Math 8245 **Homework 3, due Monday 27 October** **PJW**

GAP questions:

1. The Mathieu group M_{12} may be generated by permutations

$$(1,2,3,4,5,6)(7,8,9,10,11,12) \quad \text{and} \quad (1,9,12,7,11)(6,2,8,3,5).$$

a) Make a stabilizer chain for M_{12} and determine what the orbits $\Delta^{(i)}$ are.
b) What is the smallest possible size of a base that a group of size $|M_{12}|$ acting on 12 points can have?
c) For each i in $\{1,\ldots,12\}$ find a word in the generators of M_{12} which sends 1 to i. [You may wish to modify the function `righttransversal` which formed part of the code available in Lesson 6 so that it produces a word in the generators, or it is probably faster to do it by hand.]
d) The stabilizer of 1 in M_{12} is M_{11}. Compute a set of generators for $\mathrm{Stab}_{M_{12}}(1)$, expressing them as words in the generators of M_{12}.

2. Suppose that `grp` is a permutation group and that `sc:=StabChain(grp)` is a stabilizer chain for the group. The following is an attempt to write a function `iselement` of arguments `sc` and `g` which returns `true` precisely when `g` is a member of the permutation group `grp`.

```
iselement:=function(sc,g)
  local h,i;
  if sc.generators=[] then return g=();
  fi;
  h:=g;
  while sc.orbit[1]^h<> sc.orbit[1] do
    h:=h*sc.transversal[1^h];
  od;
  iselement(sc.stabilizer,h);
end;
```

There are some deliberate mistakes and omissions in the code. Make corrections to the code so that it works, correctly determining whether each of the permutations $(1,2,3,7,11)(4,9,6,10,5)$ and $(1,12)$ belong to

$$M_{11} = \langle (1,2,3,4,5,6,7,8,9,10,11), (3,7,11,8)(4,10,5,6) \rangle.$$

Answer the following questions:
(a) Why is the line `h:=g` present in the code? Is it necessary?
(b) Explain why it is that after some small changes of a typagrophical natrue the code will run without errors, but does not produce any answer. What should be done to correct this?

Theory questions: These are taken from the end of section 3, and have the same numbering as the questions there.

1. Suppose that V is a representation of G over $\mathbb{C}$ for which $\chi_V(g) = 0$ if $g \neq 1$. Show that $\dim V$ is a multiple of $|G|$. Deduce that $V \cong \mathbb{C}G^n$ for some n. Show that if W is any representation of G over $\mathbb{C}$ then $\mathbb{C}G \otimes_{\mathbb{C}} W \cong \mathbb{C}G^{\dim W}$ as $\mathbb{C}G$-modules.

8. Show that if every element of a finite group G is conjugate to its inverse, then every character on G is real-valued.
   Conversely, show that if every character on G is real-valued, then every element of G is conjugate to its inverse.
   [Optional extra, not to be handed in: Give an example in which every element of a finite group G is conjugate to its inverse, but not every complex representation of G is equivalent to a real representation.]

9. Let $g \in G$. Prove that g lies in the centre of G if and only if $|\chi(g)| = |\chi(1)|$ for every simple complex character χ of G.

10. Here is a column of a character table:

g
1
-1
0
-1
-1
$\frac{-1+i\sqrt{11}}{2}$
$\frac{-1-i\sqrt{11}}{2}$
0
1
0

   (a) Find the order of g.
   (b) Prove that $g \notin Z(G)$.
   (c) Show that there exists an element $h \in G$ with the same order as g but not conjugate to g.
   (d) Show that there exist two distinct simple characters of G of the same degree.

Extra questions:

3. Write a function `OrbitInfo:=function(grp,i)` whose arguments are a permutation group `grp` and an integer `i`, which returns a list `[a,b]` where `a` is a list starting with `i` whose entries are the orbit containing `i` and where `b` is a list whose entries are either undefined or are taken from the given generators of `grp`, with the property that `b[j]` is defined if and only if `j` is in the same orbit as `i`, and then `i^b[j]` either appears earlier in `a` or is `a[1]`.
(You are thus asked to produce a list `[sc.orbit,sc.transversal]` where `sc` is a stabilizer chain. However, I would like you to write code for yourself, starting from scratch.)

4. The Mathieu group M_{24} may be generated by permutations

$$(1,2,3,4,5,6,7)(8,9,10,11,12,13,14,15,16,17,18,19,20,21)(22,23)$$

and

$$(1,2,5,7,15,20,14,23,21,11,16,19,24,6,8,4,17,3,10,13,18)(9,22,12).$$

a) Make a stabilizer chain for M_{24} and determine the lengths of the orbits $\Delta^{(i)}$.
b) What is the smallest size of a base for a group of size $|M_{24}|$ acting on 24 points?
[I found the following function useful:

```
orbitlengths:=function(sc)
  local a;
  a:=ShallowCopy(sc);
  while IsBound(a.stabilizer) do
    Print("Orbit of length ", Length(a.orbit), "\n");
    a:=ShallowCopy(a.stabilizer);
  od;
  return;
end;
```

I used `ShallowCopy` a couple of times. Was it strictly necessary?]