bmatrix}4\\2\end{bmatrix}_q = \frac{[4]!_q}{[2]!_q[2]!_q} = \frac{[4]_q[3]_q}{[2]_q[1]_q}$$

$$= \frac{(q^4-1)(q^3-1)}{(q^2-1)(q-1)} = 1+q+2q^2+q^3+q^4$$



$q^0 \quad + \quad q^1 \quad + \quad q^2$



$+\, q^2 \quad + \quad q^3 \quad + \quad q^4$

THEOREM (Schubert 1889)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \#\{k\text{-dimensional } \mathbb{F}_q\text{-subspaces of } \mathbb{F}_q^n\}$$

$$= \# Gr(k, \mathbb{F}_q^n)$$

finite Grassmannian

when $\mathbb{F}_q$ is a finite field with q elements (so $q = p^e$ for some prime p)

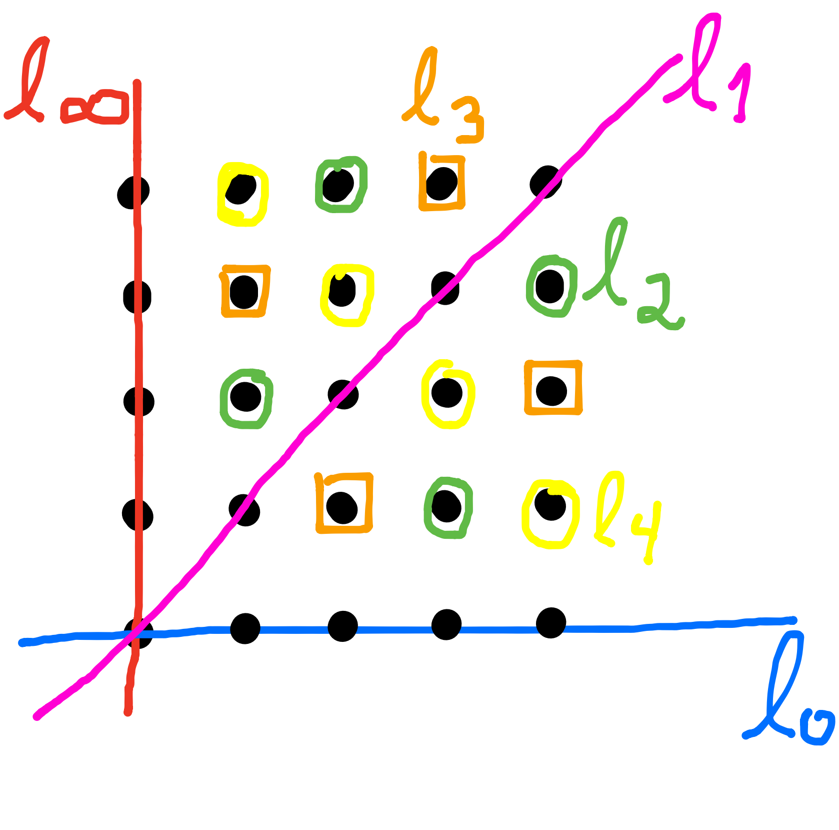
$k=1$:

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{[n]_q}{[1]_q} = 1+q+q^2+\dots+q^{n-1}$$

$$= \#\text{lines in } \mathbb{F}_q^n$$

$$\Big\downarrow {\scriptstyle n=2}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q = 1+q \overset{q=5}{\rightsquigarrow} \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q=5} = 6$$



q-Catalan

$$C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$$

$$= \frac{1}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q$$

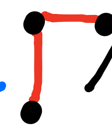
$$= \frac{[2n]_q [2n-1]_q \cdots\cdots [n+2]_q}{[n]_q [n-1]_q \cdots\cdots [2]_q}$$

$$C_3(q) = \frac{1}{[4]_q} \begin{bmatrix} 6 \\ 3 \end{bmatrix}_q = \frac{[6]_q [5]_q}{[3]_q [2]_q}$$

$$= 1 + q^2 + q^3 + q^4 + q^6$$

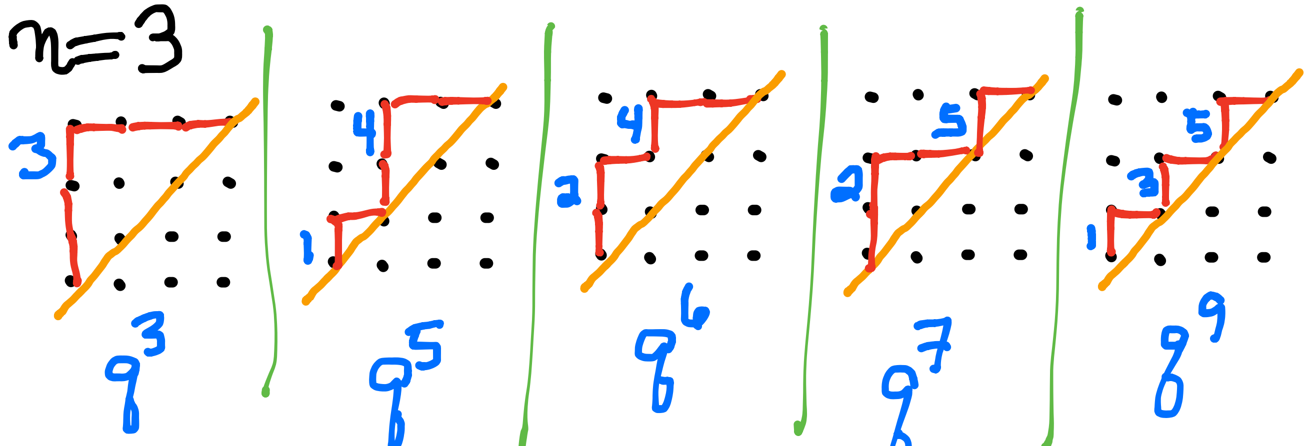
THEOREM (MacMahon 1915)

$$C_n(q) = q^{-n} \sum_{\substack{\text{paths} \\ (0,0)\to(n,n) \\ \text{above } y=x}} q^{\left(\substack{\text{sum of indices} \\ \text{on steps } i}\right)}$$



In particular, $C_n(q) \in \mathbb{N}[q]$

$n=3$



$$C_3(q) = q^{-3}(q^3+q^5+q^6+q^7+q^9)$$
$$= 1+q^2+q^3+q^4+q^6$$

$C_n(q)$ also has an $\mathbb{F}_q$ interpretation:

PROPOSITION:
$$C_n(q) = \frac{1}{[2n+1]_q} \begin{bmatrix} 2n+1 \\ n \end{bmatrix}_q$$

$= \# \, \mathbb{F}_{q^{2n+1}}^{\times}$ - orbits of n-dimensional $\mathbb{F}_q$ subspaces of $\mathbb{F}_{q^{2n+1}}$

$$= \# \, \mathbb{F}_{q^{2n+1}}^{\times} \backslash Gr_{\mathbb{F}_q}(n, \mathbb{F}_{q^{2n+1}})$$

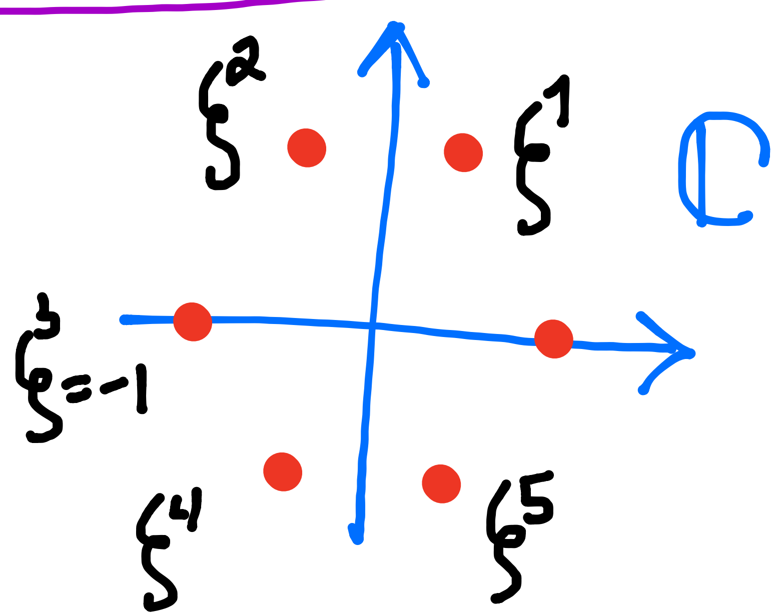
3. A cyclic symmetry phenomenon

We've seen q as a

- a polynomial variable
- a prime power $q = \#\mathbb{F}_q$

Now let's see it as a complex n^{th} root-of-unity

$n=6$



DEFINITION (Stanton-White-R.) 2004

A cyclic group $C = \mathbb{Z}/n\mathbb{Z}$ permuting a finite set X, and a polynomial $X(q)$ in q exhibit the cyclic sieving phenomenon (CSP) if every $c \in C$ of order d fixes exactly $[X(q)]_{q=\zeta}$ elements of X, where ζ is a primitive d^{th} root of unity.

EXAMPLE (The 1st one):

THEOREM:

Letting $C = \mathbb{Z}/n\mathbb{Z}$ rotate

$$\left\{ \begin{array}{c} k\text{-subsets of} \\ \{1,2,\ldots,n\} \end{array} \right\} =: X$$

modulo n (i.e. on a clock face),

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

exhibits the CSP

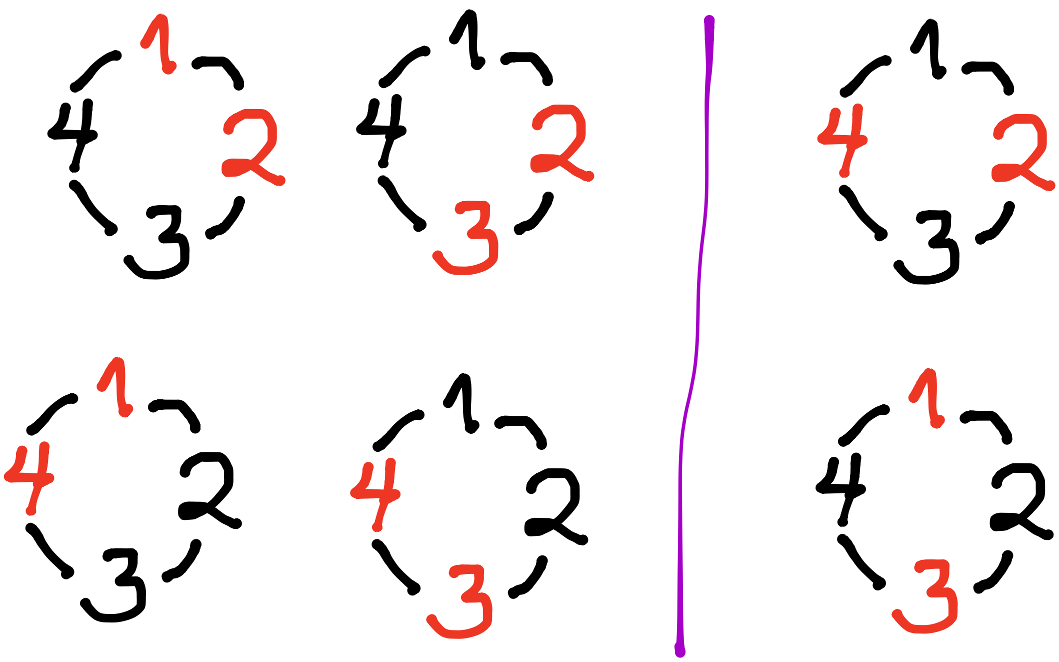
$n=4$
$k=2$

$$\begin{bmatrix}4\\2\end{bmatrix}_q = 1+q+2q^2+q^3+q^4$$

$q=1$: 6 in total

$q=-1$: $1-1+2-1+1 = 2$ have 2-fold symmetry

$q=\pm i$: $1 \pm i - 2 \mp i + 1 = 0$ have 4-fold



THEOREMS:

$X(q) =$ q-Catalan $\frac{1}{[n+1]} \begin{bmatrix} 2n \\ n \end{bmatrix}_q$

exhibits two CSP's:

- $C = \mathbb{Z}/(n+2)\mathbb{Z}$ rotating $\{\text{triangulations of } (n+2)\text{-gon}\} = X$

- $C = \mathbb{Z}/n\mathbb{Z}$ rotating $\{\text{noncrossing partitions of } \{1,2,\ldots,n\}\} = X$

$n=4$:

$$C_4(q)=\frac{1}{[5]_q}\begin{bmatrix}8\\4\end{bmatrix}_q$$

$$=\frac{[8]_q[7]_q[6]_q}{[4]_q[3]_q[2]_q}$$

$$=1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

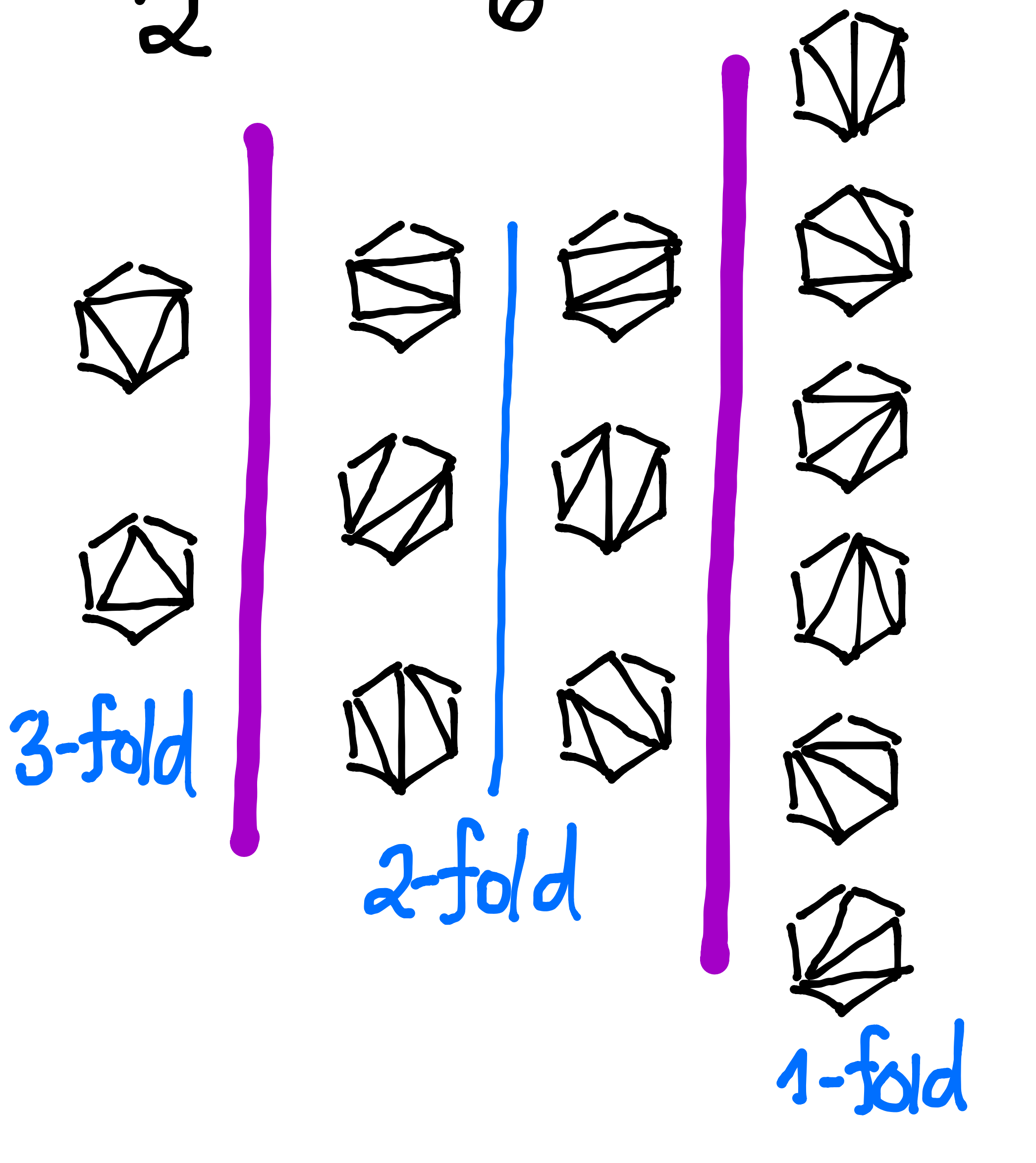
$$1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

$q=e^{\frac{2\pi i}{6}}$: 0

$q=e^{\frac{2\pi i}{3}}$: 2

$q=-1$: 6

$q=1$: 14

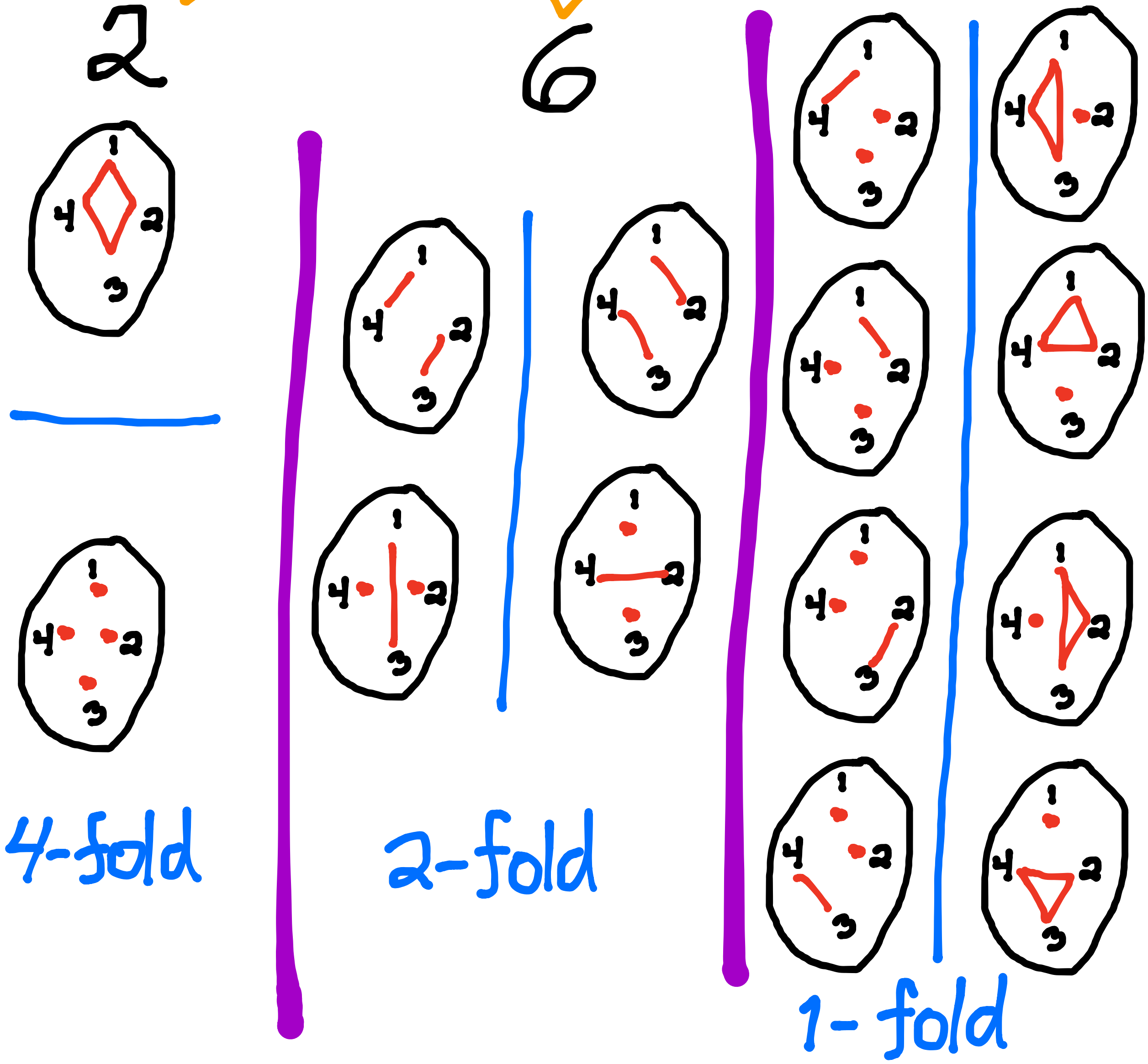


$$1+q^2+q^3+2q^4+q^5+2q^6+q^7+2q^8+q^9+q^{10}+q^{12}$$

$q=\pm i$ → 2

$q=-1$ → 6

$q=1$ → 14



Both q-Catalan CSP's generalize to finite reflection groups

- triangulations $\rightsquigarrow$ clusters of finite type

- noncrossing partitions $\rightsquigarrow$ elements on a geodesic between e = identity, c = Coxeter element in Cayley graph using reflections

4. Proof philosophies

- Brute force method —
  - count elements of X with d-fold symmetry,
  - evaluate $[X(q)]_{q=\zeta}$,
  - compare.

Not my favorite, but the only proof we know for the triangulation example!

- Algebraic method —

Find vector space V with C-action and two bases indexed by X:

- $\{v_x\}_{x\in X}$ permuted $c(v_x)=v_{c(x)}$
- $\{u_x\}_{x\in X}$ scaled $c(u_x)=\zeta^{s(x)}u_x$

with $\sum_{x\in X} q^{s(x)} = X(q)$

The equality in the CSP is the equality of trace of c in the two bases.

This algebraic approach works for

- q-binomial and subsets
- q-Catalan and non-crossing partitions (but not yet for reflection groups)
- tableaux and promotion (Rhoades)
- Brauer diagrams & rotation (Rubey & Westbury)

... but I'd like to see more!

THANKS

for your

attention!