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Fichera's method for bounding eigenvalues

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1. Introduction.

I first met Gaetano Fichera in 1950 when I had just been appointed a post-doctoral fellow at the Institute for Fluid Dynamics and Applied Mathematics of the University of Maryland. After the International Congress of Mathematicians Mauro Picone came to lecture at this Institute, and Gaetano came as his assistant. Among other things, he erased the blackboard during Professor Picone's lecture.

The next time I met Fichera was a year later, when he had become a Professor at Trieste. This time he had two assistants to erase the blackboard for him. More remarkably, they were women. Gaetano was one of the pioneers in encouraging women with mathematical talent to go into research careers.

Gaetano developed a deep friendship with Alexander Weinstein, who was the principal mathematician at the Institute for Fluid Dynamics and Applied Mathematics. Fichera paid many visits to the Institute during my stay there. Like all friendships with Weinstein, Gaetano's had its ups and downs, but on the whole it lasted until Weinstein' death.

One of the consequences (and perhaps also causes) of this friendship was that Fichera became interested in computational methods methods for finding upper and lower bounds for eigenvalues. Weinstein was one of the pioneers in this area. In 1937 he had published a monograph [Ws37] on how to compute arbitrarily close lower bounds for the eigenvalues of one problem if one knows the eigenvalues and eigenfunctions of a related problem, which is called the base problem. (As long as one deals with an operator whose eigenvalues are the successive minima of its Rayleigh quotient, upper bounds are easily obtained from the Rayleigh-Ritz method, and so it is sufficient to find lower bounds.)

There were two principal ideas. The first is that one can find the eigenvalues of a finite-dimensional perturbation of an operator whose eigenvalues and eigenfunctions are explicitly known by using what has become known as the Weinstein determinant. The second is that if two operators have the same Rayleigh quotient but the domain of the second is contained in the domain of the first, then the eigenvalues of the second operator are higher than those of the first. (This fact is called the first monotonicity principle.) It follows that if one knows the eigenvalues and eigenvectors of a Rayleigh quotient on one function space (the base problem), and if one wishes to find the eigenvalues of the restriction of the same quotient to a smaller space, one can obtain a sequence of lower bounds for the eigenvalues of the latter problem by computing the eigenvalues of the restriction of the Rayleigh quotient to the orthogonal complement of the first ℓ terms of a sequence of functions which are orthogonal to the smaller space. (See, e.g., [F65a§16,WS72,We74§4.4].)

In Weinstein's monograph, the eigenvalue problem was $\Delta^2 u = \lambda u$ on a unit square S, which has the Rayleigh quotient $\int_S (\Delta u)^2 dx / \int_S u^2 dx$. The domain of the base problem consists of the functions which vanish on the boundary. It is easily seen that the eigenvalues of this problem are the squares of the eigenvalues of the Laplace operator, and its eigenfunctions are products of sines. The domain of the second problem consists of those functions which vanish together with their normal derivatives on the boundary, so that the eigenvalues of the second problem are those of a clamped plate. Weinstein's method yields lower bounds for the latter eigenvalues.

It was noticed by N. Aronszajn [A51] that the Weinstein method could be modified so that it applies to the case of two problems in which the Rayleigh quotient of the first (the base problem) is always below that of the second. The finite-dimensional perturbation of the first operator is now obtained by replacing the first Rayleigh quotient by the second on the linear combinations of a set of finitely many functions, while keeping the first quotient unchanged on the orthogonal complement of these functions.

The Weinstein-Aronszajn methods were successfully used to compute lower bounds for the eigenvalues of a number of problems (see, e. g., [WS72]), but they had some drawbacks. In the first place, one needs to find the zeros and poles of the determinant of a matrix whose entries are transcendental functions of λ. This problem were somewhat alleviated by various adaptations of the method (see [Ba59,Ba61,BF61,BF63,Wb59]), which permitted one to work with a matrix whose entries are rational functions. In this case, one can then locate the zeros and poles of the Weinstein determinant by finding the eigenvalues of a number of members of a one-parameter family of matrices [Be87,BG95,Wb74§4.9]. Moreover, the method is computationally intensive, and it requires the explicit knowledge of at least some of the eigenvalues and eigenfunctions of the base problem. Fichera became interested in overcoming these difficulties.

In 1965 Gaetano Fichera published a remarkable monograph entitled Linear Elliptic Differential Systems and Eigenvalue Problems [F65a]. This book has had a profound effect by making the important L_2 theory of partial differential equations accessible to research workers in mechanics and other sciences. In addition, however, the last few chapters contained a new method for finding lower bounds for the eigenvalues of differential operators. This part of the book has not had as great an influence, perhaps because the presentation makes the method look more complicated than it needs to be.

Fichera's method depends on finding an arbitrarily close upper bound for the trace of the square of the Green's operator G. Instead of a knowledge some of the eigenvalues and eigenfunctions of a base problem, one only needs to know a fundamental solution of the base problem which satisfies its natural boundary conditions. The method consists of two steps. In Section 2 we shall show how to construct an explicit symmetric integral operator K with the property that the trace of K^2 gives an upper bound for the trace of G^2. In Section 3 we show how to construct a family of integral operators $K^{(\ell)}$ with the property that the trace of $[K^{(\ell)}]^2$ decreases to that of G^2. We have chosen to present a somewhat simpler version of the latter construction in the hope that this will cause the method to be more widely adopted. Section 4 outlines some applications, extensions, and further developments of this method, as well as some other contributions of Fichera to eigenvalue approximation. A simplification and extension of the construction in Section 2 will be presented elsewhere [Wb98].

2. Fichera's method of eigenvalue approximation.

Fichera presented his method in great generality, so that it could be applied to the vibration frequencies of an inhomogeneous non-isotropic elastic body. For the sake of exposition, we shall show how it is applied to the specific eigenvalue problem

$$\begin{aligned} L[u] &:= -\nabla \cdot [a(x,y)\nabla u] = \lambda u \quad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where Ω is a smooth bounded domain in two dimensions, $\partial\Omega$ is its boundary, and a is a prescribed smooth uniformly positive function.

We shall use the notation

$$(v,w) := \int_\Omega vw\,dxdy$$

for the L_2 scalar product. Integration by parts shows that the operator L is symmetric in the sense that $(v, L[w]) = (L[v], w)$ for any v and w which vanish on the boundary. It follows that the eigenvalues of L are real. Moreover, for any function $v \not\equiv 0$ which vanishes on the boundary, $(v, L[v]) = \int_\Omega a|\nabla v|^2 dxdy > 0$ because $a > 0$. We say that the operator L is positive definite.

By multiplying both sides of (2.1) by the eigenfunction u and integrating, we see that

$$\lambda = \frac{\int_\Omega a|\nabla u|^2 dxdy}{\int_\Omega u^2 dxdy}. \tag{2.2}$$

This shows that every eigenvalue is positive. The right-hand side of (2.2), defined for any u which vanishes on the boundary, is called the Rayleigh quotient of the problem (2.1). It was shown by by H. Poincaré [P90] that if $v_1, v_2, \cdots, v_n$ are any n functions which vanish on the boundary and for which (v_i, v_j) is 1 if $i = j$ and 0 otherwise, and if the eigenvalues of the $n \times n$ symmetric matrix $(v_i, L[v_j])$ are $\lambda_1^{(n)} \le \lambda_2^{(n)} \le \cdots \le \lambda_n^{(n)}$, then

$$\lambda_j^{(n)} \ge \lambda_j \qquad \text{for } j = 1, \cdots, k.$$

The upper bounds $\lambda_j^{(n)}$ were used for computational purposes by Lord Rayleigh and later by W. Ritz, and are known as the Rayleigh-Ritz upper bounds. They are easy to use in computations, and the remaining task is to find lower bounds in order to bound the error.

As is well-known, the inhomogeneous boundary value problem

$$\begin{aligned} L[v] &:= -\nabla \cdot [a(x,y)\nabla v] = f \quad \text{in } \Omega \\ v &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

with f square integrable has a unique solution, which can be written in the form

$$v(x,y) = \int_\Omega g(x,y;\xi,\eta) f(\xi,\eta)\, d\xi d\eta.$$

By definition, g is the Green's function of the operator L. This function has the symmetry property $g(x,y;\xi,\eta) = g(\xi,\eta;x,y)$, and when $(x,y) \ne (\xi,\eta)$, $L[g(\cdot,\cdot;\xi,\eta)] = 0$. There is a logarithmic singularity for (x,y) near (ξ,η).

Thus the eigenvalue equation (2.1) can be written in the equivalent form

$$G[u] := \int_\Omega g(x,y;\xi,\eta)u(\xi,\eta)d\xi d\eta = \frac{1}{\lambda}u(x,y).$$

We see by setting $v = G[f]$ that $(G[f], f) = (v, L[v]) = \int_\Omega a|\nabla v|^2 dxdy > 0$ for all $f \not\equiv 0$. That is, the integral operator G is also positive definite.

It is well-known that if K is a positive definite integral operator of the form $K[f] = \int k(x,y;\xi,\eta)f(\xi,\eta)d\xi d\eta$ with the positive eigenvalues $\mu_1 \leq \mu_2 \leq \cdots$, then

$$\int_\Omega \int_\Omega k(x,y;x,y)dxdy = \sum_{i=1}^\infty \mu_i, \tag{2.3}$$

in the sense that both sides are either finite and equal or diverge to $+\infty$. If they are finite, the operator K is said to be of trace class, and the common value is called its trace.

Because the Green's function $g(x,y;\xi,\eta)$ is infinite when $(x,y) = (\xi,\eta)$, the left-hand side of (2.3) shows that the operator G is not of trace class. However, the squared operator G^2 has the kernel

$$g^{(2)}(x,y;\xi,\eta) = \int_\Omega g(x,y;\rho,\sigma)g(\rho,\sigma;\xi,\eta)d\rho d\sigma.$$

Because of the symmetry of g, the formula (2.3) applied to G^2, whose eigenvalues are $1/\lambda_i^2$, becomes

$$\int_\Omega \int_\Omega [g(x,y;\xi,\eta)]^2 d\xi d\eta = \sum_{i=1}^\infty (1/\lambda_i^2). \tag{2.4}$$

It is easily seen that the integral on the left converges.

It was observed by Trefftz [T33] that this formula leads to lower bounds for the λ_i in the following way. Choose some $j \leq n$. The right-hand side of (2.4) is made smaller if each of the λ_i in the first n terms with the exception of the one where $i = j$ is replaced by its Rayleigh-Ritz upper bound $\lambda_i^{(n)}$, and the terms with $i > n$ are dropped. In this way one obtains the inequality

$$1/\lambda_j^2 \leq 1/(\lambda_j^{(n)})^2 + \int_\Omega \int_\Omega [g(x,y;\xi,\eta)]^2 d\xi d\eta - \sum_{i=1}^n (1/(\lambda_i^{(n)})^2). \tag{2.5}$$

If one can find the integral on the right or a good bound for it, this formula gives a lower bound for λ_j. Trefftz, however, despaired of finding such a bound, and went on to give another approximation method in the same paper. Fichera, instead, found a way to construct an arbitrarily good upper bound for the integral.

To obtain this bound, Fichera first noted that if K is an integral operator with the property that $K - G$ is positive semidefinite, then the eigenvalues of K are upper bounds for the corresponding eigenvalues of G. Therefore, the trace of K^2 is an upper bound for the trace of G^2, which is the integral in (2.5). His method consists of two parts: (1) to construct an integral operator K with a symmetric kernel such that K^2 is of trace class and $K - G$ is positive semidefinite; and (2) to improve the lower bound obtained by replacing G by K is such a way that an arbitrarily close lower bound for λ_j^2 can be obtained. We begin with step 1.

In [65a] Fichera considered the simple case in which one can find a fundamental solution $r(x, y; \xi, \eta)$ of the differential operator L. By this we mean that for each (ξ, η) in Ω, the function $w(x, y) = r(x, y; \xi, \eta) - g(x, y; \xi, \eta)$ is a smooth solution of $L[w] = 0$ in Ω which is continuously differentiable to the boundary. For example, if $a \equiv 1$, then for each (ξ, η), r is equal to $(1/2\pi)\ln(1/\sqrt{(x-\xi)^2 + (y-\eta)^2})$ plus a harmonic function of (x, y). We define the integral operator

$$R[f] := \int_\Omega r(x, y; \xi, \eta) f(\xi, \eta) d\xi d\eta.$$

It may or may not be true that the operator $R - G$ is positive definite. In fact, the fundamental solution r may not even be symmetric under the interchange of (x, y) and (ξ, η).

We introduce the Dirichlet scalar product

$$(u, v)_L := \int_\Omega a(x, y) \nabla u \cdot \nabla v dx dy. \tag{2.6}$$

This form is defined for smooth functions whether or not they vanish on the boundary, and the corresponding quadratic form is positive semidefinite for all such functions. Note that by definition, $L[R[f] - G[f]] = 0$ for any f, and recall that $G[f]$ vanishes on the boundary. Therefore integration by parts shows that $(R[f] - G[f], G[f])_L = 0$ for all f. It follows that

$$\begin{aligned} (R[f], R[f])_L &= (R[f] - G[f], R[f] - G[f])_L + (G[f], G[f])_L \\ &\geq (G[f], G[f])_L \end{aligned} \tag{2.7}$$

for all f. (Geometrically, the first line states that $R[f]$ is the hypotenuse of a right triangle of which $G[f]$ is one leg. In the language of Prager and Synge [Sy57] the solution $u = G[f]$ of the problem $-\Delta u = f$, $u_{\partial\Omega} = 0$ lies on a hypercircle of which the particular solution $w = R[f]$ of $-\Delta w = f$ is a diameter. As was pointed out by Diaz [D51], (2.7) also follows from $(R[f] - G[f], G[f])_L = 0$ and Schwarz's inequality.)

Integration by parts shows that

$$(G[f], G[f])_L = (G[f], f).$$

Fichera defined the symmetric operator K on L_2 by the formula

$$(K[f], h) := (R[f], R[h])_L \tag{2.8}$$

for all f and h in L_2. Then the inequality (2.7) takes the form

$$(K[f], f) \geq (G[f], f). \tag{2.9}$$

for all f. That is, $K - G$ is positive semidefinite. It remains to be shown that K is an integral operator with a computable kernel. For this purpose, we interchange the order of integration in (2.8), integrate by parts, and use the defining property of r to see that

$$K[f](x,y) = \int_\Omega k(x,y;\xi,\eta) f(\xi,\eta) d\xi d\eta,$$

where

$$\begin{aligned} k(x,y;\xi,\eta) &= \int_\Omega a(\rho,\sigma) \nabla_{(\rho,\sigma)}[r(\rho,\sigma;x,y)] \cdot \nabla_{(\rho,\sigma)}[r(\rho,\sigma;\xi,\eta)] d\rho d\sigma \\ &= r(x,y;\xi,\eta) + \int_{\partial\Omega} a(\rho,\sigma) r(\rho,\sigma;\xi,\eta) \frac{\partial r(\rho,\sigma;x,y)}{\partial n_{(\rho,\sigma)}} ds_{(\rho,\sigma)}. \end{aligned} \tag{2.10}$$

The first line shows that the kernel k is symmetric in the sense that $k(x,y;\xi,\eta) = k(\xi,\eta;x,y)$, even though r may not have this property.

Since $K - G$ is positive semidefinite, the eigenvalues of K are upper bounds for those of G. Thus replacing g by k in (2.5) gives a lower bound for λ_j.

The above construction of Fichera of the kernel k with the desired properties works whenever one can find a fundamental solution r of the operator L. This is always possible when the coefficient function a is constant, but is usually impossible when a is variable. For this reason, in 1971 Fichera [F71a] made the following additional observation.

Let $a_0(x,y)$ be a function with the two properties (i) $0 < a_0(x,y) \le a(x,y)$ in $\bar{\Omega}$, and (ii) a fundamental solution $r_0(x,y;\xi,\eta)$ of the base operator $L_0[v] := -\nabla \cdot (a_0 \nabla v)$ is known. For example, a_0 can be taken to be the minimum value of a, so that it is a positive constant, and then r_0 can be taken as $(1/2\pi a_0)\ln(1/\sqrt{(x-\xi)^2 + (y-\eta)^2})$. Let K now be the integral operator with the kernel k which is defined by (2.10) with r replaced by r_0 and with a replaced by a_0:

$$k(x,y;\xi,\eta) = r_0(x,y;\xi,\eta) + \int_{\partial\Omega} a_0(\rho,\sigma) r_0(\rho,\sigma;\xi,\eta) \frac{\partial r_0(\rho,\sigma;x,y)}{\partial n_{(\rho,\sigma)}} ds_{(\rho,\sigma)}. \tag{2.11}$$

Then if G_0 is the Green's operator of L_0 on Ω, the inequality (2.9) becomes

$$(K[f], f) \ge (G_0[f], f). \tag{2.12}$$

Fichera observed that because the eigenvalues of L_0 are lower bounds for those of L, the eigenvalues of G_0 are upper bounds for those of G. Therefore, the trace of K^2 is an upper bound for that of G^2, so that replacing g by k in (2.5) gives a lower bound for λ_j.

In order to unify and hence simplify the process of improving this bound, we observe that for any smooth v which vanishes on the boundary,

$$(v, (L - L_0)[v]) = \int_\Omega (a - a_0) |\nabla v|^2 dx dy \ge 0$$

so that $L - L_0$ is positive semidefinite, and apply the following simple lemma.

LEMMA. *When $L - L_0$ is positive semidefinite, $G_0 - G$ is also positive semidefinite.*

Proof. One sees from the definitions of G and G_0 that $(G_0[f], f) = (G_0[f], G_0[f])_{L_0}$ and $(G[f], f) = (G[f], G_0[f])_{L_0} = (G[f], G[f])_L$ for any function f. Therefore

$$\begin{aligned}(G_0[f], f) - (G[f], f) &= (G_0[f], G_0[f])_{L_0} - 2(G[f], G_0[f])_{L_0} + (G[f], G[f])_L \\ &= (G_0[f] - G[f], (G_0[f] - G[f])_{L_0} + (G[f], G[f])_L - (G[f], G[f])_{L_0} \\ &\geq 0,\end{aligned}$$

which proves the Lemma.

By combining the positive semidefiniteness of $G_0 - G$ with (2.12), we again obtain the inequality (2.9). That is, the operator K constructed from a fundamental solution r_0 of L_0 by means of (2.11) again has the properties that its kernel is symmetric and that $K - G$ is positive semidefinite. In either case, then, the first part of Fichera's construction yields an integral operator K with symmetric kernel k with the property that substituting k for g in (2.5) gives a lower bound for λ_j.

3. IMPROVING THE BOUNDS.

The second part of Fichera's method consists of finding a sequence of integral operators $K^{(\ell)}$ such that $\mathrm{tr}[(K^{(\ell)})^2]$ is an improved upper bound for $\mathrm{tr}[G^2]$ which decreases to $\mathrm{tr}[G^2]$ as ℓ approaches infinity. There are various ways to obtain such a family of intermediate problems. In the case where one has a fundamental solution r of the original operator L, Fichera observed that r minus any solution $\phi(\mathbf{x}, \xi)$ of $L_{\mathbf{x}}[\phi] = 0$ is again a fundamental solution. He then chose for ϕ that linear combination of solutions $\phi_1(\mathbf{x}), \cdots, \phi_\ell(\mathbf{x})$ of $L[\phi_\nu] = 0$ which makes the quadratic form of the K which comes from $r - \phi$ as small as possible. In the case where K is obtained from a fundamental solution r_0 of L_0, he used the same projection method to bridge the gap between the quadratic form of K and that of G_0, but then had to use a second projection to close the gap between the quadratic forms of G_0 and G. The second projection has the numerical advantage that one projects into a finite-dimensional space of functions which can be taken to be finite elements, so that the Gram-Schmidt matrix is sparse. This is in contrast to the first method, which uses solutions of the differential equation $L[\phi] = 0$, so that the Gram-Schmidt matrix is full. Accordingly, we shall make the process both conceptually and computationally simpler by using only the second kind of projection in all cases.

To see the basic idea, suppose that one has chosen square integrable functions $\psi_1, \psi_2, \cdots \psi_\ell$, which are subject to the conditions

$$(K[\psi_\mu] - G[\psi_\mu], \psi_\nu) = \begin{cases} 1 & \text{when } \mu = \nu \\ 0 & \text{when } \mu \neq \nu. \end{cases} \tag{3.1}$$

Define the operator

$$K^{(\ell)}[f] := K[f] - \sum_{\mu=1}^{\ell} (K[\psi_\mu] - G[\psi_\mu], f)\{K[\psi_\mu] - G[\psi_\mu]\}, \tag{3.2}$$

so that $K^{(\ell)}[f] = G[f]$ when f is any linear combination of the ψ_α, while $K^{(\ell)}[f] = K[f]$ when f is orthogonal to the functions $K[\psi_\mu] - G[\psi_\mu]$. Bessel's inequality [Wb74, p. 51] shows that $(K[f]-G[f],f) \geq (K^{(\ell)}[f]-G[f],f) \geq 0$ for all f. Therefore, the trace of $[K^{(\ell)}]^2$ is, in general, an improved upper bound for the integral in (2.5).

The operator $K^{(\ell)}$ is an integral operator with the kernel

$$k^{(\ell)}(x,y;\xi,\eta) = k(x,y;\xi,\eta) - \sum_{\mu=1}^{\ell} \{K[\psi_\mu](x,y) - G[\psi_\mu](x,y)\}\{K[\psi_\mu](\xi,\eta) - G[\psi_\mu](\xi,\eta)\}.$$

If the Green's function g is known explicitly, one does not need any machinery for computing a good bound for the integral of its square. Otherwise, this formula cannot be evaluated as it stands. Fichera suggested the following trick: Instead of choosing the functions ψ_μ, choose functions α_μ which are smooth and vanish on the boundary, and then define $\psi_\mu = L[\alpha_\mu]$. The orthonormality conditions (2.13) become

$$(K[L[\alpha_\mu]] - \alpha_\mu, L[\alpha_\nu]) = \begin{cases} 1 & \text{when } \mu = \nu \\ 0 & \text{when } \mu \neq \nu. \end{cases} \tag{3.3}$$

Observe that if one chooses any set of ℓ smooth functions $\hat{\alpha}_\mu$ which vanish on the boundary and have the property that the functions $K[L[\hat{\alpha}_\mu]] - \hat{\alpha}_\mu$ are linearly independent, the Schmidt process will produce ℓ linear combinations α_μ which satisfy (3.3).

The formula for the kernel $k^{(\ell)}$ becomes

$$k^{(\ell)}(x,y;\xi,\eta) = k(x,y;\xi,\eta) - \sum_{\mu=1}^{\ell} \{K[L[\alpha_\mu]](x,y) - \alpha_\mu(x,y)\}\{K[L[\alpha_\mu]](\xi,\eta) - \alpha_\mu(\xi,\eta)\}.$$

Substituting this kernel for g in (2.5) gives a lower bound for λ_j. This bound is nondecreasing in n and ℓ, and Fichera showed that under some completeness conditions on the sequences v_i and α_ν, it converges to λ_j as $n, \ell \to \infty$.

4. Extensions and applications.

The method presented in Sections 2 and 3 is but a part of what was in the Springer Lecture Notes [F65a].

In the first place, the trace of the square of the Green's operator G is only one of many quantities in terms of which a lower bound for an eigenvalue λ_j of L was obtained. The book gave bounds analogous to (2.5) in terms of what Fichera called the orthogonal invariants

$$\mathfrak{I}_s^q(G) = \sum_{1 \leq i_1 < i_2 < \cdots < i_s} (1/\lambda_{i_1})^q (1/\lambda_{i_2})^q \cdots (1/\lambda_{i_s})^q.$$

They may be defined as the coefficients in the Taylor series of the Fredholm function $\Pi[1-\lambda/(\lambda_j)^q]$, and can be computed as explicit integrals if the kernel g of G (the Green's function) is known. $\mathfrak{I}_0^q(G)$ is defined to be 1, and $\mathfrak{I}_1^q(G)$ is the trace of G^q. $\mathfrak{I}_s^q(G)$ is a nondecreasing function of the eigenvalues $1/\lambda_i$ of G, and the derivation of the inequality (2.5) immediately gives the generalization

$$(1/\lambda_j)^q \leq (1/\lambda_j^{(n)})^q + \frac{\mathfrak{I}_s^q(G) - \sum_{1 \leq i_1 < i_2 < \cdots < i_s \leq n} (1/\lambda_{i_1}^{(n)})^q (1/\lambda_{i_2}^{(n)})^q \cdots (1/\lambda_{i_s}^{(n)})^q}{\sum_{\substack{1 \leq i_1 < i_2 < \cdots < i_{s-1} \leq n \\ i_\alpha \neq j}} (1/\lambda_{i_1}^{(n)})^q (1/\lambda_{i_2}^{(n)})^q \cdots (1/\lambda_{i_{s-1}}^{(n)})^q}. \tag{4.1}$$

When $s = 1$, the denominator is to replaced by 1. The inequality (2.5) is the special case where $q = 2$ and $s = 1$.

Fichera not only introduced the method of orthogonal invariants in [F65]. He also proved its convergence by showing that the invariants of $K^{(\ell)}$ decrease to the corresponding invariants of G.

Following in the footsteps of his teacher Mauro Picone, Gaetano Fichera always tested his numerical methods with computations, and had his students and associates do the same. Most of the papers which present the above method and its extensions contain numerical approximations to quantities of physical interest. For instance, bounds for many vibration frequencies of a clamped square plate were computed by De Vito, Fichera, Fusciardi, and Schaerf [DFFS66], while those for a clamped circular plate were found by Mirella Schaerf [Sch69]. In a later work [F78] Fichera produced accurate upper and lower bounds for the first 49 eigenvalues of the clamped square plate, and used these numbers to demonstrate the slow convergence of the eigenvalues to their asymptotic form. Lucilla Bassotti [B78] obtained analogous results for the vibration frequencies of an elastic cube. These works are discussed in [F85].

In the same spirit, Francesco Scarpini [Sca65a,b] approximated the eigenvalues of two singular ordinary differential operators. Maria Pia Colautti showed how to apply the method to the eigenfrequencies of a vibrating incompressible elastic medium [C67], and computed a number of such frequencies for a spherical body [C67/8]. She also computed the eigenfrequencies of a beam with linearly varying cross section [C68]. Manfredi Romano applied the method to approximate vibration frequencies of elastic structures [R75,CR] and buckling loads [R76]. Flavia Lanzara [L98] has recently applied the method to some vibrating membrane problems with Dirichlet boundary conditions.

Fichera's construction of an integral operator K such that $K - G$ is positive definite is particularly simple for Dirichlet boundary conditions, which we treated above. When natural (i.e., Neumann or Robin) conditions are prescribed on part or all of the boundary, the derivation of the analog of the identity (2.6) requires that the the kernel $r(x, y; \xi, \eta)$ is not only a fundamental solution, but that it also satisfies all the natural boundary conditions. Finding a fundamental solution with this property tends to be difficult, even when the coefficients are constant. Fichera [1973] studied the Neumann problem for a second order scalar operator. He observed that if one knows a diffeomorphism from the domain Ω onto a ball, the problem can be transformed to a Neumann problem on the ball, whose Neumann function is known. Lanzara showed [L92,L93a,b] that this idea can be extended to the approximation of the vibration frequencies and the Green's function of a non-isotropic nonhomogeneous elastic body when either the displacement or the force vanishes on the whole boundary and even when the force vanishes on a part of the boundary, while the displacement vanishes on the remainder of the boundary.

Gaetano Fichera also used his method to attack some related problems. Fichera and Maria Adelaide Sneider [FS74] reduced the classical problem of finding the capacitance of a cube to finding the largest eigenvalue of a certain operator, and then applied the above methods to obtain accurate upper and lower bounds for the capacitance.

Fichera also made good use of the invariants $\mathcal{I}_s^q(G)$ for approximating quantities other than eigenvalues. For example, he observed that if p denotes the multiplicity of the largest eigenvalue $1/\lambda_1$ of the operator G, then the quantity $\lambda_1^{qs}\mathfrak{I}_s^q(G)$, decreases to $\binom{p}{s}$ as $n \to \infty$. This permits one to find an arbitrarily close upper bound for p. Lower bounds are obtained in terms of slightly more complicated expressions [F72]. By using a formula of Maria Pia Colautti [C63] for the Betti numbers of a differentiable variety, Fichera [F72] was able to construct an integral operator for each dimension k so that the Betti number b_k is bounded above by the multiplicity of the largest eigenvalue of the operator. This together with the multiplicity bound enabled him to obtain a bound for each Betti number.

Gaetano Fichera made a great many other contributions to the approximation of eigenvalues. The most important of these is a proof of the convergence of Picone's method [F55], which is still one of a very small number of ways of approximating the eigenvalues of a non-self-adjoint operator. In addition to introducing the method of orthogonal invariants, the 1965 Springer Lecture notes also proved that the method of A. Weinstein can be derived from that of N. Aronszajn by the introduction of suitable Hilbert spaces. I have tried to list all of Fichera's papers on eigenvalue approximation in the bibliography.

While Fichera's method works very well on a great many problems, it is not a panacea. In particular, it only works for an operator with a compact inverse. Thus it cannot be used for those problems in quantum mechanics which have continuous as well as discrete spectrum. As we have already mentioned, the method is more difficult to apply to problems with natural boundary conditions than to those with Dirichlet boundary conditions. Fichera's trick of mapping the domain onto a ball can only work when the domain is simply connected. Nevertheless, when the method does work, it does not require explicit knowledge of the eigenvalues or the eigenvectors of a base operator, and the computations avoid finding the zeros and poles of a Weinstein determinant, or even the finding the eigenvalues of a matrix other than the Rayleigh-Ritz matrix.

It will be shown elsewhere [Wb98] that a simpler and more general construction of the base operator K based on a more general form of the hypercircle method permits Fichera's method to be applied to a wide class of problems with arbitrary symmetric boundary conditions on arbitrary domains. Thus, Fichera's method of orthogonal invariants may well be the computational method of choice for a great many problems of interest. It deserves to be a standard weapon in the arsenal of the numerical analyst.

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