UNIVERSITY OF MINNESOTA

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GRADUATE SCHOOL

EQUIVALENCE CLASSES OF REDUCED WORDS

A THESIS
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CONTENTS

I. Definitions and Introduction.

A *permutation* of size n is a linear ordering of the set $[n] = \{1, 2, 3, \ldots n\}$, and let S_n be the *symmetric group* of all permutations of size n. It is well-known and easy to see that S_n forms a group which is generated by the *adjacent transpositions* $s_i = (i, i+1)$, where s_i swaps the numbers in positions $i, i+1$ of the permutation. A *reduced word* for a permutation w is a sequence $\underline{a} = i_1 i_2 \cdots i_l$ of minimal length l such that $w = s_{i_1} s_{i_2} \cdots s_{i_l}$. The set of all reduced words for w will be denoted $Red(w)$.

Example. Let $w = 4312$ in S_4. Then one example of a reduced word for w is $\underline{a} = 23212$, corresponding to the application of the following sequence of adjacent transpositions in going from the identity permutation 1234 to w:

$$\begin{array}{rl} & 1234 \\ s_2 \,(& \\ & 1324 \\ s_3 \,(& \\ & 1342 \\ s_2 \,(& \\ & 1432 \\ s_1 \,(& \\ & 4132 \\ s_2 \,(& \\ & 4312 \end{array}$$

In this case the set of all reduced words $Red(w)$ is

$$\{23121, 21321, 23212, 32312, 32132\}.$$

Notice that in the symmetric group S_n, one has the following *braid relations* on the generators s_i:

$$\begin{array}{rcll} s_i\, s_j & = & s_j\, s_i & \text{if } |i-j| \geq 2 \\ s_i\, s_{i+1}\, s_i & = & s_{i+1}\, s_i\, s_{i+1} & \end{array}$$

and therefore two reduced words $\underline{a}, \underline{a}'$ will correspond to the same permutation w if they differ by either of the following relations, which we call *elementary* C_1*- and* C_2*-equivalences*, respectively:

$$\begin{array}{rcll} \cdots ij \cdots & \overset{C_1}{\sim} & \cdots ji \cdots & \text{if } |i-j| \geq 2 \\ \cdots i\; i+1\; i \cdots & \overset{C_2}{\sim} & \cdots i+1\; i\; i+1 \cdots & \end{array}$$

A well-known theorem of J. Tits (see e.g. [Br]) gives a converse: any two reduced words for w are connected by a sequence of elementary C_1- and C_2-equivalences. We will say two reduced words $\underline{a}, \underline{a}'$ for w are C_1*-equivalent* if they are connected by a sequence of elementary C_1-equivalences, and C_2*-equivalence* is defined similarly.

Example. For $w = 4312$ as before, the C_1-equivalence classes of reduced words are

$$\{23121, 21321\}, \{23212\}, \{32312, 32132\}$$

and the C_2-equivalence classes of reduced words are

$$\{23121, 23212, 32312\}, \{21321\}, \{32132\}.$$

It will sometimes be convenient to draw a picture of a reduced word $\underline{a}$ for a permutation in S_n, called its *braid picture*. This picture contains n "strands" which flow across the page from left to right, with the i^{th} and $(i+1)^{st}$ strand crossing in the same order as s_i occur in the reduced word. For example, the braid picture for $\underline{a} = 23212$ from before is shown in Figure 1.

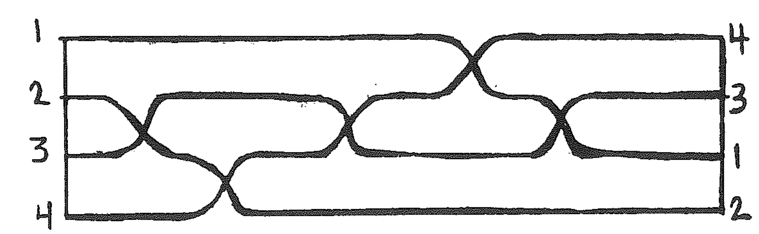


FIGURE 1. Braid picture for a reduced word

Given a reduced word $\underline{a}$, let $C_1(\underline{a}), C_2(\underline{a})$ denote its C_1-, C_2-equivalence classes respectively. The cardinality and structure of the set $Red(w)$ has been extensively studied in recent years (see e.g. [EG]). Even more recently, C_1-equivalence has received some attention (see e.g. [El]). This thesis concentrates on C_2-equivalence. In the next section, we discuss the main result (Theorem 3) which gives an encoding of the C_2-equivalence class $C_2(\underline{a})$ of a reduced word $\underline{a}$. This encoding shows that even though a $C_2(\underline{a})$ may be large, it has a very simple internal structure. As a consequence, we deduce a number of corollaries in Section III. For example, in contrast to the situation for C_1, it is easy to compute the cardinality $\#C_2(\underline{a})$ (Corollary 6). We also show that no two reduced words are simultaneously C_1- and C_2-equivalent (Corollary 10).

II. The Main Encoding Theorem.

Before we can state the main theorem, we first need some terminology related to the encoding of $C_2(\underline{a})$, and a few lemmas to show that this encoding is well-defined. Given a quadruple (l, k, n, ϵ) where

(1) l is a positive integer,
(2) k is an integer in the range $[0, l-1]$
(3) n is a positive integer
(4) ϵ is either $+, -$, or 0, and is 0 exactly when $l \leq 2$

define a particular kind of reduced word $\sigma_{l,k,n,\epsilon}$ which we call a *string* as follows: For $l \geq 3$, $\sigma_{l,k,n,+}$ is the first $2l-1$ letters in the following sequence

$$n+1, n, n+2, n+1, n+3, n+2, \cdots, n+k, n+k-1,$$

$$\mathbf{n+k}, n+k+1, n+k,$$

$$n+k+2, n+k+1, n+k+3, n+k+2, \cdots$$

and the letter $\mathbf{n+k}$ shown in boldface is called the *core* of the string. The string $\sigma_{l,k,n,-}$ is defined to be the *reverse* of the string $\sigma_{l,l-1-k,n,+}$, and has an analogously defined *core*. For $l = 1, 2$, define

$$\sigma_{2,0,n,0} = \mathbf{n}\ n+1\ n$$
$$\sigma_{2,1,n,0} = n+1\ n\ \mathbf{n+1}$$
$$\sigma_{1,0,n,0} = \mathbf{n}$$

Examples.

$$\sigma_{1,0,3,0} = 3$$

$$\sigma_{2,0,5,0} = 565$$
$$\sigma_{2,1,5,0} = 656$$

$$\sigma_{5,0,2,+} = 232435465$$
$$\sigma_{5,1,2,+} = 323435465$$
$$\sigma_{5,2,2,+} = 324345465$$
$$\sigma_{5,3,2,+} = 324354565$$
$$\sigma_{5,4,2,+} = 324354656$$

$$\sigma_{4,0,5,-} = 8786756$$
$$\sigma_{4,1,5,-} = 7876756$$
$$\sigma_{4,2,5,-} = 7867656$$
$$\sigma_{4,3,5,-} = 7867565$$

The braid pictures for $\{\sigma_{4,k,5,-}\}_{k\in[0,3]}$ are shown in Figure 2.

Note that for $l \geq 3$, the value k in the string $\sigma_{l,k,n,\epsilon}$ is the number of pairs of letters preceding the core of the string, and these pairs look like $i+1, i$ when $\epsilon = +$ and $i, i+1$ when $\epsilon = -$. Furthermore, if the core is not at the end of the string, then pairs of the same form will follow the core as well. Thus the braid picture for a string looks like a sequence of "steps", followed by the core, followed by another sequence of "steps". Notice that performing an elementary C_2-equivalence on a string may be viewed as "sliding" up or down the core of the string. In fact, from this point of view, the following Lemma is completely trivial:

Lemma 1. *The C_2-equivalence class of the string $\sigma_{l,k,n,\epsilon}$ is exactly*

$$C_2(\sigma_{l,k,n,\epsilon}) = \{\sigma_{l,k',n,\epsilon}\}_{k'\in[0,l-1]}$$

Proof. The only elementary C_2-equivalences that may be performed on $\sigma_{l,k,n,\epsilon}$ simply raise or lower k by 1. ■

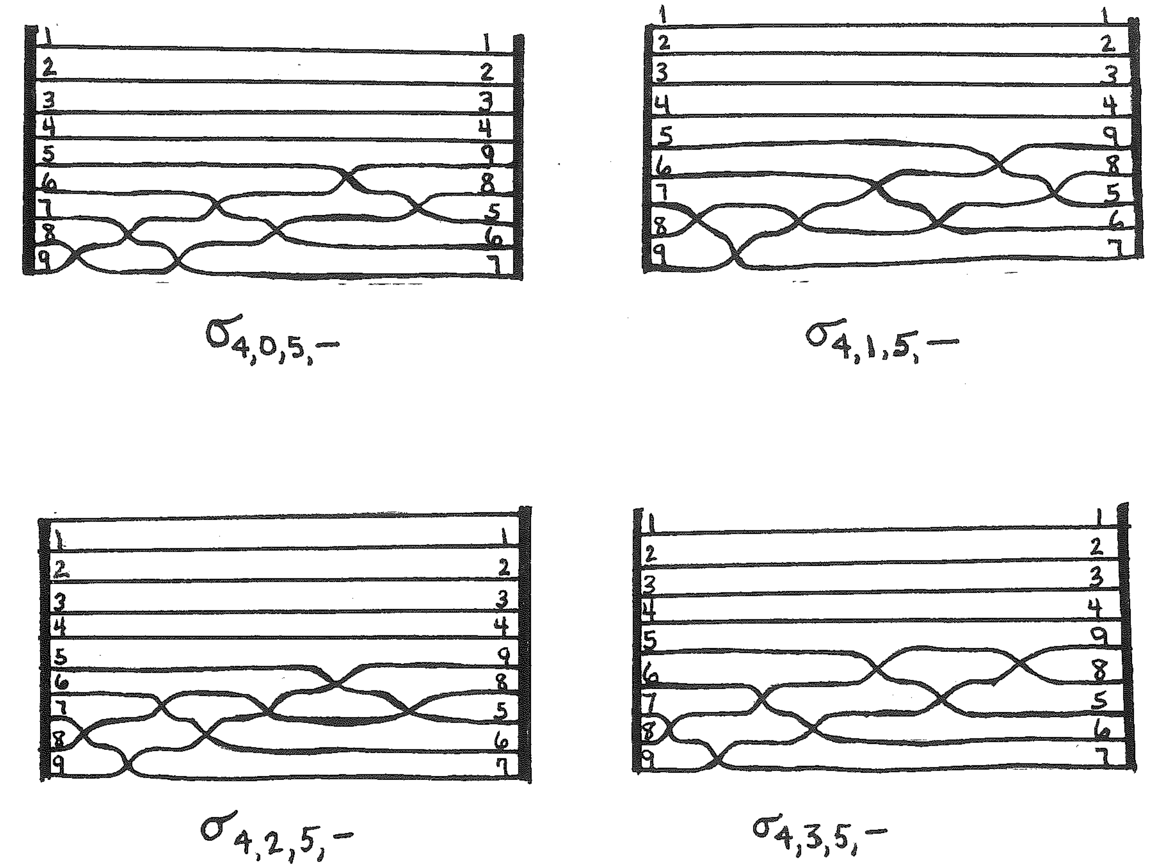


FIGURE 2. Some braid pictures of strings

Given a reduced word $\underline{a}$, a *string* in $\underline{a}$ means a consecutive subsequence of $\underline{a}$ which is equal to some string $\sigma_{l,k,n,\epsilon}$, and a *maximal string* in $\underline{a}$ is a string which is embedded within no longer string of $\underline{a}$. A decomposition

$$\underline{a} = \sigma_{l_1,k_1,n_1,\epsilon_1} \cdot \sigma_{l_2,k_2,n_2,\epsilon_2} \cdots \sigma_{l_r,k_r,n_r,\epsilon_r}$$

in which $\cdot$ denotes concatenation and each of the $\sigma_{l_i,k_i,n_i,\epsilon_i}$'s is a maximal string of $\underline{a}$ will be called a *maximal string decomposition* of $\underline{a}$.

Example. $\underline{a} = 213234356521878$ is a reduced word for $w = 534276981$ in S_9, and

$$\underline{a} = \underset{\sigma_{4,2,1,+}}{2132343} \quad \underset{\sigma_{2,0,5,0}}{565} \quad \underset{\sigma_{1,0,2,0}}{2} \quad \underset{\sigma_{1,0,1,0}}{1} \quad \underset{\sigma_{2,1,7,0}}{878}$$

is a maximal string decomposition for $\underline{a}$.

The encoding of $C_2(\underline{a})$ will be derived from the fact that maximal string decompositions are unique, which we now prove.

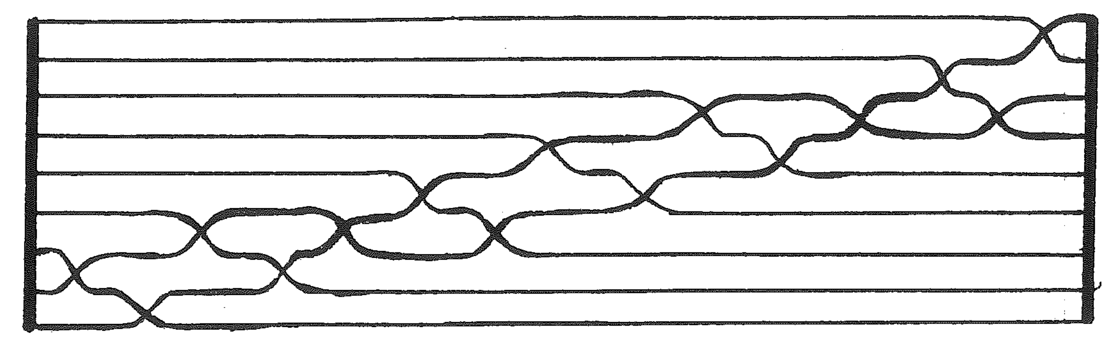
Lemma 2. *Every letter v in $\underline{a}$ is contained in a unique maximal string, and therefore every reduced word $\underline{a}$ has a unique maximal string decompostition.*

Proof. Let v be a letter in $\underline{a}$. Since $v = \sigma_{1,0,v,0}$, we have that v is itself a string which is either maximal or is contained in some maximal string. Now assume v is contained in two maximal strings, $\underline{a}_i = \sigma_{l_i,k_i,n_i,\epsilon_i}$ and $\underline{a}_j = \sigma_{l_j,k_j,n_j,\epsilon_j}$, where $\underline{a}_i \neq \underline{a}_j$. Since we know what the braid picture for a string must look like, it is easy to check that for v to be contained in two maximal strings, the braid pictures for $\underline{a}_i$ and $\underline{a}_j$ must "fit together" in one of two ways:

(1) if $\underline{a}$ contains

$$\cdots \mathbf{r}, r+1, r, r+2, r+1, \cdots, v, \cdots, r+s, r+s-1, \mathbf{r+s}, r+s+1, r+s, \cdots$$

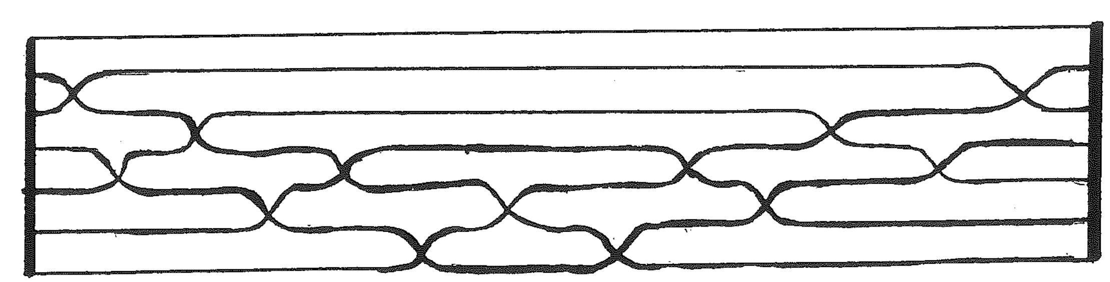
for some r, s where the cores of $\underline{a}_i, \underline{a}_j$ are in boldface respectively, then we have a braid picture that looks like:



(2) if $\underline{a}$ contains

$$\cdots r-1, r-2, r, r-1, \mathbf{r+1}, r, \mathbf{r+1}, r-1, r, r-2, r-1, \cdots$$

for some r (where this time v could be any of the letters $r+1, r, r+1$ in the middle) and the cores of $\underline{a}_i, \underline{a}_j$ appear in boldface, then we have a braid picture that looks like:



In case 1, it is easy to see by looking at the braid picture that as we perform C_2-equivalences on $\underline{a}_j$ which will slide the core of the string $\underline{a}_j$ down, this core eventually bumps into the core of $\underline{a}_i$, creating the pattern

$$\cdots r, r+1, r, r+1, r+2, r+1, \cdots$$

which is not reduced and hence cannot be contained in a reduced word.

In case 2, $\underline{a}$ contains

$$\cdots r, r-1, r+1, r, r+1, r-1, r, \cdots$$

which is again not reduced. Therefore two maximal strings cannot "fit together" in a reduced word, and thus a letter v cannot be contained in more than one maximal string. ■

As a consequence, we can now define the encoding $code(\underline{a})$ to be

$$code(\underline{a}) = ((l_1, n_1, \epsilon_1), \ldots, (l_r, n_r, \epsilon_r))$$

where the unique maximal string decomposition of $\underline{a}$ is

$$\underline{a} = \sigma_{l_1,k_1,n_1,\epsilon_1} \cdots \sigma_{l_r,k_r,n_r,\epsilon_r}.$$

Example. If $\underline{a} = 2132343\ 565\ 2\ 1\ 878$ as in the previous example, then

$$code(\underline{a}) = ((4,1,+),(2,5,0),(1,2,0),(1,1,0),(2,7,0))$$

We can now state the main result (The encoding theorem):

Theorem 3. *If $\underline{a}, \underline{a}'$ are reduced words, then $\underline{a} \overset{C_2}{\sim} \underline{a}'$ if and only if*

$$code(\underline{a}) = code(\underline{a}').$$

To prove this theorem, we need two lemmas.

Lemma 4. *If two strings $\underline{a}$ and $\underline{a}'$ are C_2-equivalent, then $code(\underline{a}) = code(\underline{a}')$.*

Proof. Let $\underline{a} = \sigma_{l_1,k_1,n_1,\epsilon_1}$ and $\underline{a}' = \sigma_{l_2,k_2,n_2,\epsilon_2}$. Since $\underline{a}$ and $\underline{a}'$ are C_2-equivalent, by Lemma 1 they are both of the form $\sigma_{l,k,n,\epsilon}$ for some fixed l, n, ϵ. Thus by definition, $code(\underline{a}) = code(\underline{a}') = (l, n, \epsilon)$. ■

Lemma 5. *Given a reduced word $\underline{a}$ with maximal string decomposition*

$$\underline{a} = \underline{a}_1 \, \underline{a}_2 \cdots \underline{a}_r$$

and a reduced word $\underline{a}'$ with maximal string decomposition

$$\underline{a}' = \underline{a}'_1 \, \underline{a}'_2 \cdots \underline{a}'_s,$$

such that $\underline{a} \overset{C_2}{\sim} \underline{a}'$, then $r = s$ and $\underline{a}_i \overset{C_2}{\sim} \underline{a}'_i$ for all $1 \leq i \leq r$.

Proof. Without loss of generality, we may assume $\underline{a}, \underline{a}'$ differ by a single elementary C_2-equivalence. Therefore, $\underline{a}$ and $\underline{a}'$ are identical except for one $j, j+1, j$ in $\underline{a}$ replaced by one $j+1, j, j+1$ in $\underline{a}'$, for some integer j. By definition, $j, j+1, j \subseteq \underline{a}$ is the string $\sigma_{2,0,j,0}$ and is thus equal to or contained in some maximal string in $\underline{a}$. Since maximal string decomposition is unique, say without loss of generality $\sigma_{2,0,j,0} \subseteq \underline{a}_p$ for some $1 \leq p \leq r$. Similarly, $j+1, j, j+1 = \sigma_{2,1,j,0}$ in $\underline{a}'$, so let $\sigma_{2,1,j,0} \subseteq \underline{a}'_q$. Since $\underline{a}$ and $\underline{a}'$ are identical prior to $\underline{a}_p$ and $\underline{a}'_q$, we have $\underline{a}_1 = \underline{a}'_1, \underline{a}_2 = \underline{a}'_2, \cdots, \underline{a}_{p-1} = \underline{a}'_{q-1}$, and therefore $p = q$. Now let $\underline{a}_p = \sigma_{l_1,k_1,n_1,\epsilon_1}$ and let $\underline{a}'_p = \sigma_{l_2,k_2,n_2,\epsilon_2}$. Assume that $l_1 \neq l_2$, so without loss of generality $l_1 > l_2$. Since $\sigma_{l_1,k_1,n_1,\epsilon_1}$ is identical to the $2l_1 - 1$ letters of $\underline{a}'$ following $\underline{a}'_{p-1}$ except for one C_2-equivalence, then these $2l_1 - 1$ letters of $\underline{a}'$ are a string in the C_2-class of $\underline{a}_p$ by Lemma 1. This string contains $\sigma_{l_2,k_2,n_2,\epsilon_2}$, which cannot then be maximal. This is a contradiction, and thus $l_1 = l_2$. Therefore $\underline{a}_p$ and $\underline{a}'_p$ are identical except for one C_2-equivalence, and so $\underline{a}_p \overset{C_2}{\sim} \underline{a}'_p$. Furthermore, since $\underline{a}$ and $\underline{a}'$ are identical outside of $\underline{a}_p$ and $\underline{a}'_p$, we have that $\underline{a}_t = \underline{a}'_t$ for any $p < t \leq r$. Therefore $\underline{a}_i \overset{C_2}{\sim} \underline{a}'_i$ for any $1 \leq i \leq r$, and it is then clear that $r = s$. ■

We are now ready to prove the main encoding theorem:

Theorem 3. *If $\underline{a}, \underline{a}'$ are reduced words, then $\underline{a} \overset{C_2}{\sim} \underline{a}'$ if and only if*

$$code(\underline{a}) = code(\underline{a}').$$

Proof.

$\Leftarrow$: Assume $code(\underline{a}) = code(\underline{a}')$. Let $\underline{a}$ have maximal string decomposition

$$\underline{a} = \underline{a}_1 \underline{a}_2 \cdots \underline{a}_r$$

and let $\underline{a}'$ have maximal string decompostition

$$\underline{a}' = \underline{a}'_1 \underline{a}'_2 \cdots \underline{a}'_s$$

where $\underline{a}_i = \sigma_{l_{i_1}, k_{i_1}, n_{i_1}, \epsilon_{i_1}}$ and $\underline{a}'_i = \sigma_{l_{i_2}, k_{i_2}, n_{i_2}, \epsilon_{i_2}}$. Since $code(\underline{a}) = code(\underline{a}')$, then $r = s$, and furthermore $l_{i_1} = l_{i_2}, n_{i_1} = n_{i_2}$ and $\epsilon_{i_1} = \epsilon_{i_2}$ for all $a \leq i \leq r$. Then by Lemma 1 we have that $\underline{a}_i \overset{C_2}{\sim} \underline{a}'_i$ for each i, and hence $\underline{a} \overset{C_2}{\sim} \underline{a}'$.

$\Rightarrow$: Now assume $\underline{a} \overset{C_2}{\sim} \underline{a}'$. Again let $\underline{a} = \underline{a}_1 \underline{a}_2 \cdots \underline{a}_r$ and $\underline{a}' = \underline{a}'_1 \underline{a}'_2 \cdots \underline{a}'_s$ be maximal string decompositions. Then by Lemma 5 we have $r = s$ and $\underline{a}_i \overset{C_2}{\sim} \underline{a}'_i$ for all $1 \leq i \leq r$. By Lemma 4 we know then that $code(\underline{a}_i) = code(\underline{a}'_i)$ for all $1 \leq i \leq r$, and therefore by definition $code(\underline{a}) = code(\underline{a}')$. ■

III. Consequences of the Encoding Theorem.

We can now state some interesting corollaries about the structure of a C_2 equivalence class.

The following is an immediate consequence of the main encoding theorem and Lemma 1.

Corollary 6. *Given a reduced word $\underline{a}$ such that*

$$code(\underline{a}) = ((l_1, n_1, \epsilon_1)(l_2, n_2, \epsilon_2) \cdots (l_r, n_r, \epsilon_r)), \text{ then}$$

$$C_2(\underline{a}) = \{\sigma_{l_1,k_1,n_1,\epsilon_1} \cdots \sigma_{l_r,k_r,n_r,\epsilon_r} : k_i \in [0, l_i - 1] \text{ for each } i\}$$

and hence

$$\#C_2(\underline{a}) = \prod_{i=1}^{r} l_i. \blacksquare$$

The next two corollaries give upper bounds on the cardinality of $C_2(\underline{a})$ in terms of the length of $\underline{a}$, and in terms of the size of the permutation w for which $\underline{a}$ is a reduced word.

Corollary 7. *Let $m(n)$ be the maximum cardinality $\#C_2(\underline{a})$ as $\underline{a}$ runs over all reduced words of length n for any permutation. Then*

$$m(n) = \begin{cases} 2^j & if \quad n = 3j \\ 2^j & if \quad n = 3j+1, \quad j \le 2 \\ 2^{j-3}3^2 & if \quad n = 3j+1, \quad j > 2 \\ 2^{j-1}3 & if \quad n = 3j+2 \end{cases}$$

Proof. Let n be a fixed non-negative integer. Given any sequence $l_1, l_2, \cdots l_r \in \mathbb{Z}^+$ such that $\sum_{i=1}^{r}(2l_i - 1) = n$, we can produce a reduced word $\underline{a}$ of length n such that $\#C_2(\underline{a}) = \prod_{i=1}^{r} l_i$ by letting $\underline{a} = \sigma_{l_1,0,n_1,+} \cdots \sigma_{l_r,0,n_r,+}$ where

$$n_r >> n_{r-1} >> \cdots >> n_1$$

i.e. the set of letters in $\sigma_{l_i,0,n_i,+}$ is disjoint from the set of letters in $\sigma_{l_j,0,n_j,+}$ for any $i \neq j$. Therefore

$$m(n) = max \left\{ \prod_{i=1}^{r} l_i : r, l_i \in \mathbb{Z}^+, \sum_{i=1}^{r}(2l_i - 1) = n \right\}$$

and thus

$$m(n) = max\left\{\prod_{i=1}^{r} l_i : r, l_i \in \mathbb{Z}^+, \sum_{i=1}^{r} l_i = \frac{n+r}{2}\right\}$$

This is now purely an arithmetic problem, and therefore proofs of all cases of Corollary 7 will not be given. Assuming the first 3 cases have been proven, we will show how the case for $n = 3j + 2$ is derived, since it is intermediate in complexity.

Case 4. *If $n = 3j + 2$, and thus $\sum_{i=1}^{r} l_i = \frac{3j+2+r}{2}$ then*

$$max\prod_{i=1}^{r} l_i = 2^{j-1} \cdot 3.$$

Proof by induction on j. For the base case $j = 1$. Then $n = 5$, so $\sum_{i=1}^{r} l_i = \frac{5+r}{2}$, and hence $r \in \{1, 3, 5\}$. Therefore $\sum_{i=}^{r} l_i = 1 + 1 + 1 + 1 + 1$ or $2 + 1 + 1$ or 3, and thus $\max(\prod_{i=1}^{r} l_i) = 3$ as desired.

Now assume the result is true for $1, 2, \ldots, j - 1$, and we will prove it for j. If there exists some subset of l_i's of size h such that the sum of that subset is $\frac{3j_1+h}{2}$, then by case 1 the maximum product of that subset of l_i's is 2^{j_1}. Furthermore, the sum of the remaining $r - h$ l_i's must be $\frac{3j_2+2+r-h}{2}$ for $j_2 < j$, where $j_1 + j_2 = j$. By our induction hypothesis, the maximum product of those remaining l_i's is $2^{j_2-1} \cdot 3$. Therefore the maximum product of all l_i's is $2^{j_1} \cdot 2^{j_2-1} \cdot 3$, which is $2^{j-1} \cdot 3$.

If there does not exist such a subset of l_i's, then one can check by consideration of remainders of $2l_i - 1 \bmod 3$ that either $r = 1$ or $r = 2$.

If $r = 1$ then $l_1 = \frac{3j+3}{2}$, and thus $\#C_2 = l_1 = \frac{3j+3}{2}$, and one can use algebra to check that $\frac{3j+3}{2} \leq 2^{j-1} \cdot 3$. If $r = 2$, then

$$l_1 = \frac{3j_1 + 2}{2}, l_2 = \frac{3j_2 + 2}{2},$$

where $j_1 + j_2 = j$. Clearly $l_1 \cdot l_2$ is maximized when $j_1 = j_2$ and thus $l_1 = l_2$. Then $j_1 = j_2 = \frac{1}{2}j$. Therefore

$$\#C_2 = \left(\frac{3\frac{j}{2} + 2}{2}\right)^2 = \left(\frac{3j+4}{4}\right)^2.$$

It is easy to check by algebra that $(\frac{3j+4}{4})^2 \leq 2^{j-1} \cdot 3$ for $j \geq 3$, and for $j = 2$, then $l_1 + l_2 = 5$, and it is clear that $\max(l_1 \cdot l_2) = 3 \cdot 2$. The case for $j = 1$ was shown in the first induction step, and thus case 4 holds for all j. ■

Corollary 8. *Let $\mu(n)$ be the maximum cardinality $\#C_2(\underline{a})$ as $\underline{a}$ runs over $Red(w)$ for all w in S_n. Then*

$$2^{\frac{n^2}{8}} \leq \mu(n) \leq 2^{\frac{n^2}{6}}$$

asymptotically in n.

Proof. Clearly $\underline{a}$ is longest when w reverses the identity permutation in S_n, in which case $(\underline{a})$ has length $\binom{n}{2}$, since every pair of numbers in [n] must be switched. By Corollary 7, we know that $\#C_2(\underline{a})$ is maximized when $\underline{a}$ is composed almost completely of strings of length 3, of which there are asymptotically $\frac{\binom{n}{2}}{3}$. Therefore

$$\mu(n) \leq 2^{\frac{\binom{n}{2}}{3}} \cong 2^{\frac{n^2}{6}}.$$

Now let n be some fixed integer. The following algorithm produces a reduced word $\underline{a}_n$ for $n\ n-1\ \cdots\ 1$, the reverse of the identity permutation, which achieves $\#C_2(\underline{a}_n) \cong 2^{\frac{n^2}{8}}$.

Algorithm for $\underline{a}_n$:

(1) Start with $w = 123\ldots n$.

(2) Let i be the smallest number such that $w_r = i$ and $r \neq n+1-i$. If there is no such i, then we are done.

(3) If $w_{r+1} < w_{r+2}$ then reverse w_r, w_{r+1}, w_{r+2} using the transpositions $s_r s_{r+1} s_r$. If $w_{r+1} > w_{r+2}$, then swap w_r, w_{r+1} using s_r.

(4) Repeat step 2.

Example of Algorithm for $n = 7$.

$\underline{w}$	$\underline{i}$	transpositions to be performed
1234567	1	$s_1 s_2 s_1$
3214567	1	$s_3 s_4 s_3$
3254167	1	$s_5 s_6 s_5$
3254761	2	s_2
3524761	2	$s_3 s_4 s_3$
3574261	2	s_5
3574621	3	$s_1 s_2 s_1$
7534621	3	$s_3 s_4 s_3$
7564321	5	s_2
7654321		

Now define t_i as the number of times this algorithm used $s_r s_{r+1} s_r$ at Step 3 when $i = w_r$. Thus in our example above, $t_1 = 3, t_2 = 1, t_3 = 2$ and $t_m = 0$ for all

$m \geq 3$. Notice that every reduced word created by this algorithm is made up of maximal strings of length 3 or length 1. Since there are two C_2-equivalent strings for every string of length 3, then the size of the C_2 class of any of these reduced words is $2^{\sum t_i}$.

If n is odd, say $n = 2j - 1$ for some j, one can check that componentwise

$$(t_1, t_2, \cdots, t_{j-1}) \geq (j-1, j-3, j-2, j-5, j-4, \cdots).$$

If n is even, say $n = 2j$ for some j, one can check that componentwise

$$\begin{aligned}(t_1, t_2, \cdots, t_{j-1}) &\geq (j-1, j-2, j-2, j-4, j-4, j-6, j-6, \cdots) \\ &\geq (j-1, j-3, j-2, j-5, j-4, \cdots)\end{aligned}$$

Thus for n odd or even,

$$\sum_{i>1} t_i \geq \sum_{i=2}^{j-1} i = \frac{(j+1)(j-2)}{2}$$

where $j = \lceil \frac{n}{2} \rceil$. One can then check that

$$\frac{(\lceil \frac{n}{2} \rceil + 1)(\lceil \frac{n}{2} \rceil - 2)}{2} \geq \begin{cases} \frac{n^2-4n-8}{8} & \text{for n odd} \\ \frac{n^2-2n-11}{8} & \text{for n even.} \blacksquare \end{cases}$$

Besides the cardinality of C_2-classes, it is also interesting to consider the possible relationships between C_1- and C_2-equivalence, for instance, is it possible for two reduced words $\underline{a}, \underline{a}'$ to be both C_1- and C_2-equivalent when $\underline{a} \neq \underline{a}'$? A priori, this is not obviously impossible, since it is possible for two different C_2-equivalent words to have the same content, such as

$$1213243212 \overset{C_2}{\sim} 2123243121.$$

In order to resolve this question, we need to understand one way in which C_1-equivalence is encoded.

Consider a permutation $w = w_1 w_2 \cdots w_n \in S_n$, and let $\underline{a} \in \text{Red(w)}$. Let

$$Trip(w) = \{(i, j, k) : i < j < k, w_i > w_j > w_k\}.$$

Since w reverses the order of i, j, k, and since $\underline{a}$ is reduced, then the sequence of transpositions represented by the letters of $\underline{a}$ must swap each pair in $\{i, j, k\}$

exactly once. Furthermore, since j is between i and k, then the first switch in the set $\{i,j,k\}$ must be (i,j) or (j,k), moving j either left or right. One can see then that there are only two possible sequences of switching pairs in this set:

(1) $(i,j),(i,k),(j,k)$ OR
(2) $(j,k),(i,k),(i,j)$.

Now let $f_{\underline{a}} : Trip(w) \to \{L,R\}$ be the map which assigns to each $(i,j,k) \in Trip(w)$ an L if $\underline{a}$ switches pairs as in (1) above, or an R is $\underline{a}$ switches pairs as in (2) above. The following Lemma [Zi] shows that $f_{\underline{a}}$ characterizes the C_1-class of $\underline{a}$.

Lemma 9. *Given $\underline{a},\underline{a}'$ reduced words of some permutation $w \in S_n$, then $\underline{a} \overset{C_1}{\sim} \underline{a}'$ if and only if $f_{\underline{a}} = f_{\underline{a}'}$.*

For our purposes, we will only need the easier implication, namely that $\underline{a} \overset{C_1}{\sim} \underline{a}'$ implies $f_{\underline{a}} = f_{\underline{a}'}$. For a proof that $f_{\underline{a}} = f_{\underline{a}'}$ implies $\underline{a} \overset{C_1}{\sim} \underline{a}'$, see [Zi].

Proof. Assume $\underline{a} \overset{C_1}{\sim} \underline{a}'$, and furthermore assume $\underline{a}$ and $\underline{a}'$ differ by exactly one elementary C_1-equivalence. Thus we can say

$$\begin{aligned} \underline{a} &= a_1 a_2 \cdots a_r a_s \cdots a_t \\ \underline{a}' &= a_1 a_2 \cdots a_s a_r \cdots a_t \qquad \text{where } |a_s - a_r| \geq 2. \end{aligned}$$

Fix $(i,j,k) \in Trip(w)$. Clearly the relative positions of i,j,k during the processes $\underline{a}$ and $\underline{a}'$ are the same up to the performance of a_r, a_s. Since $|a_r - a_s| \geq 2$, we know a_s switches two letters in w and a_r switches two entirely different letters in w.

First assume a_r switches two letters in $\{i,j,k\}$, so without loss of generality say a_r switches (i,j). Let a_s switch (k,q), where q is some other letter in w. Then k will still have the same relative position to the now switched pair i,j, regardless of the order of a_r, a_s, since k and q are clearly left or right of i and j.

If neither a_r nor a_s switch more than one letter in $\{i,j,k\}$, then a_r and a_s have no effect on the relative positions of $\{i,j,k\}$, since a_r and a_s cannot move one past the other.

Therefore i,j,k are in the same order after the performance of a_r, a_s or a_s, a_r. Furthermore, the permutations are identical after a_r, a_s, and thus $f_{\underline{a}}(i,j,k) = f_{\underline{a}'}(i,j,k)$. ■

We can now use this Lemma to prove our final result:

Corollary 10. *Let $\underline{a}$ and $\underline{a}'$ be reduced words of some permutation $w = w_1 w_2 \cdots w_n$. Then $\underline{a} \overset{C_2}{\sim} \underline{a}'$ and $\underline{a} \overset{C_1}{\sim} \underline{a}'$ if and only if $\underline{a} = \underline{a}'$.*

Proof. $\Rightarrow$: It is trivial that when $\underline{a} = \underline{a}'$, then both $\underline{a} \overset{C_1}{\sim} \underline{a}'$ and $\underline{a} \overset{C_2}{\sim} \underline{a}'$.

$\Leftarrow$: Assume $\underline{a} \overset{C_1}{\sim} \underline{a}'$ and $\underline{a} \overset{C_2}{\sim} \underline{a}'$ where

$$\underline{a} = a_1 a_2 a_3 \ldots a_j \ldots a_m$$

$$\underline{a}' = a_1' a_2' a_3' \ldots a_j' \ldots a_m'.$$

Then $a_i = a_i'$ up to some point, say $a_j \neq a_j'$, where $a_i = a_i'$ for all $i < j$. Since $\underline{a} \overset{C_2}{\sim} \underline{a}'$, we know a_j and a_j' are in C_2-equivalent maximal strings by Lemma 5, and furthermore $|a_j - a_j'| = 1$, since sliding the core of the strings containing a_j and a_j' can change any letter in the string by at most 1. Without loss of generality, say $a_j' = a_j + 1$. Since $\underline{a}$ and $\underline{a}'$ are identical up to a_{j-1}, then the sequences of permutations that result as the transpositions of $\underline{a}$ and $\underline{a}'$ are performed are identical up to the $(j-1)^{st}$ permutation. Let this $(j-1)^{st}$ permutation be $v = v_1 v_2 \cdots v_n$. By performing an s_{a_j} on v, this becomes $v_1 v_2 \cdots v_{a_j+1} v_{a_j} v_{a_j+2} \cdots v_n$, and by performing $s_{a_{j'}}$ on v it becomes $v_1 v_2 \cdots v_{a_j} v_{a_j+2} v_{a_j+1} \cdots v_n$, since $a_{j'} = a_j + 1$. Since $a_j' \neq a_j$, the core of some string must be precisely at a_j or a_j', and so without loss of generality we can say

$$(a_j, a_{j+1}, a_{j+2}) = (h, h+1, h)$$

for some h. Thus $v_{a_j}, v_{a_j+1}, v_{a_j+2}$ are reversed in $\underline{a}$ and $\underline{a}'$, and therefore $(v_{a_j}, v_{a_j+1}, v_{a_j+2}) \in Trip(w)$. Furthermore, $f_{\underline{a}}(v_{a_j}, v_{a_j+1}, v_{a_j+2}) = L$ and $f_{\underline{a}'}(v_{a_j}, v_{a_j+1}, v_{a_j+2}) = R$. Thus by the contrapositive of Lemma 9, $\underline{a}$ and $\underline{a}'$ cannot be C_1-equivalent if $\underline{a}, \underline{a}'$ are C_2-equivalent. ■

IV. Open questions. There are many question about C_2-equivalence which we have not answered, and we list some here.

Let $w_0^{(n)}$ be the permutation in S_n which reverses the identity permutation.

(1) For $Red(w_0^{(n)})$, how many different C_2-classes are there? By implementing the main encoding theorem (Theorem 3) in the computer language C, we have computed these values for $n \leq 5$:

n	$\#Red(w_0^{(n)})/\overset{C_2}{\approx}$	$\#Red(w_0^{(n)})$
1	1	1
2	1	1
3	1	1
4	8	16
5	432	768

Is it true that

$$\lim_{n \to \infty} \frac{\#Red(w_0^{(n)})/\overset{C_2}{\approx}}{\#Red(w_0^{(n)})} = 1?$$

(2) Let G_n be the graph with a vertex for each C_2-class in $Red(w_0^{(n)})$ and an edge between two classes $C_2(\underline{a}), C_2(\underline{a}')$ if they have representatives $\underline{a}, \underline{a}'$ which differ by an elementary C_1-equivalence. By Tits' Theorem (see Introduction), G_n is connected. For example, G_4 is depicted in Figure 3.

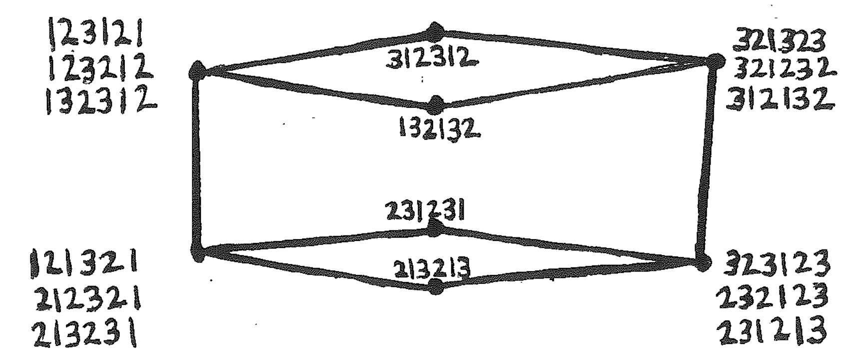


FIGURE 3

Note that this graph always has two commuting symmetries coming from

$$a_1\ a_2 \cdots a_l \leftrightarrow a_l \cdots a_2\ a_1$$

$$a_1\ a_2 \cdots a_l \leftrightarrow n - a_l \cdots n - a_2\ n - a_1.$$

What else can one say about this graph? For example, what is its diameter as a function of n?

(3) Let $\nu(n)$ be the number of $\underline{a} \in Red(w_0^{(n)})$ which have $\#C_2(\underline{a}) = 1$. The table below shows some of the first few values for $\nu(n)$.

n	$\nu(n)$	$\#Red(w_0^{(n)})$
1	1	1
2	1	1
3	0	2
4	4	16
5	256	768
6	100208	292864

Can one find a generating function or asymptotic formula for $\nu(n)$? Computations with random reduced words in Mathematica suggest that

$$\lim_{n \to \infty} \frac{\nu(n)}{\#Red(w_0^{(n)})} = c$$

where c is a constant approximately equal to $\frac{1}{e}$.

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