Math 8245 **Homework 5, due Monday December 8** **PJW**

Some of the questions on this sheet are new questions which I have added to the text, and I am eager to try them out. I am not sure if they are free of errors, or are reasonable questions to ask. Any feedback you can give me about this will be greatly appreciated.

I think 7 questions is too many, so instead please do this: select 5 of the 7 questions and do the 5 which you choose. I particularly hope that some people will do question 11, because I am not sure if it is correct.

These are some new questions taken from the end of **Section 4**, and have the same numbering as the questions there.

8. Let V be a representation of G over a field k of characteristic zero. Prove that the symmetric power $S^n(V)$ is isomorphic as a kG-module to the space of symmetric tensors in $V^{\otimes n}$.

9. Let U, V be kG-modules where k is a field, and suppose we are given a non-degenerate bilinear pairing
$$\langle\ ,\ \rangle : U \times V \to k$$
which has the property $\langle u, v\rangle = \langle gu, gv\rangle$ for all $u \in U$, $v \in V$, $g \in G$. If U_1 is a subspace of U let $U_1^\perp = \{v \in V \mid \langle u, v\rangle = 0 \text{ for all } u \in U_1\}$ and if V_1 is a subspace of V let $V_1^\perp = \{u \in U \mid \langle u, v\rangle = 0 \text{ for all } v \in V_1\}$.
   (a) Show that $V \cong U^*$ as kG-modules, and that there is an identification of V with U^* so that $\langle\ ,\ \rangle$ identifies with the canonical pairing $U \times U^* \to k$.
   (b) Show that if U_1 and V_1 are kG-submodules, then so are $U_1^\perp$ and $V_1^\perp$.
   (c) Show that if $U_1 \subseteq U_2$ are kG-submodules of U then
$$U_1^\perp / U_2^\perp \cong (U_2/U_1)^*$$
   as kG-modules.
   (d) Show that the composition factors of U^* are the duals of the composition factors of U.

11. Let V be a kG-module where k is a field, and let $\langle\ ,\ \rangle : V \times V^* \to k$ be the canonical pairing between V and its dual, so $\langle v, f\rangle = f(v)$.
   (a) Show that the specification $\langle v_1 \otimes \cdots \otimes v_n, f_1 \otimes \cdots \otimes f_n\rangle = f_1(v_1)\cdots f_n(v_n)$ determines a non-degenerate bilinear pairing $\langle\ ,\ \rangle : V^{\otimes n} \times (V^*)^{\otimes n} \to k$ which is invariant both for the diagonal action of G and the action of S_n given by permuting the positions of the tensors.
   (b) Let I and J be the subspaces of $V^{\otimes n}$ which appear in the definitions of the symmetric and exterior powers, so $S^n(V) = V^{\otimes n}/I$ and $\Lambda^{\otimes n} = V^{\otimes n}/J$. Show that $I^\perp$ equals the space of symmetric tensors in $(V^*)^{\otimes n}$, and that $J^\perp$ equals the space of skew-symmetric tensors in $(V^*)^{\otimes n}$ (at least, when $\operatorname{char} k \neq 2$).

(c) Show that $(S^n(V))^* \cong \mathrm{ST}^n(V^*)$, and that $(\Lambda^n(V))^* \cong \mathrm{SST}^n(V^*)$, where ST^n denotes the symmetric tensors, and in general we define the skew-symmetric tensors $\mathrm{SST}^n(V^*)$ to be $J^\perp$.

12. Let $G = C_2 \times C_2$ be the Klein four group with generators a and b, and $k = \mathbb{F}_2$ the field of two elements. Let V be a 3-dimensional space on which a and b act via the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Show that $S^2(V)$ is not isomorphic to either $ST^2(V)$ or $ST^2(V)^*$, where ST denotes the symmetric tensors. [Hint: Compute the dimensions of the spaces of fixed points of these representations.]

The following are taken from the new end of **Section 6**, and have the same numbering as the questions there.

3. Prove that if N is a normal subgroup of G and k is a field then $\mathrm{Rad}(kN) = kN \cap \mathrm{Rad}(kG)$.

7. Let H be a subgroup of G.
   (a) Let $\overline{H} = \sum_{h \in H} h$ be the sum of the elements of H, as an element of RG. Show that $RG \cdot \overline{H} \cong R \uparrow_H^G$ as RG-modules.
   (b) Let IH be the augmentation ideal of RH, as a subset of RG. Show that $RG \cdot IH \cong IH \uparrow_H^G$ as RG-modules, and that $RG/(RG \cdot IH) \cong R \uparrow_H^G$ as RG-modules.

15. Let Ω be a transitive G-set and k a field. Let $k\Omega$ be the corresponding permutation module. There is a homomorphism of kG-modules $\epsilon : k\Omega \to k$ defined as $\epsilon(\sum_{\omega \in \Omega} a_\omega \omega) = \sum_{\omega \in \Omega} a_\omega$. Let $\overline{\Omega} = \sum_{\omega \in \Omega} \omega \in k\Omega$.
   (a) Show that every kG-module homomorphism $k\Omega \to k$ is a scalar multiple of ϵ.
   (b) Show that the fixed points of G on $k\Omega$ are $k\Omega^G = k \cdot \overline{\Omega}$.
   (c) Show that $\epsilon(\overline{\Omega}) = 0$ if and only if $\mathrm{char}\, k \mid |\Omega|$, and that if this happens then $\overline{\Omega} \in \mathrm{Rad}\, k\Omega$ and the trivial module k occurs as a composition factor of $k\Omega$ with multiplicity ≥ 2.
   (d) Show that if $\epsilon(\overline{\Omega}) \neq 0$ then ϵ is a split epimorphism and $\overline{\Omega} \notin \mathrm{Rad}\, k\Omega$.
   (e) Show that kG is semisimple if and only if the regular representation kG has the trivial module k as a direct summand (i.e. k is a projective module).